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Irreversible *k*-threshold conversion number of some graphs

Ramy Shaheen, Suhail Mahfud and Ali Kassem
Department of Mathematics, Tishreen University, Lattakia, Syria

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Abstract

Purpose – This paper aims to study Irreversible conversion processes, which examine the spread of a one way change of state (from state 0 to state 1) through a specified society (the spread of disease through populations, the spread of opinion through social networks, etc.) where the conversion rule is determined at the beginning of the study. These processes can be modeled into graph theoretical models where the vertex set V(G) represents the set of individuals on which the conversion is spreading.

Design/methodology/approach – The irreversible *k*-threshold conversion process on a graph *G*=(*V*,*E*) is an iterative process which starts by choosing a set S_0?*V*, and for each step *t* (*t* = 1, 2,. . .,), S_t is obtained from S_(*t*−1) by adjoining all vertices that have at least *k* neighbors in S_(*t*−1). S_0 is called the seed set of the *k*-threshold conversion process and is called an irreversible *k*-threshold conversion set (IkCS) of *G* if S_t = *V*(*G*) for some *t* = 0. The minimum cardinality of all the IkCSs of *G* is referred to as the irreversible *k*-threshold conversion number of *G* and is denoted by C_k (*G*).

Findings – In this paper the authors determine C_k (*G*) for generalized Jahangir graph J_(*s*,*m*) for 1 < *k* = *m* and *s*, *m* are arbitraries. The authors also determine C_k (*G*) for strong grids P_2? P_n when *k* = 4, 5. Finally, the authors determine C_2 (*G*) for P_n? P_n when *n* is arbitrary.

Originality/value – This work is 100% original and has important use in real life problems like Anti-Bioterrorism.

Keywords Jahangir graph, Strong grid graph, Graph conversion process, *k*-threshold conversion set

Paper type Research paper

1. Introduction

As usual $n = |V|$ and $m = |E|$ denote the numbers of vertices and edges at a graph $G(V, E)$, respectively. Let $Y \subseteq V$ and let F be a subset of E such that F consists of all edges of G which have endpoints in Y, then $H = (Y, F)$ is called an induced subgraph of G by Y and is denoted by G_Y. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$ while the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is denoted by $\deg(v)$ and $\deg(v) = |N(v)|$. An independent vertex set of a graph $G(V, E)$ is a subset of V such that no two vertices in the subset represent and edge of G. The independence number, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set of G. The term irreversible k-threshold conversion problem on graphs refers to the process of finding the least number of vertices we need to initially convert in step $t = 0$ in order to get an irreversible k-threshold conversion process, which is an iterative process that starts by choosing a seed set $S_0 \subseteq V$, and for each step $t(t = 1, 2, \ldots,)$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}.We call S_0 the seed set of the k-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $C_k(G)$) is the minimum cardinality of all the

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IkCSs of G. It is obvious that $1 \leq k \leq \Delta(G)$ and $C_1(G) = 1$ for connected graphs. The first graph model of the Irreversible k-threshold conversion problem was presented by Dreyer and Roberts in Ref. [1] where they determined the value of $C_2(G)$ for paths and cycles. They also determined $C_2(G)$ and $C_3(G)$ for grid graphs $P_3 \boxtimes P_n$. In Ref. [2] Kynčl *et al.* found an upper bound for $C_k(G)$ of toroidal grids of size $m \times n$ if $m = 4$ or $n = 4$. In Ref. [3] Adams *et al.* presented an upper bound for $C_k(G)$ of the tensor product of two arbitrary graphs G and H. In Ref. [4] Mynhardt and Wodlinger presented a lower bound for $C_k(G)$ of graphs of maximum degree $k+1$. Frances *et al.* [5] studied the relationship between IkCSs and minimum decycling sets. An upper bound for $C_k(G)$ of regular graphs was presented by Mynhardt and Wodlinger in Ref. [6]. In Ref. [7] Shaheen *et al.* studied irreversible k-threshold conversion processes on circulant graphs. In Ref. [8] Shaheen *et al.* determined $C_2(G)$ and $C_3(G)$ for the strong grid graphs $P_m \boxtimes P_n$ when $m = 2, 3$. For further information on the irreversible k-threshold conversion problem on graphs see Centeno *et al.* [9], Takaoka and Ueno [10], Kynčl *et al.* [11]. A generalized Jahangir graph $J_{s,m}$ *for* $m \geq 2$ is a graph on $sm+1$vertices, i.e. a graph consisting of a cycle $C_{sm}$with one additional vertex which is adjacent to m vertices of C_{sm} at distance s from each other on C_{sm}, see Ref. [12] for more information on Jahangir graph. Let v_{sm+1} be the label of the central vertex and $v_1, v_2, \ldots, v_{sm}$ be the labels of the vertices that incident clockwise on cycle C_{sm} so that $\deg(v_1) = 3$. We will use this labeling for the rest of the article. The vertices that are adjacent to v_{sm+1} have the labels $v_1, v_{1+s}, v_{1+2s}, \ldots v_{1+(m-1)s}$. Let P_m, P_n be two paths, we define the strong product of P_m and P_n (also called strong grid graph) as the graph $P_m \boxtimes P_n$ such that $V(P_m \boxtimes P_n) = \{(i,j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices (i_1, j_1), (i_2, j_2) are adjacent if and only if $max\{|i_2 - i_1|, |j_2 - j_1|\} = 1$. See Ref. [13] for more information on strong grids.

Proposition 1.1. [3] For $n \geq 2$; $C_2(P_n) = \frac{n+1}{2}$.

Proposition 1.2. [3] For $n \geq 3$; $C_2(C_n) = \frac{n}{2}$.

Proposition 1.3. [13] For $n \geq 2$; $C_2(P_2 \boxtimes P_n) = 2$.

Proposition 1.4. [13] For $n \geq 2$; $C_3(P_2 \boxtimes P_n) = n+1$.

Proposition 1.5. [6] For $m, n \geq 2$; $\alpha(P_m \boxtimes P_n) = \frac{m}{2}\frac{n}{2}$.

Note 1: As an immediate consequence of the definition, $C_k(G) \geq k$ for any graph G**.**

Note 2: As an immediate consequence of the definition, when studying an irreversible k-threshold conversion process on a graph $G(V, E)$ all vertices $\{v \in V\ ;\ \deg(v) < k\}$ must be included in the seed set S_0, otherwise the process will fail because none of these vertices can satisfy the conversion rule.

