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Research article

Stability results for neutral fractional stochastic differential equations

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Abstract: Many techniques have been recently employed by researchers to address the challenges posed by fractional differential equations. In this paper, we investigate the concept of Ulam-Hyers stability for a class of neutral fractional stochastic differential equations by using the Banach fixed point theorem and the stochastic analysis techniques. An example is presented at the end of the paper to show the interest and the applicability of the results.

Keywords: fractional calculus; stochastic systems
Mathematics Subject Classification: 26A33, 39B82, 60H10

1. Introduction

The purpose of fractional analysis is to extend derivatives with integer orders to non-integer orders. In the literature, many dynamical systems are described by a fractional-order dynamical model, generally according to the notion of fractional differentiation or integration. The study of fractional order systems is more delicate than the study of their counterparts in classical derivatives.

Classical calculus is based on the differentiation and integration of integer order. However, the concept of fractional calculus has enormous potential to change the way we see, model and control the world around us. Several theoretical and experimental studies show that certain electrochemical [1–3], thermal [4–6], viscoelastic [7–9] and mechanical systems [10–12] are governed by differential equations with non-integer derivatives. Therefore, the use of the classical models based on the derivation with integer order is not appropriate. Thus, models based on differential equations with non-integer derivatives have been developed, like physical models [13–16] and mechanical models [4, 9].

The origins of fractional calculus dates back to the late 17th century, when Newton and Leibniz developed the foundations of differential and integral calculus, but it was only during the last three decades that fractional calculus has had the most interest and applications [4, 8, 9].

In 1940, Ulam has introduced the stability question of the solutions of functional equations [17]. Then, Hyers gave the first answer to Ulam's problem in Banach spaces in 1941 [18]. After Hyers's answer, many scientists were interested in the Ulam-Hyers stability (UHS) [19, 20], the Ulam-Hyers-Rassias stability [21, 22] and for the case when $\alpha \in (\frac{1}{2}, 1)$, see [23].

In the literature, there is no existing work which investigates the qualitative study like the EU and UHS for a class of neutral fractional stochastic differential equations (CNFSDE). So, it is an interesting challenge to cover this gap. In this sense, this paper generalizes the works in [19, 21] to the neutral case. Different from the results in [19–21], the highlights of this paper are as follows:

(*i*) Investigate the existence, uniqueness (EU) and UHS of solution of CNFSDE by employing the Banach fixed point theorem (BFT) and the techniques of stochastic calculus like the Cauchy-Schwartz inequality and Itô's isometry formula.

(*ii*) The neutral term and the fractional operator make our systems much more sophisticated.

(*iii*) A numerical example is presented, as an application of the theoretical obtained results.

The paper is organized as follows: Section 2 is devoted to the basic classical notions and results. Section 3 is devoted to the fundamental results about the EU and UHS of CNFSDE. We present an example in Section 4 to make our results much more applicable. We give a conclusion in Section 5 to give perspective and present an idea for a future work.

2. Preliminaries

Let $\alpha > 0$. Then,

$$\{\Omega, \mathbb{F}, \mathcal{F} := (\mathbb{F}_\varrho)_{0\leq\varrho\leq\alpha}, \mathbb{P}\},$$

is a complete probability space and $W(\varrho)$ is an d-dimensional Brownian motion.

Let

$$\mathcal{X}_\varrho = L^2(\Omega, \mathbb{F}_\varrho, \mathbb{P}) \qquad (\forall\ \varrho \in [0, \alpha])$$

be the family of all $\mathbb{F}_\varrho$-measurable and mean-square integrable functions $h = (h_1, ..., h_d)^T : \Omega \to \mathbb{R}^d$.

Definition 2.1 ([9]). *Given* $0 < \varpi < 1$. *The CFD is given by,*

$$^{C}D^{\varpi} g(s) = \frac{1}{\Gamma(1-\varpi)} \frac{d}{ds} \int_0^s (s-\omega)^{-\varpi} (g(\omega) - g(a)) d\omega. \tag{2.1}$$

Definition 2.2 ([9]). *The MLF is defined by:*

$$E_\varpi(s) = \sum_{q=0}^{+\infty} \frac{s^q}{\Gamma(q\varpi + 1)},$$

where $\varpi > 0$ *and* $s \in \mathbb{C}$.

