Large deviations for the stochastic functional integral equation with nonlocal condition

Gopal Shruthi and Murugan Suvinthra
Department of Applied Mathematics, Bharathiar University, Coimbatore, India

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Abstract

Purpose – The purpose of this paper is to study large deviations for the solution processes of a stochastic equation incorporated with the effects of nonlocal condition.

Design/methodology/approach – A weak convergence approach is adopted to establish the Laplace principle, which is same as the large deviation principle in a Polish space. The sufficient condition for any family of solutions to satisfy the Laplace principle formulated by Budhiraja and Dupuis is used in this work.

Findings – Freidlin–Wentzell type large deviation principle holds good for the solution processes of the stochastic functional integral equation with nonlocal condition.

Originality/value – The asymptotic exponential decay rate of the solution processes of the considered equation towards its deterministic counterpart can be estimated using the established results.

Keywords Large deviations, Nonlocal condition, Stochastic integral equation

Paper type Research paper

1. Introduction

Differential equations are applied in a wide variety of disciplines, including physics, chemistry, engineering, economics and biology by many researchers. An initial value problem describes the evolution of a physical system. An improvement of initial/local condition is done by imposing a nonlocal condition into the problem. Nonlocal conditions take values/measurements at more places and are more precise than the local condition. That is, if we consider the differential equation

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0,$$

for $t \geq t_0$ with $x(t) \in \mathbb{R}$ then the corresponding nonlocal differential equation evolves when the initial condition is replaced by the following nonlocal condition:

$$x(t_0) + g(x(\cdot)) = x_0,$$

where $x(\cdot)$ denotes the solution at some specific times $t \geq t_0$ and g defines a mapping consisting of certain functions on some space. Nonlocal conditions have several applications in real life situations, for example, in the diffusion phenomenon/dripping of a small amount of coloured water in a transparent tube filled with colourless water.

JEL Classification — 60F10, 34B10, 60H20

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Here there will be too little coloured water at the initial time t_0, so the measurement $x(t_0) + g(x(\cdot))$ will be more precise compared to $x(t_0)$. One more example is from the field of kinematics, to determine the location of a physical object with an evolution $t \to x(t)$ where the initial position holds the nonlocal condition.

Today, nonlocal condition has enormous applications in the field of quantum mechanics, continuum mechanics, damage mechanics, sub-surface flows, image recognition, peri-dynamic models, environment–human coupled systems like AI and decision making problems. Some of the applications are cited in References [1–5]. Recently, in Reference [6], the authors have discussed in detail different mathematical models with non-local initial condition and their applications.

But, since the late 1970s, non-local condition has been studied by many researchers, including Kerefov, who proposed non-local boundary value problems for parabolic equations [7]. Vabishchevich [8] in 1981, studied parabolic problems with non-local condition and problem related to the inverse of heat conduction. Charbowski [9] was the first to propose nonlocal initial-boundary value problems for linear parabolic equations and also investigated their existence and uniqueness. Byszewski, motivated by the physical problems in 1991, investigated nonlocal problems for nonlinear parabolic equations and also found the existence and uniqueness of solutions for non-local hyperbolic equations [10, 11]. Byszewski also found the existence and uniqueness of the solution for semi-linear evolution non-local Cauchy problems [12]. Inspired by the works of Byszewski, Jackson [13] also contributed to the generalization of the classical Cauchy problem to the nonlocal Cauchy problem yielding better results in many physical systems. The research works [14–16] are few more contributions on non-local Cauchy problems. Byszewski and Lakshmikantham were the first to introduce the study of non-local condition in Banach spaces [17], which paved the way for many researchers to study integro-differential equations with non-local condition in Banach spaces (for instance, References [18, 19]). Ntouyas has given a detailed survey about nonlocal initial and boundary problems in Reference [20]. The existence and uniqueness of solutions for nonlocal stochastic differential equations was studied by Lorenz [21]. A Volterra type non-local random integral equation was studied by Abdou *et al.* [22] by using admissibility of integral operator theory. A study of the Volterra–Itô–Doob type non-local stochastic functional integral equation is developed by Elborai and Youssef [23] using the fixed point technique.

The preceding works in the field of nonlocal condition and its variant applications in real life have motivated us to investigate large deviations for nonlocal stochastic functional differential equations. The subject of the large deviation theory is about controlling the probabilities of atypical events. It is a sub-discipline of probability theory that studies the exponential decline of probability measures of particular kinds of tail events. It has a rich history of development, starting with the works of a Swedish mathematician Cramer in the 1930s for insurance business modelling. In the year 1954, Petrov generalized Cramer's limit theorem. In 1966, Varadhan [24] developed the large deviation principle in a unique manner, making way for many more applications in a more convincing manner, like entropy calculation in statistical mechanics. Using Varadhan's contraction principle, Freidlin and Wentzell [25] developed the large deviation principle (LDP) for differential equations with small stochastic perturbation. Large deviations are established for stochastic differential delay equations by Mo and Luo [26]. Large deviations for stochastic integro-differential equations are studied in Reference [27] and for stochastic functional differential equations with infinite delay in Reference [28]. Large deviations for stochastic partial differential equations driven by a Poisson random measure is worked in Reference [29] and for the mean reflected stochastic differential equation with jumps in Reference [30].