Note 3: For Jahangir graph $J_{s,m}$, we denote the set of vertices of degree 3 which consists of $v_1, v_{1+s}, \ldots v_{1+(m-1)s}$ by R. So, $R = \{v_{1+is}\ : i = 0, 1, \ldots, m-1\}$

Note 4: In every figure of this article, we assign the black color to the converted vertices and the white color to unconverted ones.

2. Main results

In this section we determine $C_k(G)$ for generalized Jahangir graph $J_{s,m}$ for $1 < k \leq m$ and s, m are arbitraries. We also determine $C_k(G)$ for strong grids $P_2 \boxtimes P_n$ when $k = 4, 5$. Then we determine $C_2(G)$ for $P_n \boxtimes P_n$ when n is arbitrary.

2.1 $C_k(J_{s,m})$

In this sub-section we find $C_k(G)$ of generalized Jahangir graph $J_{s,m}$ for $1 < k \leq m$ and s, m are arbitraries**.**

Theorem 2.1. *For $s, m \geq 2$, $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$.*

Proof. Let $J_{s,m}$ be a Jahangir graph on which an irreversible 2-threshold conversion process is being studied with a seed set S_0, let $U \subseteq V - S_0$ and $v_{sm+1} \notin U$, then U is a 2-unconvertible set of $J_{s,m}$ if it satisfies the following condition:

$$\text{For all } u \in U : \ |N(u) \cap U| \geq 2.$$

Which means each vertex $u \in U$ is unconverted and is adjacent to at least 2 vertices of U at $t = 0$. Since $\deg(u) \leq 3$ then $|N(u) \cap S_0| < 2$ and the conversion cannot reach any vertex of U during any step of the process. Therefore, we try to avoid having any version of U on $J_{s,m}$ when choosing S_0. We imply that the following sets $A_1 = \{a, b, c\}$ with ($\deg(a) = \deg(c) = 2$ and $\deg(b) = 3$), $B_1 = \{x, y\}$ (with $\deg(x) = \deg(y) = 2$) are 2-unconvertible sets on $J_{s,m}$. Both A_1 and B_1 are represented in Figure 1.

By Proposition 1.1, we have $C_2(P_n) = \frac{n+1}{2}$. It is obvious that the $\frac{n+1}{2}$ – IkCS of a path P_n must contain the end vertices $\{v_1, v_n\}$ otherwise the spread will never reach them. The vertices of R divide C_{sm} into m paths (each of which consists of $s-1$ vertices and they are separated by the vertices of R). We denote these paths as follows:

$$\begin{aligned} P_{s-1}^{(1)} &= v_2 \ldots v_s, \\ P_{s-1}^{(2)} &= v_{2+s} \ldots v_{2s}, \\ &\vdots \\ P_{s-1}^{(m)} &= v_{2+(m-1)s} \ldots v_{ms}. \end{aligned}$$

We consider the following subcases:

Case 1. s is even.

In this case, each path $P_{s-1}^{(i)} : 1 \leq i \leq m$ contains an odd number of vertices. We divide $V(P_{s-1}^{(i)})$ into two sets:

$$EP_i = \{v_{2+(i-1)s}, v_{4+(i-1)s}, \ldots, v_{is}\} \text{ which consists of } \frac{s-1}{2} \text{ vertices.}$$

$$OP_i = \begin{cases} \varnothing \ \textit{if } s = 2; \\ \{v_{3+(i-1)s}, v_{5+(i-1)s}, \ldots, v_{is-1}\} \textit{ which consists of } \dfrac{s-1}{2} \textit{ vertices if } s \geq 4. \end{cases}$$

We define a family of sets $D = \left\{ D_i : 1 \leq i \leq m, \text{ where } D_i = \begin{Bmatrix} EP_i \textit{ if } i \textit{ is odd}; \\ OP_i \textit{ if } i \textit{ is even}. \end{Bmatrix} \right.$

The process goes as follows:

$t = 0$: We convert the vertices of $S_0 = D \cup \{v_{sm+1}\}$.

$t = 1$: The conversion spreads to:

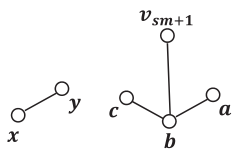

Figure 1. $A_1 = \{a, b, c\}$, $B_1 = \{x, y\}$ are 2-unconvertible on $J_{s,m}$

- The vertices of $\{OP_i : i\, is\, odd\}$.
- The vertices of $\{EP_i - \{v_{2+(i-1)s}, v_{is}\} : i\, is\, even\}$.
- The vertices of degree 3 (vertices of R).

$t = 2$: The conversion spreads to $\{v_{2+(i-1)s}, v_{is} : i\, is\, even\}$.

By the end of step $t = 2$, the conversion is spread to $V(J_{s,m})$ entirely and the process succeeds. It is obvious that $|S_0| = \begin{cases} \frac{m}{2}\left(\frac{s-1}{2} + \frac{s-1}{2}\right) + 1\, if\, m\, is\, even; \\ \frac{m}{2}\frac{s-1}{2} + \frac{m}{2}\frac{s-1}{2} + 1\, if\, m\, is\, odd. \end{cases}$

Which means:

$$C_2(J_{s,m}) \leq \frac{m(s-1)}{2} + 1 \quad (1)$$

Figure 2 shows that $C_2(J_{6,4}) \leq 11$.

We imply that the sets $A_1 = \{a, b, c\}, B_1 = \{x, y\}$ represented in Figure 1 are 2-unconvertible on $J_{s,m}$. We notice that D is the only $\frac{m(s-1)}{2}$ - seed set that does not leave any versions of A_1 or B_1 on C_{sm} and every k-seed set with $k < \frac{m(s-1)}{2}$ will leave some versions of A_1 or B_1 on C_{sm}. Let us assume that D_0 is a IkCS of cardinality $\frac{m(s-1)}{2}$, we consider the following subcases:

Case 1.a. $v_{sm+1} \notin D_0$, which means $D_0 \subseteq V(C_{sm})$, and since $|D_0| = \frac{m(s-1)}{2}$, then $D_0 = D$ as we found earlier. However, since $C_2(C_{sm}) = \frac{sm}{2}$ by Proposition 1.2, it is impossible to convert all the vertices of C_{sm} depending only on D. Therefore, we need to convert v_{sm+1} at some point and benefit from it being adjacent to m vertices of C_{sm}. To achieve that we need at step $t = 0$ to choose one of three strategies:

- Convert 2 vertices of R (e.g. v_1, v_{1+s}). However, that leaves $\frac{m(s-1)}{2} - 2$ vertices in D_0 which means we end up with at least two versions of B_1 and the process fails as shown in Figure 3(a). Without loss of generality, any choice of the two vertices of R leads to the same result.
- Convert 1 vertex of R (e.g. v_1), and 2 vertices that are adjacent to a vertex of R (for example v_{2s}, v_{2+2s}), by converting the remaining $\frac{m(s-1)}{2} - 3$ vertices in D_0 in a similar way to D, we end up with two versions of B_1 and the process fails as shown in

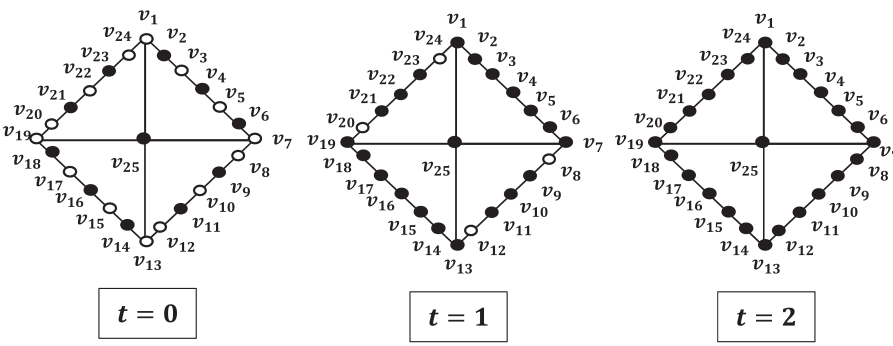

Figure 2.
$C_2(J_{6,4}) \leq 11$

Figure 3(b). Without loss of generality, any choice of the one vertex of R and the two vertices that are adjacent to a vertex of R leads to the same result.

- Convert 2 pairs of vertices each of which is adjacent to one vertex of R (e.g.$v_2, v_{sm}, v_s, v_{2+s}$), by converting the remaining $\frac{m(s-1)}{2} - 4$ vertices in D_0 in a similar way to D, we end up with two versions of B_1 and the process also fails as shown in Figure 3(c). Without loss of generality, the same result is obtained for whatever 4 vertices that each of which is adjacent to a vertex of R we choose to initially convert.

All strategies end up with two versions of B_1 on C_{sm}, and without loss of generality, we get the same results by choosing different vertices that satisfy the conditions mentioned in the three strategies above, therefore $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ when s is even and $v_{sm+1} \notin D_0$.

Case 1.b. $v_{sm+1} \in D_0$, by converting $\frac{m(s-1)}{2}$ vertices of C_{sm} we end up with two versions of B_1 (as shown in Figure 3(d)), and the process fails. Therefore, $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ in this case as well.

From Case 1.a and Case 1.b we conclude that:

$$\text{For } s \text{ is even, } C_2(J_{s,m}) > \frac{m(s-1)}{2}. \tag{2}$$

From (1) and (2) we conclude that for s is even; $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$.

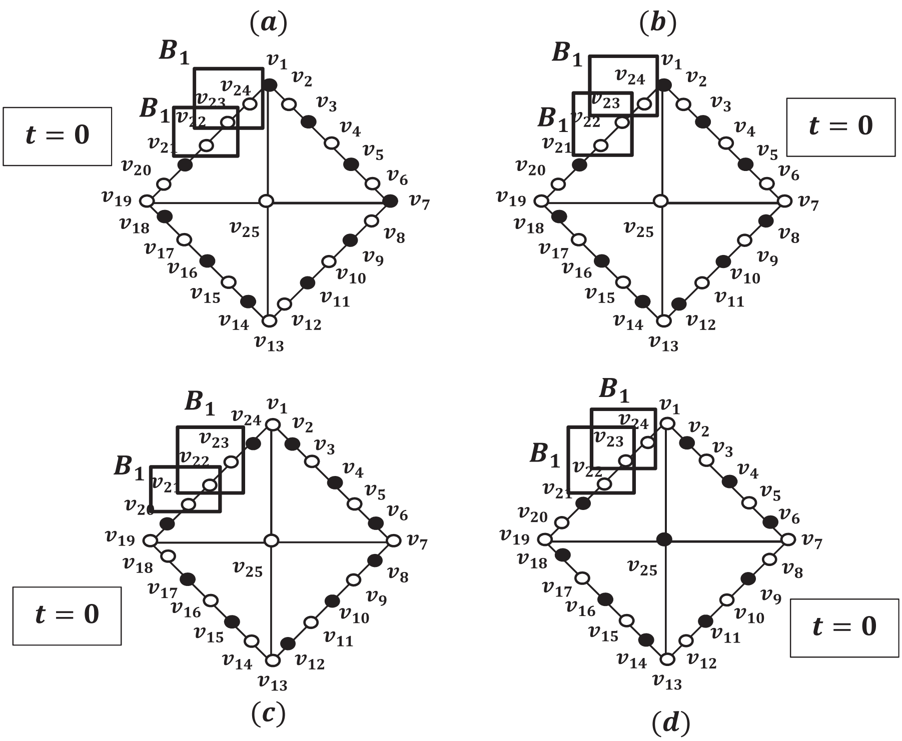

Figure 3. $C_2(J_{6,4}) > 10$

Case 2. s is odd.

In this case, each path $P_{s-1}^{(i)} : 1 \leq i \leq m$ contains an even number of vertices. We define a family of sets $D = \{D_i : 1 \leq i \leq m\}$ where $D_i = \{v_{2+(i-1)s}, v_{4+(i-1)s}, \ldots, v_{is-1}\}$ which contains $\frac{m(s-1)}{2}$ vertices. The process goes as follows:

(1) $t = 0$: We convert the vertices of $S_0 = D \cup \{v_{sm+1}\}$.

(2) $t = 1$: The conversion spreads to the vertices of $\{D_i - \{v_{is-1}\} : 1 \leq i \leq m\}$.

(3) $t = 2$: The conversion spreads to $\{v_{is-1} : 1 \leq i \leq m\}$.

By the end of step $t = 2$, the conversion is spread to $V(J_{s,m})$ entirely and the process succeeds.

It is obvious that $|S_0| = \frac{m(s-1)}{2} + 1$ which means $C_2(J_{s,m}) \leq \frac{m(s-1)}{2} + 1$. In a similar way to Case 1, S_0 is the only IkCS of cardinality $\frac{m(s-1)}{2} + 1$ because D is the only set of cardinality $\frac{m(s-1)}{2}$ that does not leave any versions of A_1 or B_1 on C_{sm}. By following the same discussion in Case 1 we conclude that $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ if s is odd, which means $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$ if s is odd.