Consider the following CNFSDE with respect to the Caputo derivative:

$$ {}^{C}D^{\varpi_2}\zeta(\varrho) - {}^{C}D^{\varpi_1}h(\varrho,\zeta(\varrho)) = f(\varrho,\zeta(\varrho)) + g(\varrho,\zeta(\varrho))\frac{\mathrm{d}W(\varrho)}{\mathrm{d}\varrho}, \qquad (0 \le \varrho \le \alpha), \tag{2.2} $$

with initial condition $\zeta(0) = \omega$, $0 < \varpi_1 < \frac{1}{2}$, $\frac{1}{2} + \varpi_1 < \varpi_2 < 1$ and $f, g, h : [0,\alpha] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are measurable.

The associated integral equation of CNFSDE (2.2) is given by:

$$ \begin{aligned} \zeta(\varrho) = \omega &- \frac{\varrho^{\varpi_2-\varpi_1}}{\Gamma(\varpi_2-\varpi_1+1)}h(0,\omega) + \frac{1}{\Gamma(\varpi_2-\varpi_1)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-\varpi_1-1}h(\nu,\zeta(\nu))\mathrm{d}\nu \\ &+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-1}f(\nu,\zeta(\nu))\mathrm{d}\nu \\ &+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-1}g(\nu,\zeta(\nu))\mathrm{d}W(\nu). \end{aligned} \tag{2.3} $$

Consider the following assumptions:

$\mathcal{H}_1$**:** There is $L > 0$ satisfying

$$ \|h(\varrho,\zeta_1) - h(\varrho,\zeta_2)\| + \|f(\varrho,\zeta_1) - f(\varrho,\zeta_2)\| + \|g(\varrho,\zeta_1) - g(\varrho,\zeta_2)\| \le L\|\zeta_1 - \zeta_2\|, \tag{2.4} $$

for all

$$ (\varrho,\zeta_1,\zeta_2) \in [0,\alpha] \times \mathbb{R}^d \times \mathbb{R}^d. $$

$\mathcal{H}_2$**:** $h(\cdot,0)$, $f(\cdot,0)$ and $g(\cdot,0)$ verify

$$ \|g(\cdot,0)\|_\infty = \operatorname{ess}\sup_{\nu\in[0,\alpha]}\|g(\nu,0)\| < \infty, \tag{2.5} $$

$$ \int_0^{\alpha}\|h(\nu,0)\|^2\,\mathrm{d}\nu < \infty, $$

and

$$ \int_0^{\alpha}\|f(\nu,0)\|^2\,\mathrm{d}\nu < \infty. $$

3. Main results

Let $\mathbb{G}^2([0,\alpha],\mathbb{R}^d)$ be the set of all processes ζ which are $\mathcal{F}$-adapted and measurable such that

$$ \|\zeta\|_{\mathbb{G}^2} = \sup_{0\le\mu\le\alpha}\|\zeta(\mu)]|_{ms} < \infty. $$

Thus $(\mathbb{G}^2([0,\alpha],\mathbb{R}^d), \|\cdot\|_{\mathbb{G}^2})$ is a Banach space.

Now, we give the definition of UHS.

Definition 3.1. *Equation* (2.2) *is Ulam-Hyers stable with respect to ε (UHS with respect to ε) if there is a constant $V > 0$ such that, for each $\varepsilon > 0$ and for each solution z of*

$$\begin{aligned}
&\mathbb{E}\Bigg\|z(\varrho) - z(0) + \frac{\varrho^{\varpi_2-\varpi_1}}{\Gamma(\varpi_2-\varpi_1+1)}h(0,z(0)) \\
&- \frac{1}{\Gamma(\varpi_2-\varpi_1)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}\, h(\mu,z(\mu))d\mu \\
&- \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\, f(\mu,z(\mu))d\mu \\
&- \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\, g(\mu,z(\mu))\mathrm{d}W(\mu)\Bigg\|^2 \\
&\leq \varepsilon,\ \forall \varrho \in [0,\alpha],
\end{aligned} \tag{3.1}$$

there is a solution $\zeta \in \mathbb{G}^2([0,\alpha],\mathbb{R}^d)$ *of* (2.2)*, with* $\zeta(0) = z(0)$*, which satisfies*

$$\mathbb{E}\|z(\varrho) - \zeta(\varrho)\|^2 \leq V\varepsilon, \qquad (\forall\ \varrho \in [0,\alpha]).$$