In this paper, the large deviation theory is studied for the stochastic functional integral equation with non-local initial condition by adopting the weak convergence technique, which was formulated by Dupuis and Ellis [31] using Fleming's [32] stochastic control approach. The basic idea behind Dupuis and Ellis's formulation is that, under Polish space, the Laplace principle and the LDP are equivalent. The sufficient conditions for any family of solutions to satisfy the Laplace principle formulated by Budhiraja and Dupuis [33] are precisely used in this work.

2. Preliminaries

We consider the following nonlocal stochastic functional differential equation:

$$\begin{aligned} dX(t) &= f(t, X(t), AX(t))\mathrm{d}t + g(t, X(t), B(t)X(t))\mathrm{d}W(t), \\ X(0) &= X_0 + h(X(\cdot)), \end{aligned} \tag{2.1}$$

where $t \in J := [0, T]$, $T < \infty$, and the nonlocal condition $h(X(\cdot))$ is used in the sense that in the place of "$\cdot$"; we can substitute only elements of the set $\{t_1, t_2, \ldots, t_p\}$, where $0 \le t_1 < t_2 < \ldots < t_p \le T$, $p \in \mathbb{N}$.

Let $(\Omega, \mathbb{F}, P)$ be a complete probability space with a filtration $\{\mathbb{F}_t\}_{t \in J}$ where Ω is a nonempty set known as the sample space, $\mathbb{F} = \mathbb{F}_T$ is a σ - algebra of events of Ω occurring during the time interval J, P is a complete probability measure and $\{\mathbb{F}_t\}_{t \in J}$ is an increasing family of sub σ algebras $\mathbb{F}_t \subset \mathbb{F}$ satisfying the usual conditions. Let $\{X^\epsilon, \epsilon > 0\}$ be a sequence of random variables taking values in a Polish space $\mathcal{Z}$ and defined in $(\Omega, \mathbb{F}, P)$. Also, let $\mathbb{C} := \mathbb{C}(J, L_2(\Omega, \mathbb{F}, P))$ be the space of all continuous stochastic processes which are adapted to the filtration $\{\mathbb{F}_t\}_{t \in J}$. The following definitions and results are needed for this work.

Definition 2.1. [33]: A function $I : \mathcal{Z} \to [0, +\infty]$ is called a rate function if I is lower semi-continuous. A rate function I is called a good rate function if for each $a < \infty$, the level set $\{f \in \mathcal{Z} : I(f) \le a\}$ is compact.

Definition 2.2. [31]: Let I be a rate function on $\mathcal{Z}$. We say the sequence $\{X^\epsilon, \epsilon > 0\}$ satisfies the LDP with rate function I if the following two conditions hold:

(1) *Large deviation upper bound: For each closed subset F of $\mathcal{Z}$,*

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbf{P}(X^\epsilon \in F) \le -I(F).$$

(2) *Large deviation lower bound: For each open subset G of $\mathcal{Z}$,*

$$\liminf_{\epsilon \to 0} \epsilon \log \mathbf{P}(X^\epsilon \in G) \ge -I(G).$$

Definition 2.3. [33]: Let I be a rate function on $\mathcal{Z}$. We say $\{X^\epsilon\}$ satisfies the Laplace principle with rate function I if for all real valued bounded continuous functions h defined on $\mathcal{Z}$,

$$\lim_{\epsilon \to 0} \epsilon \log E\left\{ \exp\left[-\frac{1}{\epsilon} h(X^\epsilon) \right] \right\} = -\inf_{f \in \mathcal{Z}} \{h(f) + I(f)\}.$$

Theorem 2.4. [31] *The family $\{X^\epsilon\}$ satisfies the Laplace principle with good rate function on a Polish space if and only if $\{X^\epsilon\}$ satisfies the LDP with the same rate function.*

Equation (2.1) is equivalent to the following Volterra–Itô type stochastic functional integral equation with nonlocal condition:

$$X(t) = X_0 + h(X(\cdot)) + \int_0^t f(\tau, X(\tau), AX(\tau))\mathrm{d}\tau + \int_0^t g(\tau, X(\tau), B(\tau)X(\tau))\mathrm{d}W(\tau), \quad (2.2)$$

where $X_0 \in R$, and the function $h(X(\cdot))$ is random and defined on $\mathbb{R}$ with values in the space $\mathbb{R}$.

In the above equation the first integral is a mean square Riemann integral and the second is an Itô integral. $W(t)$ is a real valued Brownian motion adapted to the filtration $\{\mathbb{F}_t\}_{t \in J}$. The operator A is closed, linear and defined on $\mathbb{C}$ with values in $\mathbb{C}$. The operators $\{B(t) : t \in J\}$ are linear, bounded and defined on $\mathbb{C}$ into $\mathbb{C}$. By the closed graph theorem, we get that

$$|AX(t)| \le \beta |X(t)|, \ t \in J \ \text{ and } \ |B(t)X(t)| \le \gamma(t)|X(t)|, \ t \in J, \quad (2.3)$$

in such a way that $\gamma(t)$ is square integrable on J and $\beta \ge 0$ is a real constant. The functions f and g are real, measurable and defined on $J \times \mathbb{C} \times \mathbb{C}$ with values in the space $\mathbb{C}$.

Let $|\cdot|$ denote the Euclidean norm. The functions f, g and h will be specified with the conditions below.

H1. The functions $f(t, x, y)$ and $g(t, x, y)$ are mean square continuous in (x, y) for each $t \in J$.