Figure 4 illustrates a 2-conversion process on $J_{7,4}$ starting with $|S_0| = 13$.

From Case 1 and Case 2 we conclude that $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$ for $s \geq 2$.

Theorem 2.2. *For $m \geq 2$, $C_3(J_{s,m}) = m(s-1) + 1$.*

Proof. By definition all vertices with degree lower than 3 need to be added to the seed set S_0. However, in order to convert a vertex of degree 3, we need it to be adjacent to three converted vertices which means the conversion will not spread unless v_{sm+1} is initially converted. The process goes as follows:

$t = 0$: We convert the vertices of $S_0 = \{V(C_{sm}) - R\} \cup \{v_{sm+1}\}$, we implied that this set is unique.

$t = 1$: The conversion spreads to the vertices of R.

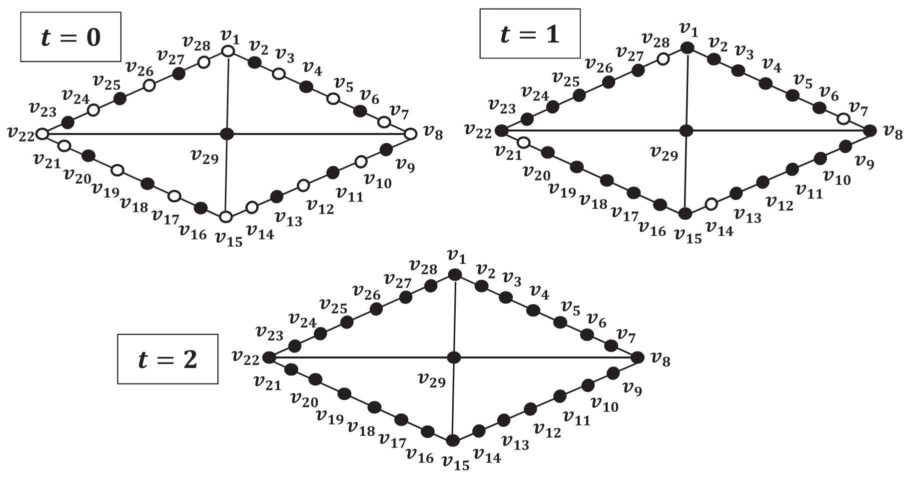

Figure 4.
A 2-conversion process on $J_{7,4}$ starting with $|S_0| = 13$

The process succeeds and $J_{s,m}$ is entirely converted by the end of step 2 which means that $C_3(J_{s,m}) \leq m(s-1)+1$, since S_0 is unique and none of its vertices can be removed, then $C_3(J_{s,m}) = m(s-1)+1$.

Theorem 2.3. *For* $4 \leq k \leq m$, $C_k(J_{s,m}) = sm$.

Proof. By definition all vertices with degree lower than 4 need to be included in S_0 which means $S_0 = V(C_{sm})$ and this set is unique. The process goes as follows:

$t = 0$: We convert the vertices of $S_0 = V(C_{sm})$.

$t = 1$: The conversion spreads to v_{sm+1}.

The process succeeds and $J_{s,m}$ is entirely converted by the end of step 2 which means that $C_k(J_{s,m}) \leq sm$, since S_0 is unique and none of its vertices can be removed, we conclude that $C_k(J_{s,m}) = sm$.

2.2 $C_k(P_m \boxtimes P_n)$

In this sub-section we determine $C_k(G)$ for strong grids $P_2 \boxtimes P_n$ when $k = 4, 5$. Then we determine $C_2(G)$ for $P_n \boxtimes P_n$ when n is arbitrary.

Theorem 2.4. For $n \geq 3$; $C_4(P_2 \boxtimes P_n) = \begin{cases} n+1 \, if \, n \, is \, odd; \\ n+2 \, if \, n \, is \, even. \end{cases}$

Proof. Let $P_2 \boxtimes P_n$ be a strong grid graph on which an irreversible 4-threshold conversion process is being studied with a seed set S_0, let $U \subseteq V - S_0$ and $\{(1,1),(1,n),(2,1),(2,n)\} \cap U = \emptyset$, then U is a 2-unconvertible set of $P_2 \boxtimes P_n$ if it satisfies the following condition:

$$\text{For all } u \in U : \ |N(u) \cap U| \geq 2.$$

Which means each vertex $u \in U$ is unconverted and is adjacent to at least 2 vertices of U at $t = 0$. Since $\deg(u) = 5$ then $|N(u) \cap S_0| \leq 3$ and the conversion cannot reach any vertex of U during any step of the process. Therefore, we try to avoid having any version of U on $P_2 \boxtimes P_n$ when choosing S_0. For $2 \leq j \leq n-1$, we imply that the following sets are 4-unconvertible:

$X_1 = \{(1,j-1),(1,j),(2,j)\}$, $X_2 = \{(2,j-1),(1,j),(2,j)\}$, $X_3 = \{(1,j),(2,j),(1,j+1)\}$, $X_4 = \{(1,j),(2,j),(2,j+1)\}$. Figure 5 shows that for $1 \leq i \leq 4$: if $X_i \cap S_0 = \emptyset$ on $P_2 \boxtimes P_6$ then none of the vertices of X_i can be converted and the process fails even if $S_0 = V - X_i$. In order to avoid having any version of X_1, X_2, X_3 or X_4 on $P_2 \boxtimes P_n$, every two adjacent columns must include at least two vertices of S_0 at $t = 0$.

We consider the following cases:

Case 1. n is odd.