Lemma 3.2 ([22])**.** *Let* $d : \mathbb{G}^2([0,\alpha],\mathbb{R}^d) \times \mathbb{G}^2([0,\alpha],\mathbb{R}^d) \to \mathbb{R}_+$ *be the function such that*

$$d^2(\zeta_1,\zeta_2) = \inf\Big\{\Lambda \in [0,+\infty), \frac{\mathbb{E}\|\zeta_1(\varrho)-\zeta_2(\varrho)\|^2}{h_1(\varrho)} \leq \Lambda h_2(\varrho), \forall\varrho \in [0,\alpha]\Big\},$$

where $h_1,\ h_2 \in C([0,\alpha],\mathbb{R}_+^*)$*. Then,* $(\mathbb{G}^2([0,\alpha],\mathbb{R}^d), d)$ *is a complete metric space.*

Theorem 3.3 ([24])**.** *Suppose that* (T,d) *is a complete metric space and* $Q : T \to T$ *is a contraction (with* $\nu \in [0,1)$*). Also, let* $\vartheta \in T$*,* $\sigma > 0$ *and* $d(\vartheta, Q(\vartheta)) \leq \sigma$*. Then, there exists a unique* $y \in T$ *that satisfies* $y = Q(y)$*. Moreover, we have*

$$d(\vartheta, y) \leq \frac{\sigma}{1-\nu}. \tag{3.2}$$

Theorem 3.4. *Suppose that the hypotheses* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *hold true. Then Eq* (2.2) *is Ulam-Hyers stable with respect to* ε*.*

Proof. We consider the operator $\mathcal{A} : \mathbb{G}^2([0,\alpha],\mathbb{R}^d) \to \mathbb{G}^2([0,\alpha],\mathbb{R}^d)$ defined by

$$\begin{aligned}
(\mathcal{A}\zeta)(\varrho) = z(0) &- \frac{\varrho^{\varpi_2-\varpi_1}}{\Gamma(\varpi_2-\varpi_1+1)}h(0,z(0)) + \frac{1}{\Gamma(\varpi_2-\varpi_1)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-\varpi_1-1}\, h(\nu,\zeta(\nu))\mathrm{d}\nu \\
&+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-1}\, f(\nu,\zeta(\nu))\mathrm{d}\nu \\
&+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\nu)^{\varpi_2-1}\, g(\nu,\zeta(\nu))\mathrm{d}W(\nu), \quad \forall\varrho \in [0,\alpha].
\end{aligned} \tag{3.3}$$

We will divide our proof into two steps:

Step 1: First, we will prove that $\mathcal{A}$ is well defined. Let $\zeta \in \mathbb{G}^2([0,\alpha],\mathbb{R}^d)$, we have

$$\begin{aligned}
\|\mathcal{A}\zeta(\varrho)\|_{ms}^2 \leq 5\|z(0)\|_{ms}^2 &+ \frac{5}{\Gamma(\varpi_2-\varpi_1+1)^2}\alpha^{2(\varpi_2-\varpi_1)}\|h(0,z(0))\|_{ms}^2 \\
&+ \frac{5}{\Gamma(\varpi_2-\varpi_1)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}\, h(\mu,\zeta(\mu))\mathrm{d}\mu\right\|^2\right)
\end{aligned}$$