H2. f and g have the following restriction on growth:

$$|f(t,x,y)| \le \alpha \sqrt{(1 + |x|^2 + |y|^2)},$$
$$|g(t,x,y)| \le \alpha \sqrt{(1 + |x|^2 + |y|^2)},$$

for all $t \in J$, $x, y \in \mathbb{R}$, where the constant $\alpha > 0$.

H3. There exist constants $\alpha_1 > 0$, $0 < \alpha_2 < \frac{1}{2}$ such that

$$|f(t,x_2,y_2) - f(t,x_1,y_1)| \le \alpha_1 \sqrt{\left(|x_2 - x_1|^2 + |y_2 - y_1|^2\right)},$$
$$|g(t,x_2,y_2) - g(t,x_1,y_1)| \le \alpha_1 \sqrt{\left(|x_2 - x_1|^2 + |y_2 - y_1|^2\right)},$$
$$|h(t,x_2) - h(t,x_1)| \le \alpha_2 |x_2 - x_1|,$$

for all $t \in J$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

The existence and uniqueness of solution of equation (2.2) has been established by Elborai and Youseff [23]. In this work we study large deviation principle for Equation (2.2).

Consider Equation (2.2) stochastically perturbed by a small parameter $\epsilon > 0$

$$X(t) = X_0 + h(X(\cdot)) + \int_0^t f(\tau, X(\tau), AX(\tau))\mathrm{d}\tau + \sqrt{\epsilon} \int_0^t g(\tau, X(\tau), B(\tau)X(\tau))\mathrm{d}W(\tau). \quad (2.4)$$

Let $\{X^\epsilon\}$ denote the solution of the perturbed Equation (2.4). Since $\{X^\epsilon\}$ is a strong solution to Equation (2.4), there exists a Borel-measurable function $G^\epsilon : \mathbb{C}(J; \mathbb{R}) \to \mathbb{C}(J; \mathbb{R})$ such that $X^\epsilon(\cdot) = G^\epsilon(W(\cdot))$ a.s. by the Yamada Watanabe theorem [34].

Let $\mathcal{A} = \left\{v : v \text{ is a real valued } \mathbb{F}_t \text{ predictable process and } \int_0^T |v(\tau)|^2 \mathrm{d}\tau < \infty \ a.s.\right\}$, $S_N = \left\{v \in L_2(J; \mathbb{R}) : \int_0^T \left|v(\tau)\right|^2 d\tau \le N\right\}$, where $L_2(J; \mathbb{R})$ is the space of all real valued square integrable functions on J. Then S_N endowed with the weak topology in $L_2(J; \mathbb{R})$ is a compact Polish space. Let us also define $\mathcal{A}_N = \{v \in \mathcal{A} : v(\omega) \in S_N \ \ P - \text{a.s}\}$.

The sufficient conditions framed by Budhiraja and Dupuis [33] under which the Laplace principle holds for the family $\{X^\epsilon\}$ are the following:

Proposition 2.5. *Suppose that there exists a measurable map* $G^0: \mathbb{C}(J;\mathbb{R}) \mapsto \mathbb{C}(J;\mathbb{R})$ *such that the following two conditions hold:*

(1) *Compactness : For each* $N<\infty$, *the set* $K_N = \{G^0(\int_0^\cdot v(\tau)\mathrm{d}\tau) : v \in S_N\}$ *is a compact subset of* $\mathbb{C}(J;\mathbb{R})$.

(2) *Weak convergence: Let* $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ *for some* $N<\infty$. *If* v^ϵ *converge to* v *in distribution as* S_N *valued random elements, then* $G^\epsilon\left(W(\cdot)+\frac{1}{\sqrt{\epsilon}}\int_0^\cdot v^\epsilon(\tau)\mathrm{d}\tau\right)$ $\to G^0\left(\int_0^\cdot v(\tau)\mathrm{d}\tau\right)$ *in distribution as* $\epsilon \to 0$.

Then the family $\{X^\epsilon, \epsilon>0\}$ *satisfies the Laplace principle in* $\mathbb{C}(J;\mathbb{R})$ *with the rate function* I *given by*

$$I(\phi) = \inf\left\{\frac{1}{2}\int_0^T |v(\tau)|^2\mathrm{d}\tau;\ \phi = G^0\left(\int_0^\cdot v(\tau)\mathrm{d}\tau\right) \ \text{and}\ \ v\in L_2(J;\mathbb{R})\right\}$$

for each $\phi \in \mathbb{C}(J;\mathbb{R})$.

Theorem 2.6. [31] *Let* $\{Y_n, n\in\mathbb{N}\}$ *be a sequence of real-valued random variables that are defined on a sequence of probability spaces* $\{(\Omega_n, \mathbb{F}_n, P_n), n\in\mathbb{N}\}$. *If* $Y_n \xrightarrow{D} Y$ *and* $f \in \mathcal{C}_b(X)$, *then* $f(Y_n)\xrightarrow{D} f(Y)$.

3. The Large deviation principle

Here we establish the LDP for the family of solution processes of Equation (2.4) by using Proposition 2.5.