Let S_0 be a seed set of an irreversible 4-threshold conversion process on $P_2 \boxtimes P_n$, since each vertex of $W = \{(1,1),(1,n),(2,1),(2,n)\}$ is of degree 3, then $W \subset S_0$, otherwise the process fails. Since we are trying to avoid having two adjacent columns that include less than two

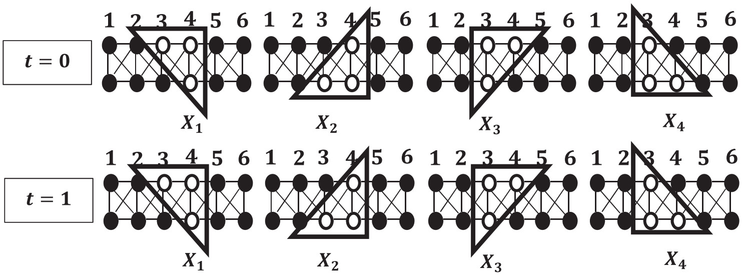

Figure 5. X_1, X_2, X_3 and X_4 are 4-unconvertible on $P_2 \boxtimes P_6$

vertices of S_0 at $t=0$, we choose $S_0=\{(1,2l+1),(2,2l+1)\colon 0\le l\le\frac{n-1}{2}\}$, which means S_0 contains all the vertices of the odd indexed columns of $P_2\boxtimes P_n$ and $|S_0|=n+1$. The process goes as follows:

$$t=0\colon\ S_0=\left\{(1,2l+1),(2,2l+1)\colon 0\le l\le\frac{n-1}{2}\right\}.$$

$$t=1\colon\ S_1=S_0\cup\left\{(1,2l),(2,2l)\colon 1\le l\le\frac{n-1}{2}\right\}=V(P_2\boxtimes P_n).$$

This means S_0 is an I4CS on $P_2\boxtimes P_n$. Therefore, if n is odd then:

$$C_4(P_2\boxtimes P_n)\le n+1 \tag{3}$$

Figure 6 illustrates that $C_4(P_2\boxtimes P_9)\le 10$.

Now let us assume that D_0 is a 4-conversion seed set of cardinality n on $P_2\boxtimes P_n$. Since W must be contained in D_0, this means we need to distribute the remaining $n-4$ vertices of D_0 on the remaining $n-2$ columns $(2,3,\ldots,n-1)$ without having two adjacent columns that include less than two vertices of D_0 which is impossible. Therefore, we end up with at least one version of X_1,X_2,X_3 or X_4 on $P_2\boxtimes P_n$ and the process fails. This means if n is odd:

$$C_4(P_2\boxtimes P_n)>n \tag{4}$$

From (3) and (4) we conclude that $C_4(P_2\boxtimes P_n)=n+1$ if n is odd.

Case 2. n is even.

In a similar way to Case 1, the vertices of W must be contained in the seed set S_0. We choose $S_0=\{(1,2l+1),(2,2l+1)\colon 0\le l\le\frac{n}{2}-1\}\cup\{(1,n),(2,n)\}$ which is of cardinality $n+2$. The process goes as follows:

$$t=0\colon\ S_0=\left\{(1,2l+1),(2,2l+1)\colon 0\le l\le\frac{n}{2}-1\right\}\cup\{(1,n),(2,n)\}.$$

$$t=1\colon\ S_1=S_0\cup\left\{(1,2l),(2,2l)\colon 1\le l\le\frac{n}{2}-1\right\}=V(P_2\boxtimes P_n).$$

This means S_0 is an I4CS on $P_2\boxtimes P_n$. Therefore, if n is even then:

$$C_4(P_2\boxtimes P_n)\le n+2 \tag{5}$$

Figure 7 shows that $C_4(P_2\boxtimes P_{10})\le 12$. According to the same 4-threshold conversion process, let D_0 be an I4CS of cardinality $n+1$. Since W must be contained in D_0, it is impossible to distribute the remaining $n-3$ vertices of D_0 on the $n-2$ unconverted columns at $t=0$ without having at least two adjacent columns that include less than two vertices of D_0, which means a version of X_1,X_2,X_3 or X_4 will definitely be created on $P_2\boxtimes P_n$ and the process fails. We conclude that if n is even then:

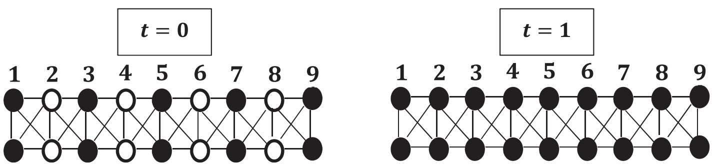

Figure 6.
$C_4(P_2\boxtimes P_9)\le 10$

$$C_4(P_2 \boxtimes P_n) > n+1 \tag{6}$$

From (5) and (6) we conclude that $C_4(P_2 \boxtimes P_n) = n+2$ if $n \geq 4$ and n is even. From Case 1 and Case 2 we conclude the requested.

Theorem 2.5. For $n \geq 3$; $C_5(P_2 \boxtimes P_n) = \begin{cases} \frac{3n+1}{2} \textit{ if n is odd;} \\ \frac{3n}{2}+1 \textit{ if n is even.} \end{cases}$

Proof. In a similar way to Theorem 2.4, the vertices of $W = \{(1,1),(1,n),(2,1),(2,n)\}$ must be included in the seed set S_0. Now we try to determine which vertices of $M = V - W$ we need to include in S_0. Since $M = \{(1,j),(2,j): 2 \leq j \leq n-1\}$, every vertex of M is of degree 5 which means there cannot be two adjacent vertices $v_1, v_2 \in M - S_0$. Otherwise, the process will fail because neither v_1 nor v_2 will be converted at any step of the conversion process. We conclude that $M - S_0$ must be an independent set. In order to make S_0 as small as possible, we try to make $M - S_0$ as large as possible. We notice that M represents the vertices of a strong grid $P_2 \boxtimes P_{n-2}$ with the difference that the end vertices of M: $(1,2),(2,2),(1,n-1),(2,n-1)$ are of degree 5 while the end vertices of a usual $P_2 \boxtimes P_{n-2}$ strong grid: $(1,1),(2,1),(1,n),$ $(2,n)$ are of degree 3, but this difference does not change that $\alpha(G_M) = \alpha(P_2 \boxtimes P_{n-2})$ which means from Proposition 1.5, $\alpha(G_M) = \frac{n-2}{2}$. We conclude that the minimum cardinality of S_0 that does not allow having two adjacent unconverted vertices is:

$|S_0| = |M| - \alpha(G_M) + |W| = 2(n-2) - \frac{n-2}{2} + 4 = 2n - \frac{n-2}{2}$. We consider the following cases for n:

Case 1. n is odd.
Since n is odd then $\alpha(G_M) = \frac{n-1}{2}$. Therefore, $C_5(P_2 \boxtimes P_n) = 2n - \frac{n-1}{2} = \frac{3n+1}{2}$.
Case 2. n is even.
Since n is even then $\alpha(G_M) = \frac{n-2}{2}$. Therefore, $C_5(P_2 \boxtimes P_n) = 2n - \frac{n-2}{2} = \frac{3n}{2} + 1$.
From Case 1 and case 2 we conclude the requested.

Theorem 2.6. For $n \geq 3$; $C_2(P_n \boxtimes P_n) = 2$.