$$+\frac{5}{\Gamma(\varpi_2)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}f(\mu,\zeta(\mu))\mathrm{d}\mu\right\|^2\right)$$
$$+\frac{5}{\Gamma(\varpi_2)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}g(\mu,\zeta(\mu))\mathrm{d}W(\mu)\right\|^2\right). \tag{3.4}$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}h(\mu,\zeta(\mu))\mathrm{d}\mu\right\|^2\right)\\ &\le\left(\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\mathrm{d}\mu\right)\mathbb{E}\left(\int_0^{\varrho}\left\|h(\mu,\zeta(\mu))\right\|^2\mathrm{d}\mu\right)\\ &\le\frac{\varrho^{2(\varpi_2-\varpi_1)-1}}{2(\varpi_2-\varpi_1)-1}\mathbb{E}\left(\int_0^{\varrho}\left\|h(\mu,\zeta(\mu))\right\|^2\mathrm{d}\mu\right).\end{aligned} \tag{3.5}$$

Using hypothesis $\mathcal{H}_1$, we get

$$\left\|h(\mu,\zeta(\mu))\right\|^2\le 2L^2\left\|\zeta(\mu)\right\|^2+2\left\|h(\mu,0)\right\|^2. \tag{3.6}$$

Then, we have

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}h(\mu,\zeta(\mu))\mathrm{d}\mu\right\|^2\right)\\ &\qquad\le 2L^2\frac{\alpha^{2(\varpi_2-\varpi_1)-1}}{2(\varpi_2-\varpi_1)-1}\sup_{\mu\in[0,\alpha]}\mathbb{E}\left(\|\zeta(\mu)\|^2\right)+2\frac{\alpha^{2(\varpi_2-\varpi_1)-1}}{2(\varpi_2-\varpi_1)-1}\int_0^{\alpha}\|h(\mu,0)\|^2\mathrm{d}\mu<\infty.\end{aligned} \tag{3.7}$$

Similar to (3.7), we get

$$\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}f(\mu,\zeta(\mu))\mathrm{d}\mu\right\|^2\right)<\infty.$$

According to Itô's isometry formula, we obtain

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}g(\mu,\zeta(\mu))\mathrm{d}W(\mu)\right\|^2\right)\\ &\qquad=\mathbb{E}\left(\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\left\|g(\mu,\zeta(\mu))\right\|^2\mathrm{d}\mu\right).\end{aligned} \tag{3.8}$$

Hence, by using hypothesis $\mathcal{H}_1$, we find that

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}g(\mu,\zeta(\mu))\mathrm{d}W(\mu)\right\|^2\right)\\ &\qquad\le 2L^2\frac{\alpha^{2\varpi_2-1}}{2\varpi_2-1}\sup_{\mu\in[0,\alpha]}\mathbb{E}\left(\|\zeta(\mu)\|^2\right)+2\frac{\alpha^{2\varpi_2-1}}{2\varpi_2-1}\|g(\cdot,0)\|_{\infty}^2<\infty.\end{aligned} \tag{3.9}$$

Therefore, $\mathcal{A}$ is well defined.

Step 2: Consider $d_{\eta_1,\eta_2}:\mathbb{G}^2([0,\alpha],\mathbb{R}^d)\times\mathbb{G}^2([0,\alpha],\mathbb{R}^d)\to\mathbb{R}_+$ such that

$$d^2_{\eta_1,\eta_2}(\zeta_1,\zeta_2)=\sup_{\varrho\in[0,\alpha]}\frac{\mathbb{E}\|\zeta_1(\varrho)-\zeta_2(\varrho)\|^2}{\psi(\varrho)},$$

where

$$\psi(\varrho) = E_{2(\varpi_2-\varpi_1)-1}(\eta_1\varrho^{2(\varpi_2-\varpi_1)-1})E_{2\varpi_2-1}(\eta_2\varrho^{2\varpi_2-1}).$$

We will prove that $\mathcal{A}$ is contractive for some $\eta_1, \eta_2 > 0$.