Theorem 3.1. *With the assumptions (H1) to (H3), the family* $\{X^\epsilon\}$ *of solutions of Equation* (2.4) *satisfies the LDP (equivalently, the Laplace principle) in* $\mathbb{C}(J;\mathbb{R})$ *with good rate function*

$$I(\phi) = \inf\left\{\frac{1}{2}\int_0^T |v(\tau)|^2\mathrm{d}\tau;\ X_v = \phi\right\},$$

where $v\in L_2(J;\mathbb{R})$ *and* X_v *denotes the solution of the controlled equation*

$$X_v(t) = X_0 + h(X_v(\cdot)) + \int_0^t f(\tau, X_v(\tau), AX_v(\tau))\mathrm{d}\tau + \int_0^t g(\tau, X_v(\tau), B(\tau)X_v(\tau))v(\tau)\mathrm{d}\tau \tag{3.1}$$

with the convention that the infimum of an empty set is infinity.

The above theorem gets proved with the proof of the following two lemmas:

Lemma 3.2. *(Compactness): Define* $G^0 : \mathbb{C}(J;\mathbb{R}) \to \mathbb{C}(J;\mathbb{R})$ *by*

$$G^0(g) := \begin{cases} X_v, & \text{if}\ \ g = \int_0^\cdot v(\tau)\mathrm{d}\tau \ \ \text{for some}\ \ v\in L_2(J;\mathbb{R}) \\ 0, & \text{otherwise,} \end{cases}$$

where X_v *denotes the solution of Equation* (3.1). *Then for each* $N<\infty$, *the set* $K_N = \{G^0(\int_0^\cdot v(\tau)\mathrm{d}\tau) : v\in S_N\}$ *is a compact subset of* $\mathbb{C}(J;\mathbb{R})$.

Proof: Consider a sequence $\{v_n\} \in S_N$ such that $v_n \to v$ weakly in $L_2(J;\mathbb{R})$ as $n \to \infty$. Let X_{v_n} denote the solution of the controlled Equation (3.1) with v replaced by v_n. That is

$$X_{v_n}(t)=X_0 + h(X_{v_n}(\cdot)) + \int_0^t f(\tau,X_{v_n}(\tau),AX_{v_n}(\tau))\mathrm{d}\tau + \int_0^t g(\tau,X_{v_n}(\tau),B(\tau)X_{v_n}(\tau))v_n(\tau)\mathrm{d}\tau. \quad (3.2)$$

From Equations (3.1) and (3.2),

$$\begin{aligned} X_{v_n}(t) - X_v(t) = {} & h(X_{v_n}(\cdot)) - h(X_v(\cdot)) \\ & + \int_0^t f(\tau,X_{v_n}(\tau),AX_{v_n}(\tau)) - f(\tau,X_v(\tau),AX_v(\tau))\mathrm{d}\tau \\ & + \int_0^t g(\tau,X_{v_n}(\tau),B(\tau)X_{v_n}(\tau))v_n(\tau) - g(\tau,X_v(\tau),B(\tau)X_v(\tau))v(\tau)\mathrm{d}\tau. \end{aligned}$$

Taking modulus and by using Hölder's inequality

$$\begin{aligned} |X_{v_n}(t) - X_v(t)| \le {} & |h(X_{v_n}(\cdot)) - h(X_v(\cdot))| \\ & + \left(\int_0^t |f(\tau,X_{v_n}(\tau),AX_{v_n}(\tau)) - f(\tau,X_v(\tau),AX_v(\tau))|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \times T^{\frac{1}{2}} \\ & + \left(\int_0^t |g(\tau,X_{v_n}(\tau),B(\tau)X_{v_n}(\tau)) - g(\tau,X_v(\tau),B(\tau)X_v(\tau))|^2 d\tau\right)^{\frac{1}{2}} \\ & \times \left(\int_0^t |v_n(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}} + \left|\int_0^t g(\tau,X_v(\tau),B(\tau)X_v(\tau))(v_n(\tau) - v(\tau))\mathrm{d}\tau\right|. \end{aligned} \quad (3.3)$$

Let $\zeta^n(t) = \int_0^t g(\tau,X_v(\tau),B(\tau)X_v(\tau))(v_n(\tau) - v(\tau))\mathrm{d}\tau$. Squaring both sides of Equation (3.3) and using conditions (H2) and (H3),

$$\begin{aligned} |X_{v_n}(t) - X_v(t)|^2 \le {} & 4\alpha_2^2|X_{v_n}(\cdot) - X_v(\cdot)|^2 \\ & + 4\alpha_1^2 T\int_0^t \left(|X_{v_n}(\tau) - X_v(\tau)|^2 + |AX_{v_n}(\tau) - AX_v(\tau)|^2\right)\mathrm{d}\tau \\ & + 4\alpha_1^2 N\int_0^t \left(|X_{v_n}(\tau) - X_v(\tau)|^2 + |B(\tau)X_{v_n}(\tau) - B(\tau)X_v(\tau)|^2\right)\mathrm{d}\tau + 4|\zeta^n(t)|^2. \end{aligned}$$

Using Equation (2.3),

$$|X_{v_n}(t)-X_v(t)|^2\leq 4\alpha_2^2|X_{v_n}(\cdot)-X_v(\cdot)|^2 + 4\alpha_1^2T\int_0^t |X_{v_n}(\tau)-X_v(\tau)|^2+\beta^2\Big|X_{v_n}(\tau)-X_v(\tau)|^2 \mathrm{d}\tau + 4\alpha_1^2N\int_0^t \Big(|X_{v_n}(\tau)-X_v(\tau)|^2+\gamma^2(\tau)\Big|X_{v_n}(\tau)-X_v(\tau)|^2\Big)\mathrm{d}\tau+4|\zeta^n(t)|^2. \tag{3.4}$$