Proof. It is known by definition that $C_k(G) \geq k$ for any G. Therefore, $C_2(P_n \boxtimes P_n) \geq 2$. Now we prove that $C_2(P_n \boxtimes P_n) \leq 2$ by finding an I2CS of cardinality 2 on $P_n \boxtimes P_n$. In order to make the conversion steps as few as possible, we start from the middle by choosing the seed set to be $S_0 = \left\{\left(\frac{n-1}{2}, \frac{n+1}{2}\right), \left(\frac{n+3}{2}, \frac{n+1}{2}\right)\right\}$. The process goes as follows:

$$t=0\text{: } S_0 = \left\{\left(\frac{n-1}{2}, \frac{n+1}{2}\right), \left(\frac{n+3}{2}, \frac{n+1}{2}\right)\right\}.$$

$$t=1\text{: } S_1 = S_0 \cup \left\{\left(\frac{n+1}{2}, \frac{n-1}{2}\right), \left(\frac{n+1}{2}, \frac{n+1}{2}\right), \left(\frac{n+1}{2}, \frac{n+3}{2}\right)\right\}.$$

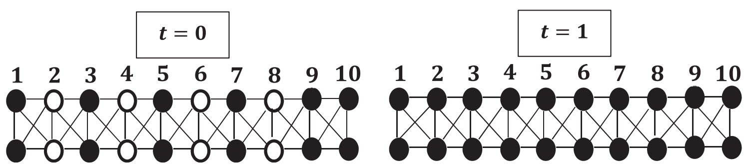

Figure 7.
$C_4(P_2 \boxtimes P_{10}) \leq 12$

$$t=2\colon\ S_2=S_1\cup\left\{\left(\frac{n-1}{2},\frac{n-1}{2}\right),\left(\frac{n-1}{2},\frac{n+3}{2}\right),\left(\frac{n+3}{2},\frac{n-1}{2}\right),\left(\frac{n+3}{2},\frac{n+3}{2}\right)\right\}.$$

$$t=3\colon\ S_3=S_2\cup\left\{\left(\frac{n-3}{2},\frac{n-1}{2}\right),\left(\frac{n-3}{2},\frac{n+1}{2}\right),\left(\frac{n-3}{2},\frac{n+3}{2}\right),\left(\frac{n-1}{2},\frac{n-3}{2}\right),\right.$$

$$\left(\frac{n+1}{2},\frac{n-3}{2}\right),\left(\frac{n+3}{2},\frac{n-3}{2}\right),\left(\frac{n+5}{2},\frac{n-1}{2}\right),\left(\frac{n+5}{2},\frac{n+1}{2}\right),\left(\frac{n+5}{2},\frac{n+3}{2}\right),$$

$$\left.\left(\frac{n-1}{2},\frac{n+5}{2}\right),\left(\frac{n+1}{2},\frac{n+5}{2}\right),\left(\frac{n+3}{2},\frac{n+5}{2}\right)\right\}.$$

$$t=4\colon\ S_4=S_3\cup\left\{\left(\frac{n-5}{2},\frac{n-1}{2}\right),\left(\frac{n-5}{2},\frac{n+1}{2}\right),\left(\frac{n-5}{2},\frac{n+3}{2}\right),\left(\frac{n-1}{2},\frac{n-5}{2}\right),\right.$$

$$\left(\frac{n+1}{2},\frac{n-5}{2}\right),\left(\frac{n+3}{2},\frac{n-5}{2}\right),\left(\frac{n+7}{2},\frac{n-1}{2}\right),\left(\frac{n+7}{2},\frac{n+1}{2}\right),\left(\frac{n+7}{2},\frac{n+3}{2}\right),$$

$$\left(\frac{n-1}{2},\frac{n+7}{2}\right),\left(\frac{n+1}{2},\frac{n+7}{2}\right),\left(\frac{n+3}{2},\frac{n+7}{2}\right),\left(\frac{n-3}{2},\frac{n-3}{2}\right),$$

$$\left.\left(\frac{n-3}{2},\frac{n+5}{2}\right),\left(\frac{n+5}{2},\frac{n-3}{2}\right),\left(\frac{n+5}{2},\frac{n+5}{2}\right)\right\}.$$

$$5\le t\le\frac{n+1}{2}\colon\ S_t=S_{t-1}\cup\left\{\left(\frac{n-2t+3}{2},\frac{n-1}{2}\right),\left(\frac{n-2t+3}{2},\frac{n+1}{2}\right),\right.$$

$$\left(\frac{n-2t+3}{2},\frac{n+3}{2}\right),\left(\frac{n-1}{2},\frac{n-2t+3}{2}\right),\left(\frac{n+1}{2},\frac{n-2t+3}{2}\right),\left(\frac{n+3}{2},\frac{n-2t+3}{2}\right),$$

$$\left(\frac{n+2t-1}{2},\frac{n-1}{2}\right),\left(\frac{n+2t-1}{2},\frac{n+1}{2}\right),\left(\frac{n+2t-1}{2},\frac{n+3}{2}\right),\left(\frac{n-1}{2},\frac{n+2t-1}{2}\right),$$

$$\left(\frac{n+1}{2},\frac{n+2t-1}{2}\right),\left(\frac{n+3}{2},\frac{n+2t-1}{2}\right),\left(\frac{n-2l+5}{2},\frac{n-2t+2l-3}{2}\right),$$

$$\left(\frac{n-2t+2l-3}{2},\frac{n+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n-2t+2l-3}{2}\right),$$

$$\left.\left(\frac{n+2l-3}{2},\frac{n+2t-2l+5}{2}\right)\colon 4\le l\le t\right\}.$$

We notice that at $t=\frac{n+1}{2}$, the spread reaches its limits horizontally and vertically (the three middle vertices of each of the first row, the last row, the first column and the last column are converted). Therefore, in the remaining steps, the conversion spreads only diagonally as follows:

$$t=\frac{n+1}{2}+1:\ S_{\frac{n+1}{2}+1}=S_{\frac{n+1}{2}}\cup\left\{\left(\frac{n-2l+5}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n-2t+2l-3}{2},\frac{n+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n+2t-2l+5}{2}\right):4\le l\le t\right\}.$$

$$t=\frac{n+1}{2}+2:\ S_{\frac{n+1}{2}+2}=S_{\frac{n+1}{2}+1}\cup\left\{\left(\frac{n-2l+5}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n-2t+2l-3}{2},\frac{n+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n+2t-2l+5}{2}\right):5\le l\le t-1\right\}.$$