Let $\zeta_1,\ \zeta_2 \in \mathbb{G}^2([0,\alpha],\mathbb{R}^d)$, we have $\forall\varrho \in [0,\alpha]$

$$\begin{aligned}(\mathcal{A}\zeta_1)(\varrho) - (\mathcal{A}\zeta_2)(\varrho) &\\ &= \frac{1}{\Gamma(\varpi_2-\varpi_1)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}\left[h(\mu,\zeta_1(\mu)) - h(\mu,\zeta_2(\mu))\right]\mathrm{d}\mu\\ &+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[f(\mu,\zeta_1(\mu)) - f(\mu,\zeta_2(\mu))\right]\mathrm{d}\mu\\ &+ \frac{1}{\Gamma(\varpi_2)}\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[g(\mu,\zeta_1(\mu)) - g(\mu,\zeta_2(\mu))\right]\mathrm{d}W(\mu).\end{aligned} \tag{3.10}$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}\|(\mathcal{A}\zeta_1)(\varrho) - (\mathcal{A}\zeta_2)(\varrho)\|^2 &\\ &\le \frac{3}{\Gamma(\varpi_2-\varpi_1)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}\left[h(\mu,\zeta_1(\mu)) - h(\mu,\zeta_2(\mu))\right]d\mu\right\|^2\right)\\ &+ \frac{3}{\Gamma(\varpi_2)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[f(\mu,\zeta_1(\mu)) - f(\mu,\zeta_2(\mu))\right]d\mu\right\|^2\right)\\ &+ \frac{3}{\Gamma(\varpi_2)^2}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[g(\mu,\zeta_1(\mu)) - g(\mu,\zeta_2(\mu))\right]\mathrm{d}W(\mu)\right\|^2\right).\end{aligned} \tag{3.11}$$

Now, by using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-\varpi_1-1}\left[h(\mu,\zeta_1(\mu)) - h(\mu,\zeta_2(\mu))\right]\mathrm{d}\mu\right\|^2\right) &\\ &\le L^2\left(\int_0^{\varrho}\mathrm{d}\mu\right)\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\,\mathbb{E}\,\|\zeta_1(\mu)-\zeta_2(\mu)\|^2\,d\mu\\ &\le L^2\alpha\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\,\mathbb{E}\,\|\zeta_1(\mu)-\zeta_2(\mu)\|^2\,d\mu,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[f(\mu,\zeta_1(\mu)) - f(\mu,\zeta_2(\mu))\right]\mathrm{d}\mu\right\|^2\right) &\\ &\le L^2\alpha\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\,\mathbb{E}\,\|\zeta_1(\mu)-\zeta_2(\mu)\|^2\,d\mu.\end{aligned} \tag{3.13}$$

By the Itô isometry formula, we obtain

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^{\varrho}(\varrho-\mu)^{\varpi_2-1}\left[g(\mu,\zeta_1(\mu)) - g(\mu,\zeta_2(\mu))\right]\mathrm{d}W(\mu)\right\|^2\right) &\\ &= \mathbb{E}\left(\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\left\|g(\mu,\zeta_1(\mu)) - g(\mu,\zeta_2(\mu))\right\|^2\,\mathrm{d}\mu\right)\end{aligned}$$

$$\leq L^2 \int_0^{\varrho} (\varrho-\mu)^{2\varpi_2-2}\,\mathbb{E}\left\|\zeta_1(\mu)-\zeta_2(\mu)\right\|^2\ \mathrm{d}\mu. \tag{3.14}$$