Consider

$$|\zeta^n(t)| = \left|\int_0^t g(\tau,X_v(\tau),B(\tau)X_v(\tau))(v_n(\tau)-v(\tau))\mathrm{d}\tau\right|.$$

Applying Hölder's inequality and taking supremum to the above equation leads to,

$$\begin{aligned}\sup_{\tau\in J}|\zeta^n(\tau)| &\leq \alpha\left(\int_0^T\Big(1+|X_v|^2_{\mathbb{C}(J;\mathbb{R})}(1+\gamma^2(\tau))\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^T|v_n(\tau)-v(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\leq \alpha\sqrt{N}\left[T+\|X_v\|^2_{\mathbb{C}(J;\mathbb{R})}\left(\int_0^T(1+\gamma^2(\tau))\mathrm{d}\tau\right)\right]^{\frac{1}{2}}\\ &\leq C<\infty, \text{where } C \text{ is a constant.}\end{aligned} \tag{3.5}$$

Observe that $\{\zeta^n(t)\}$ is a family of linear, continuous, real valued functions mapping S_N to $\mathbb{C}(J;\mathbb{R})$. By noting that $\sup_{0\leq\tau\leq t}|\zeta^n(\tau)|<C$, the constant C not depending on n, we get that the family $\{\zeta^n\}$ is uniformly bounded by C. It can be concluded that $\{\zeta^n\}$ is equi-continuous by use of the fact that between Banach spaces the family of point-wise bounded continuous linear functions is equi-continuous. Since $v_n \rightharpoonup v$ in $L_2(J;\mathbb{R})$, $\zeta^n(t)\to 0$ point-wise for $t\in J$.

By applying a version of the Arzela – Ascoli theorem immediately implies that $\zeta^n \to 0$ uniformly in $\mathbb{C}(J;\mathbb{R})$. Hence

$$\lim_{n\to\infty}\sup_{\tau\in J}|\zeta^n(\tau)|=0. \tag{3.6}$$

Set $\kappa^n(t)=\sup_{0\leq\tau\leq t}|X_{v_n}(\tau)-X_v(\tau)|^2$. Then from Equation (3.4)

$$\kappa^n(t)\leq\frac{4\alpha_1^2}{1-4\alpha_2^2}\int_0^T\kappa^n(\tau)\big[\big(1+\beta^2\big)T+\big(1+\gamma^2(\tau)\big)N\big]\mathrm{d}\tau+\frac{4}{1-4\alpha_2^2}\sup_{0\leq\tau\leq t}|\zeta^n(\tau)|^2.$$

Now by using Gronwall's lemma,

$$\kappa^n(t)\leq C_1\sup_{0\leq\tau\leq t}|\zeta^n(\tau)|^2e^{C_2T}$$

where C_1 and C_2 are constants depending on β, $\gamma(\tau)$, α_1, α_2, N and T. Hence

$$|X_{v_n}-X_v|^2_{\mathbb{C}(J;\mathbb{R})} = \sup_{\tau\in J}|X_{v_n}(\tau)-X_v(\tau)|^2\leq C_1\sup_{t\in J}|\zeta^n(t)|^2e^{C_2T}$$

and so $X_{v_n} \to X_v$ in $\mathbb{C}(J;\mathbb{R})$ by Equation (3.6). Since the space S_N is compact, it follows the set $K_N = \left\{ G^0\left(\int_0^{\cdot} v(\tau)\mathrm{d}\tau\right) : v \in S_N \right\}$ for $N < \infty$ is compact.

We consider the following stochastic integral equation for verifying the weak convergence condition of Proposition 2.5.

$$\begin{aligned} X^{\epsilon}_{v^{\epsilon}}(t) &= X_0 + h\big(X^{\epsilon}_{v^{\epsilon}}(\cdot)\big) + \int_0^t f\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), AX^{\epsilon}_{v^{\epsilon}}(\tau)\big)\mathrm{d}\tau \\ &+ \int_0^t g\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), B(\tau)X^{\epsilon}_{v^{\epsilon}}(\tau)\big)v^{\epsilon}(\tau)\mathrm{d}\tau \\ &+ \sqrt{\epsilon}\int_0^t g\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), B(\tau)X^{\epsilon}_{v^{\epsilon}}(\tau)\big)\mathrm{d}W(\tau). \end{aligned} \tag{3.7}$$

From Girsanov's theorem, the existence of the above Equation (3.7) follows. So we move on to the weak convergence result.