For $2\le m\le\frac{n-3}{2}$ which means for $\frac{n+1}{2}+2\le t\le n-1$:

$$S_{\frac{n+1}{2}+m}=S_{\frac{n+1}{2}+m-1}\cup\left\{\left(\frac{n-2l+5}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n-2t+2l-3}{2},\frac{n+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n-2t+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n+2t-2l+5}{2}\right):m+3\le l\le t-m+1\right\}.$$

When we reach step $t=n-1$, we have $m=\frac{n-3}{2}$ which means:

$$S_{n-1}=S_{n-2}\cup\left\{\left(\frac{n-2l+5}{2},\frac{n-2(n-1)+2l-3}{2}\right),\left(\frac{n-2(n-1)+2l-3}{2},\frac{n+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n-2(n-1)+2l-3}{2}\right),\left(\frac{n+2l-3}{2},\frac{n+2(n-1)-2l+5}{2}\right):\frac{n+3}{2}\le l\le\frac{n+3}{2}\right\}$$
$$=S_{n-2}\cup\{(1,1),(1,n),(n,1),(n,n)\}=V(P_n\boxtimes P_n)$$

we conclude that S_0 is an I2CS of cardinality 2 on $P_n\boxtimes P_n$. Therefore, $C_2(P_n\boxtimes P_n)\le 2$ which means $C_2(P_n\boxtimes P_n)=2$ if n is odd. Figure 8 illustrates that $C_2(P_9\boxtimes P_9)=2$.

Case 2. n is even.

In a similar way to Case 1, we need to prove that $C_2(P_n\boxtimes P_n)\le 2$by finding an I2CS of cardinality 2 on $P_n\boxtimes P_n$. We start from the middle to make the conversion steps as few as possible. We choose the seed set to be $S_0=\left\{\left(\frac{n}{2},\frac{n}{2}\right),\left(\frac{n}{2}+1,\frac{n}{2}+1\right)\right\}$. The process goes as follows:

$$t=0:\ S_0=\left\{\left(\frac{n}{2},\frac{n}{2}\right),\left(\frac{n}{2}+1,\frac{n}{2}+1\right)\right\}.$$

$$t=1:\ S_1=S_0\cup\left\{\left(\frac{n}{2},\frac{n}{2}+1\right),\left(\frac{n}{2}+1,\frac{n}{2}\right)\right\}.$$

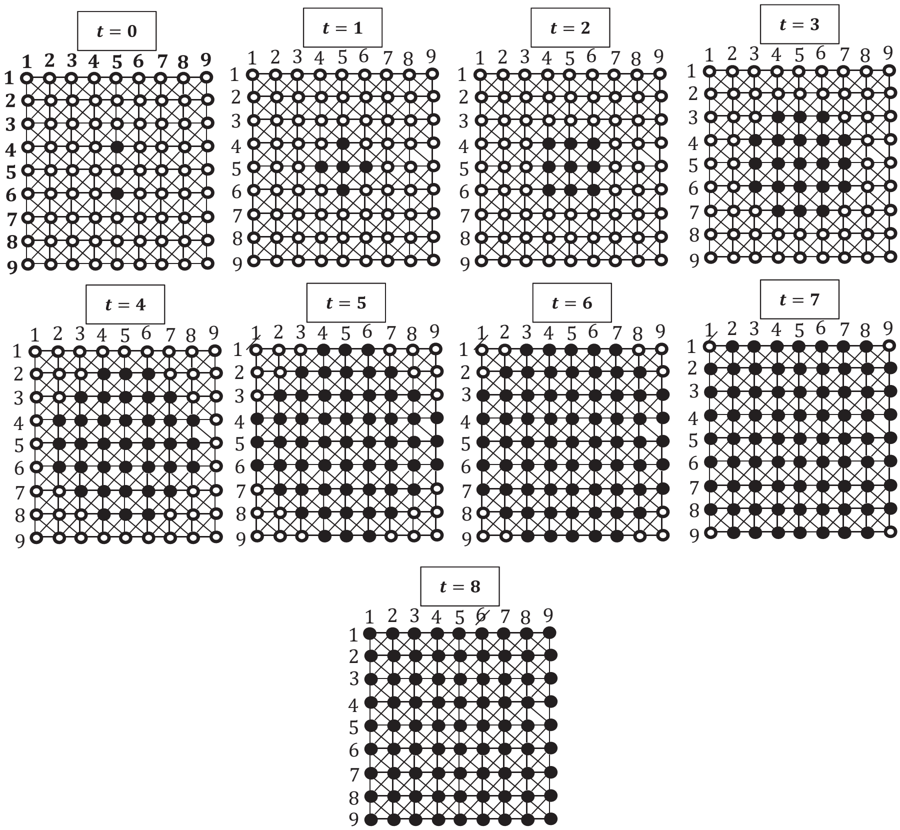

Figure 8.
$C_2(P_9 \boxtimes P_9) = 2$

$$t=2\colon\ S_2 = S_1 \cup \left\{ \left(\frac{n}{2}-1,\ \frac{n}{2}\right), \left(\frac{n}{2}-1,\ \frac{n}{2}+1\right), \left(\frac{n}{2}, \frac{n}{2}-1\right), \left(\frac{n}{2}+1,\ \frac{n}{2}-1\right), \left(\frac{n}{2}+2,\ \frac{n}{2}\right), \left(\frac{n}{2}+2,\ \frac{n}{2}+1\right), \left(\frac{n}{2}, \frac{n}{2}+2\right), \left(\frac{n}{2}+1, \frac{n}{2}+2\right) \right\}.$$

$$t=3\colon\ S_3 = S_2 \cup \left\{ \left(\frac{n}{2}-2,\ \frac{n}{2}\right), \left(\frac{n}{2}-2,\ \frac{n}{2}+1\right), \left(\frac{n}{2}, \frac{n}{2}-2\right), \left(\frac{n}{2}+1,\ \frac{n}{2}-2\right), \left(\frac{n}{2}+3,\ \frac{n}{2}\right), \left(\frac{n}{2}+3,\ \frac{n}{2}\right), \left(\frac{n}{2}, \frac{n}{2}+3\right), \left(\frac{n}{2}+1, \frac{n}{2}+3\right), \left(\frac{n}{2}-1, \frac{n}{2}-1\right), \left(\frac{n}{2}+2, \frac{n}{2}-1\right), \left(\frac{n}{2}+2, \frac{n}{2}+2\right), \left(\frac{n}{2}-1, \frac{n}{2}+2\right) \right\}.$$