Then, we get

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{A}\zeta_1)(\varrho)-(\mathcal{A}\zeta_2)(\varrho)\|^2\\
&\leq \frac{3}{\Gamma(\varpi_2-\varpi_1)^2}L^2\alpha\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\,\mathbb{E}\left\|\zeta_1(\mu)-\zeta_2(\mu)\right\|^2\ \mathrm{d}\mu\\
&+\frac{3}{\Gamma(\varpi_2)^2}L^2\,(\alpha+1)\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\,\mathbb{E}\left\|\zeta_1(\mu)-\zeta_2(\mu)\right\|^2\ \mathrm{d}\mu\\
&\leq \frac{3}{\Gamma(\varpi_2-\varpi_1)^2}L^2\alpha\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\,\frac{\mathbb{E}\left\|\zeta_1(\mu)-\zeta_2(\mu)\right\|^2}{\psi(\mu)}\psi(\mu)\ \mathrm{d}\mu\\
&+\frac{3}{\Gamma(\varpi_2)^2}L^2\,(\alpha+1)\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\,\frac{\mathbb{E}\left\|\zeta_1(\mu)-\zeta_2(\mu)\right\|^2}{\psi(\mu)}\psi(\mu)\ \mathrm{d}\mu\\
&\leq \frac{3}{\Gamma(\varpi_2-\varpi_1)^2}L^2\alpha d^2_{\eta_1,\eta_2}(\zeta_1,\zeta_2)E_{2\varpi_2-1}(\eta_2\varrho^{2\varpi_2-1})\int_0^{\varrho}(\varrho-\mu)^{2(\varpi_2-\varpi_1)-2}\,E_{2(\varpi_2-\varpi_1)-1}(\eta_1\mu^{2(\varpi_2-\varpi_1)-1})\mathrm{d}\mu\\
&+\frac{3}{\Gamma(\varpi_2)^2}L^2\,(\alpha+1)\,d^2_{\eta_1,\eta_2}(\zeta_1,\zeta_2)E_{2(\varpi_2-\varpi_1)-1}(\eta_1\varrho^{2(\varpi_2-\varpi_1)-1})\int_0^{\varrho}(\varrho-\mu)^{2\varpi_2-2}\,E_{2\varpi_2-1}(\eta_2\mu^{2\varpi_2-1})\mathrm{d}\mu\\
&\leq \frac{3}{\eta_1\Gamma(\varpi_2-\varpi_1)^2}L^2\alpha d^2_{\eta_1,\eta_2}(\zeta_1,\zeta_2)\Gamma(2(\varpi_2-\varpi_1)-1)\psi(\varrho)\\
&+\frac{3}{\eta_2\Gamma(\varpi_2)^2}L^2(\alpha+1)d^2_{\eta_1,\eta_2}(\zeta_1,\zeta_2)\Gamma(2\varpi_2-1)\psi(\varrho).
\end{aligned} \tag{3.15}$$

Then, we get

$$d_{\eta_1,\eta_2}(\mathcal{A}\zeta_1,\mathcal{A}\zeta_2)\leq k d_{\eta_1,\eta_2}(\zeta_1,\zeta_2), \tag{3.16}$$

where

$$k=\sqrt{\frac{3L^2\alpha\Gamma(2(\varpi_2-\varpi_1)-1)}{\eta_1\Gamma(\varpi_2-\varpi_1)^2}+\frac{3L^2\,(\alpha+1)\,\Gamma(2\varpi_2-1)}{\eta_2\Gamma(\varpi_2)^2}}.$$

Thus, clearly, $\mathcal{S}\zeta$ is a contractive mapping on $\mathbb{G}^2([0,\alpha],\mathbb{R}^d)$ for some $\eta_1,\eta_2>0$.

Consider a function z that satisfies (3.1). We have

$$\frac{\mathbb{E}\|z(\varrho)-\mathcal{S}z(\varrho)\|^2}{\psi(\varrho)}\leq\varepsilon, \tag{3.17}$$

for all $\varrho\in[0,\alpha]$. Then,

$$d_{\eta_1,\eta_2}(z,\mathcal{A}z)\leq\sqrt{\varepsilon}. \tag{3.18}$$

In view of Theorem 3.3, there exists a unique solution ($\zeta(0)=z(0)$) such that

$$d_{\eta_1,\eta_2}(\zeta,z)\leq\frac{\sqrt{\varepsilon}}{1-k}. \tag{3.19}$$

Consequently, $\forall\varrho\in[0,\alpha]$, we have

$$\mathbb{E}\|\zeta(\varrho)-z(\varrho)\|^2\leq\frac{\psi(\alpha)}{(1-k)^2}\varepsilon. \tag{3.20}$$