Lemma 3.3. *(Weak Convergence): Let $\{v^{\epsilon} : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. Assume v^{ϵ} converges to v in distribution as S_N-valued random elements. Then $G^{\epsilon}\big(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^{\cdot} v^{\epsilon}(\tau)\mathrm{d}\tau\big) \to G^0\left(\int_0^{\cdot} v(\tau)\mathrm{d}\tau\right)$ in distribution as $\epsilon \to 0$.*

Proof: Applying Itô's formula

$$\begin{aligned} |X^{\epsilon}_{v^{\epsilon}}(t) - X_v(t)|^2 &= |h\big(X^{\epsilon}_{v^{\epsilon}}(\cdot)\big) - h(X_v(\cdot))|^2 \\ &+ 2\int_0^t \big(f\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), AX^{\epsilon}_{v^{\epsilon}}(\tau)\big) - f(\tau, X_v(\tau), AX_v(\tau))\big) \\ &\times \big(X^{\epsilon}_{v^{\epsilon}}(\tau) - X_v(\tau)\big)\mathrm{d}\tau \\ &+ 2\int_0^t \big(g\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), B(\tau)X^{\epsilon}_{v^{\epsilon}}(\tau)\big) - g(\tau, X_v(\tau), B(\tau)X_v(\tau))\big) \\ &\times \big(X^{\epsilon}_{v^{\epsilon}}(\tau) - X_v(\tau)\big)v^{\epsilon}(\tau)\mathrm{d}\tau \\ &+ 2\int_0^t (X^{\epsilon}_{v^{\epsilon}}(\tau) - X_v(\tau))g(\tau, X_v(\tau), B(\tau)X_v(\tau))(v^{\epsilon}(\tau) - v(\tau))\mathrm{d}\tau \\ &+ 2\sqrt{\epsilon}\int_0^t (X^{\epsilon}_{v^{\epsilon}}(\tau) - X_v(\tau))g\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), B(\tau)X^{\epsilon}_{v^{\epsilon}}(\tau)\big)\mathrm{d}W(\tau) \\ &+ \epsilon\int_0^t |g\big(\tau, X^{\epsilon}_{v^{\epsilon}}(\tau), B(\tau)X^{\epsilon}_{v^{\epsilon}}(\tau)\big)|^2\mathrm{d}\tau. \end{aligned} \tag{3.8}$$

Define

$$\zeta^{\epsilon}(t) = \int_0^t g(\tau, X_v(\tau), B(\tau)X_v(\tau))(v^{\epsilon}(\tau) - v(\tau))\mathrm{d}\tau. \tag{3.9}$$

Taking supremum and expectation we get $E\big(\sup_{t\in J}|\zeta^{\epsilon}(t)|\big) \le C_3 < \infty$, where C_3 is a constant.

By virtue of Itô's formula

$$
\begin{aligned}
&2\int_0^t (X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau))g(\tau,X_v(\tau),B(\tau)X_v(\tau))(v^\epsilon(\tau)-v(\tau))\mathrm{d}\tau\\
&=2\big(X^\epsilon_{v^\epsilon}(t)-X_v(t)\big)(\zeta^\epsilon(t))-2\int_0^t \big(h\big(X^\epsilon_{v^\epsilon}(\cdot)\big)-h(X_v(\cdot))\big)\zeta^\epsilon(\tau)\mathrm{d}\tau\\
&\quad-2\int_0^t \big(f\big(\tau,X^\epsilon_{v^\epsilon}(\tau),AX^\epsilon_{v^\epsilon}(\tau)\big)-f(\tau,X_v(\tau),AX_v(\tau))\big)\zeta^\epsilon(\tau)\mathrm{d}\tau \qquad (3.10)\\
&\quad-2\int_0^t \big(g\big(\tau,X^\epsilon_{v^\epsilon}(\tau),B(\tau)X^\epsilon_{v^\epsilon}(\tau)\big)v^\epsilon(\tau)-g(\tau,X_v(\tau),B(\tau)X_v(\tau))v(\tau)\big)\\
&\quad\times\zeta^\epsilon(\tau)\mathrm{d}\tau-2\sqrt{\epsilon}\int_0^t g\big(\tau,X^\epsilon_{v^\epsilon}(\tau),B(\tau)X^\epsilon_{v^\epsilon}(\tau)\big)\zeta^\epsilon(\tau)\mathrm{d}W(\tau)\\
&=I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

By the help of Young's inequality and (H3), we get

$$I_1 \le \frac{1}{2}|X^\epsilon_{v^\epsilon}(t)-X_v(t)|^2+2|\zeta^\epsilon(t)|^2. \qquad (3.11)$$

$$I_2 \le \frac{1}{2}\alpha_2^2\int_0^t |X^\epsilon_{v^\epsilon}(\cdot)-X_v(\cdot)|^2\mathrm{d}\tau+2\int_0^t |\zeta^\epsilon(\tau)|^2\mathrm{d}\tau. \qquad (3.12)$$

$$I_3 \le \frac{1}{2}\alpha_1^2\int_0^t\Big[|X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau)|^2+|AX^\epsilon_{v^\epsilon}(\tau)-AX_v(\tau)|^2\Big]\mathrm{d}\tau+2\int_0^t|\zeta^\epsilon(\tau)|^2\mathrm{d}\tau. \qquad (3.13)$$

$$
\begin{aligned}
I_4 &\le \frac{1}{2}\alpha_1^2\int_0^t\Big[|X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau)|^2+|B(\tau)X^\epsilon_{v^\epsilon}(\tau)-B(\tau)X_v(\tau)|^2\Big]\mathrm{d}\tau\\
&\quad+2\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|^2\int_0^t|v^\epsilon(\tau)|^2\mathrm{d}\tau+2\Big(\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|\Big)^2\\
&\le\frac{1}{2}\alpha_1^2\int_0^t\Big[|X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau)|^2+|B(\tau)X^\epsilon_{v^\epsilon}(\tau)-B(\tau)X_v(\tau)|^2\Big]\mathrm{d}\tau \qquad (3.14)\\
&\quad+2N\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|^2+2\Big(\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|\Big)^2.
\end{aligned}
$$