$$t=4\colon\ S_4 = S_3 \cup \left\{ \left(\frac{n}{2}-3,\ \frac{n}{2}\right), \left(\frac{n}{2}-3,\ \frac{n}{2}+1\right), \left(\frac{n}{2}, \frac{n}{2}-3\right), \left(\frac{n}{2}+1,\ \frac{n}{2}-3\right), \left(\frac{n}{2}+4,\ \frac{n}{2}\right), \left(\frac{n}{2}+4,\ \frac{n}{2}+1\right), \left(\frac{n}{2}, \frac{n}{2}+4\right), \left(\frac{n}{2}+1, \frac{n}{2}+4\right), \left(\frac{n}{2}-2, \frac{n}{2}-1\right), \left(\frac{n}{2}-1, \frac{n}{2}-2\right), \left(\frac{n}{2}+3, \frac{n}{2}-1\right), \left(\frac{n}{2}+2, \frac{n}{2}-2\right), \left(\frac{n}{2}-2, \frac{n}{2}+2\right), \left(\frac{n}{2}-1, \frac{n}{2}+3\right) \right\}.$$

$$2\le t\le \frac{n}{2}:\ S_t = S_{t-1}\cup\Big\{\Big(\frac{n}{2}-t+1,\ \frac{n}{2}\Big),\ \Big(\frac{n}{2}-t+1,\ \frac{n}{2}+1\Big),\ \Big(\frac{n}{2},\ \frac{n}{2}-t+1\Big),$$
$$\Big(\frac{n}{2}+1,\frac{n}{2}-t+1\Big),\ \Big(\frac{n}{2}+t,\ \frac{n}{2}\Big),\ \Big(\frac{n}{2}+t,\ \frac{n}{2}+1\Big),\ \Big(\frac{n}{2},\frac{n}{2}+t\Big),\ \Big(\frac{n}{2}+1,\frac{n}{2}+t\Big),$$
$$\Big(\frac{n}{2}-l+2,\frac{n}{2}-t+l-1\Big),\ \Big(\frac{n}{2}+l-1,\frac{n}{2}-t+l-1\Big),\ \Big(\frac{n}{2}-t+l-1,\frac{n}{2}+l-1\Big),$$
$$\Big(\frac{n}{2}+l-1,\frac{n}{2}+t-l+2\Big):\ 3\le l\le t\Big\}.$$

We notice that at $t=\frac{n}{2}$, the spread reaches its limits horizontally and vertically (the three middle vertices of each of the first row, the last row, the first column and the last column are converted). Therefore, in the remaining steps, the conversion spreads only diagonally as follows:

For $1\le m\le \frac{n}{2}-1$ which means for $\frac{n+1}{2}+1\le t\le n-1$:

$$S_{\frac{n}{2}+m} = S_{\frac{n}{2}+m-1}\cup\Big\{\Big(\frac{n}{2}-l+2,\frac{n}{2}-t+l-1\Big),\ \Big(\frac{n}{2}+l-1,\frac{n}{2}-t+l-1\Big),$$
$$\Big(\frac{n}{2}-t+l-1,\frac{n}{2}+l-1\Big),\ \Big(\frac{n}{2}+l-1,\frac{n}{2}+t-l+2\Big):\ m+2\le l\le t-m+1\Big\}.$$

When we reach step $t=n-1$, we have $m=\frac{n}{2}-1$ which means $l=\frac{n}{2}+1$ therefore:

$$S_{n-1} = S_{\frac{n}{2}+\frac{n}{2}-1} = S_{\frac{n}{2}+\frac{n}{2}-2}\cup\Big\{\Big(\frac{n}{2}-\Big(\frac{n}{2}+1\Big)+2,\frac{n}{2}-(n-1)+\Big(\frac{n}{2}+1\Big)-1\Big),$$
$$\Big(\frac{n}{2}+\Big(\frac{n}{2}+1\Big)-1,\frac{n}{2}-(n-1)+\Big(\frac{n}{2}+1\Big)-1\Big),\ \Big(\frac{n}{2}-(n-1)+\Big(\frac{n}{2}+1\Big)-1,\frac{n}{2}+\Big(\frac{n}{2}+1\Big)-1\Big),$$
$$\Big(\frac{n}{2}+\Big(\frac{n}{2}+1\Big)-1,\frac{n}{2}+(n-1)-\Big(\frac{n}{2}+1\Big)+2\Big)\Big\}$$
$$= S_{\frac{n}{2}+\frac{n}{2}-2}\cup\{(1,1),(n,1),(1,n),(n,n)\} = V(P_n\boxtimes P_n).$$

We conclude that S_0 is an I2CS and $C_2(P_n\boxtimes P_n)\le 2$ which means $C_2(P_n\boxtimes P_n)=2$ if n is even. Figure 9 illustrates that $C_2(P_8\boxtimes P_8)=2$. From Case 1 and Case 2 we conclude the requested.

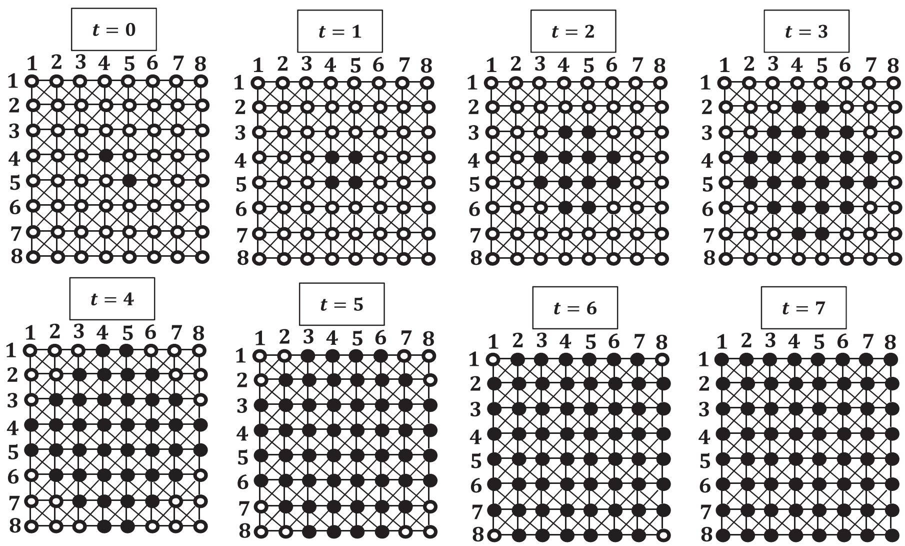

Figure 9.

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Further reading

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Corresponding author
Ali Kassem can be contacted at: ali2007.kasem@gmail.com

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