Therefore, Eq (2.2) is UHS with respect to ε. □

4. Numerical example

Now, we illustrate the interest of our results by providing a theoretical example. Consider Eq (2.2) for

$$\zeta(\varrho) \in \mathbb{G}^2([0,\alpha],\mathbb{R}),$$

$$h(\varrho,\zeta(\varrho)) = \cos^2(\zeta(\varrho)),$$

$$f(\varrho,\zeta(\varrho)) = \sin^2(\zeta(\varrho)),$$

and

$$g(\varrho,\zeta(\varrho)) = \zeta(\varrho).$$

Let $(\varrho,\zeta_1,\zeta_2) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Then,

$$\|h(\varrho,\zeta_1) - h(\varrho,\zeta_2)\| \leq 2\|\zeta_1 - \zeta_2\|, \tag{4.1}$$

$$\|f(\varrho,\zeta_1) - f(\varrho,\zeta_2)\| \leq 2\|\zeta_1 - \zeta_2\|, \tag{4.2}$$

and

$$\|g(\varrho,\zeta_1) - g(\varrho,\zeta_2)\| \leq \|\zeta_1 - \zeta_2\|. \tag{4.3}$$

Consequently, the assumption $\mathcal{H}_1$ holds true. Moreover, we have

$$\|g(\cdot,0)\|_\infty = \operatorname{ess}\sup_{\varrho\in[0,\alpha]} \|g(\varrho,0)\| = 0, \tag{4.4}$$

$$\int_0^\alpha \|h(\varrho,0)\|^2 \mathrm{d}\varrho \leq \alpha. \tag{4.5}$$

and

$$\int_0^\alpha \|f(\varrho,0)\|^2 \mathrm{d}\varrho = 0. \tag{4.6}$$

Hence, assumption $\mathcal{H}_2$ holds true. Then, applying Theorem 3.4, the equation is UHS with respect to ε.

For Eq (2.2), we conduct a simulation based on Euler-Maruyama scheme with a step size 10^{-6}. Set $\varpi_1 = 0.15$, $\varpi_2 = 0.85$ and the initial data $\zeta(0) = -2$. Then, the simulations results of ζ and z with the same initial data of Eq (2.2) are shown in Figure 1. We can see from Figure 1 that the solution trajectory of the inequations (3.1) almost coincides with that of Eq (2.2). It follows that the distance between $\zeta(\varrho)$ and $z(\varrho)$ is less than a constant, which shows that Eq (2.2) is UHS according to Definition 3.1.

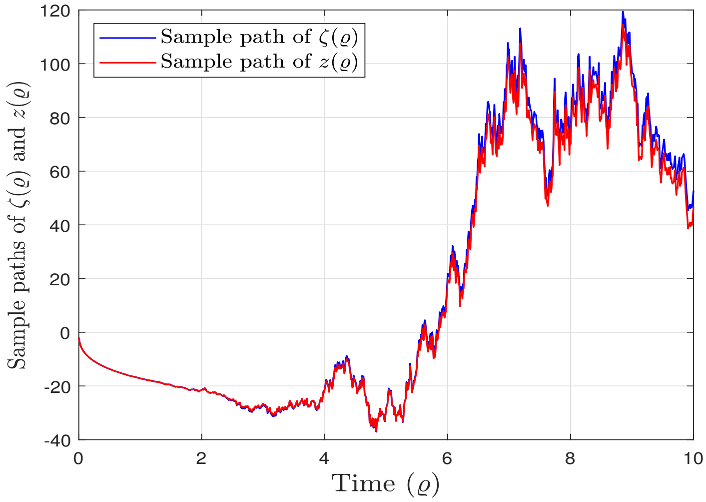

Figure 1. Trajectory simulation of $\zeta(\varrho)$ and $z(\varrho)$ on the interval $[0, 10]$ for $\varpi_1 = 0.15$ and $\varpi_2 = 0.85$.

5. Conclusions

This article investigated the EU problem for CNFSDE, and obtained the UHS result. It has been achieved that these results can be shown by using the BFT (Theorem 3.3) and some stochastic calculus techniques like the Itô isometry formula. Finally, a numerical example has been presented to illustrate the effectiveness of the theoretical results. In a future paper, it would be interesting to extend this work to the neutral case with time delay effects.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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