Using Young's inequality, condition (H3) and substituting the estimates (3.11)–(3.14) of I_1, I_2, I_3 and I_4 in Equation (3.8), we get,

$$|X^\epsilon_{v^\epsilon}(t)-X_v(t)|^2\le |h\big(X^\epsilon_{v^\epsilon}(\cdot)\big)-h(X_v(\cdot))|^2+\int_0^t |X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau)|^2\mathrm{d}\tau$$

$$+ \quad \alpha_1^2 \int_0^t \Big[|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + |AX_{v^\epsilon}^\epsilon(\tau) - AX_v(\tau)|^2\Big] \mathrm{d}\tau$$

$$+ \quad \int_0^t |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 |v^\epsilon(\tau)|^2 \mathrm{d}\tau$$

$$+ \alpha_1^2 \int_0^t \Big[|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + |B(\tau)X_{v^\epsilon}^\epsilon(\tau) - B(\tau)X_v(\tau)|^2\Big] \mathrm{d}\tau$$

$$+ \frac{1}{2}|X_{v^\epsilon}^\epsilon(t) - X_v(t)|^2 + 2|\zeta^\epsilon(t)|^2 + \frac{1}{2}\alpha_2^2 \int_0^t |X_{v^\epsilon}^\epsilon(\cdot) - X_v(\cdot)|^2 \mathrm{d}\tau$$

$$+ 4\int_0^t |\zeta^\epsilon(\tau)|^2 \mathrm{d}\tau + \frac{1}{2}\alpha_1^2 \int_0^t \Big[|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + |AX_{v^\epsilon}^\epsilon(\tau) - AX_v(\tau)|^2\Big] \mathrm{d}\tau$$

$$+ \frac{1}{2}\alpha_1^2 \int_0^t \Big[|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + |B(\tau)X_{v^\epsilon}^\epsilon(\tau) - B(\tau)X_v(\tau)|^2\Big] \mathrm{d}\tau$$

$$+ 2\sqrt{\epsilon}\Big| \int_0^t (X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau) - \zeta^\epsilon(t)) g\big(\tau, X_{v^\epsilon}^\epsilon(\tau), B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big) |\mathrm{d}W(\tau)\Big|$$

$$+ \epsilon \int_0^t |g\big(\tau, X_{v^\epsilon}^\epsilon(\tau), B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big)|^2 \mathrm{d}\tau + 2(N+1)\left(\sup_{\tau\in[0,t]} |\zeta^\epsilon(\tau)|\right)^2. \quad (3.15)$$

Using conditions (H2) and (H3) and simplifying, we get

$$|X_{v^\epsilon}^\epsilon(t) - X_v(t)|^2 \le \frac{2}{1 - 2\alpha_2^2} \int_0^t |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 \mathrm{d}\tau$$

$$+ \frac{2\alpha_1^2}{1 - 2\alpha_2^2} \int_0^t \Big(|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + \beta^2 |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2\Big) \mathrm{d}\tau$$

$$+ \frac{2}{1 - 2\alpha_2^2} \int_0^t |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 |v^\epsilon(\tau)|^2 \mathrm{d}\tau$$

$$+ \frac{2\alpha_1^2}{1 - 2\alpha_2^2} \int_0^t \Big(|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + \gamma^2(\tau) |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2\Big) \mathrm{d}\tau$$

$$+ \frac{4}{1 - 2\alpha_2^2} |\zeta^\epsilon(t)|^2 + \frac{\alpha_2^2}{1 - 2\alpha_2^2} \int_0^t |X_{v^\epsilon}^\epsilon(\cdot) - X_v(\cdot)|^2 \mathrm{d}\tau$$

$$+ \frac{8}{1 - 2\alpha_2^2} \int_0^t |\zeta^\epsilon(\tau)|^2 \mathrm{d}\tau$$

$$+ \frac{\alpha_1^2}{1 - 2\alpha_2^2} \int_0^t \Big(|X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2 + \beta^2 |X_{v^\epsilon}^\epsilon(\tau) - X_v(\tau)|^2\Big) \mathrm{d}\tau$$

$$+\frac{\alpha_1^2}{1-2\alpha_2^2}\int_0^t\Big(|X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)|^2+\gamma^2(\tau)|X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)|^2\Big)\mathrm{d}\tau$$
$$+\frac{4}{1-2\alpha_2^2}(N+1)\left(\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|\right)^2$$
$$+\frac{2\epsilon\alpha^2}{1-2\alpha_2^2}\int_0^t\Big[1+|X_{v^\epsilon}^\epsilon(\tau)|^2+\gamma^2(\tau)|X_{v^\epsilon}^\epsilon(\tau)|^2\Big]\mathrm{d}\tau$$
$$+\frac{4\sqrt{\epsilon}}{1-2\alpha_2^2}\Big|\int_0^t\big(X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)-\zeta^\epsilon(\tau)\big)g\big(\tau,X_{v^\epsilon}^\epsilon(\tau),B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big)\mathrm{d}W(\tau)\Big|. \tag{3.16}$$

Denote

$$\kappa^\epsilon(t)=\sup_{0\le\tau\le t}|X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)|^2.$$

Taking supremum of Equation (3.16)

$$\kappa^\epsilon(t)\le M_1\int_0^t\Big(1+|v^\epsilon(\tau)|^2\Big)\kappa^\epsilon(\tau)\mathrm{d}\tau+M_2\int_0^t\kappa^\epsilon(\tau)\big(1+\gamma^2(\tau)\big)\mathrm{d}\tau$$
$$+M_3\left(\sup_{\tau\in[0,t]}|(\zeta^\epsilon(\tau))|\right)^2+\frac{2\epsilon\alpha^2}{1-2\alpha_2^2}\int_0^t\Big[1+|X_{v^\epsilon}^\epsilon(\tau)|^2\big(1+\gamma^2(\tau)\big)\Big]\mathrm{d}\tau$$
$$+\frac{4\sqrt{\epsilon}}{1-2\alpha_2^2}\sup_{s\in[0,t]}\Big|\int_0^s\big(X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)-\zeta^\epsilon(\tau)\big)g\big(\tau,X_{v^\epsilon}^\epsilon(\tau),B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big)\mathrm{d}W(\tau)\Big|.$$

Here M_1, M_2 and M_3 are constants depending on β, N, α, α_1, α_2 and T. By using Gronwall's lemma

$$\kappa^\epsilon(t)\le\left(M_3\left(\sup_{\tau\in[0,t]}|\zeta^\epsilon(\tau)|\right)^2+\frac{2\epsilon\alpha^2}{1-2\alpha_2^2}\int_0^t\Big[1+\big|X_{v^\epsilon}^\epsilon(\tau)|^2\big(1+\gamma^2(\tau)\big)\Big]\mathrm{d}\tau\right.$$
$$\left.+\frac{4\sqrt{\epsilon}}{1-2\alpha_2^2}\sup_{s\in[0,t]}\left|\int_0^s\big(X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)-\zeta^\epsilon(\tau)\big)g\big(\tau,X_{v^\epsilon}^\epsilon(\tau),B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big)\mathrm{d}W(\tau)\right|\right)$$
$$\times\exp\left(M_1\int_0^t\Big(1+|v^\epsilon(\tau)|^2\Big)\mathrm{d}\tau+M_2\int_0^t\big(1+\gamma^2(\tau)\big)\mathrm{d}\tau|\right).$$

Simplifying further

$$\kappa^\epsilon(T)\le C\left(M_3\left(\sup_{\tau\in J}|\zeta^\epsilon(\tau)|\right)^2+\frac{2\epsilon\alpha^2}{1-2\alpha_2^2}\Big[T+\sup_{\tau\in J}|X_{v^\epsilon}^\epsilon(\tau)|^2\int_0^T\big(1+\gamma^2(\tau)\big)\mathrm{d}\tau\Big]\right.$$
$$\left.+\frac{4\sqrt{\epsilon}}{1-2\alpha_2^2}\sup_{s\in J}\Big|\int_0^s\big(X_{v^\epsilon}^\epsilon(\tau)-X_v(\tau)-\zeta^\epsilon(\tau)\big)g\big(\tau,X_{v^\epsilon}^\epsilon(\tau),B(\tau)X_{v^\epsilon}^\epsilon(\tau)\big)\mathrm{d}W(\tau)\Big|\right). \tag{3.17}$$

For showing convergence of $\zeta^\epsilon(\tau)$, we define

$$F(u)=\int_0^t g(\tau,X_v(\tau),B(\tau)X_v(\tau))u(\tau)\mathrm{d}\tau,u\in\mathcal{A}_N.$$

Recall that v^ϵ converges to v in distribution as S_N valued random elements which is endowed with the weak topology. By linear growth of g, we could observe that F as a mapping from S_N to $\mathbb{C}(J;\mathbb{R})$ is bounded and continuous. Now applying Theorem 2.6, we get $\zeta^\epsilon\to 0$ in distribution as $\epsilon\to 0$.

Using Burkholder–Davis–Gundy inequality, it can be easily verified that

$$\mathbf{E}\left(\sup_{s\in J}\left|\int_0^s\left(X^\epsilon_{v^\epsilon}(\tau)-X_v(\tau)-\zeta^\epsilon(\tau)\right)g\left(\tau,X^\epsilon_{v^\epsilon}(\tau),B(\tau)X^\epsilon_{v^\epsilon}(\tau)\right)\mathrm{d}W(\tau)\right|\right)\leq C.$$

With the above bound and with the distributional convergence of ζ^ϵ to 0, it follows from (3.17) that $\kappa^\epsilon(T)\to 0$ in distribution as $\epsilon\to 0$. Thus the lemma is established, thereby proving the main theorem.

4. Conclusion

Here, a control system for the corresponding original system is taken, and the compactness of the solution is proved. And with the help of the estimates of the solution, it is proved successfully that, for a controlled deterministic system, a controlled stochastic system converges weakly. Thus, the stochastic functional integral equation with a nonlocal condition under Polish space satisfies the Laplace principle, and thereby the large deviation principle is also proved. In the future, one can move to study exit time problems, the uniform large deviation principle, and the moderate deviation principle for the equation considered in this work, and also study the LDP and moderate deviation principle for coupled stochastic integral equations with nonlocal condition.

References

1. Bruno L, Tosin A, Tricerri P, Venuti F. Non-local first-order modelling of crowd dynamics: a multidimensional framework with applications. Appl Math Model. 2011; 35: 426-45.
2. Emmrich E, Lehoucq RB, Puhst D. Peridynamics: a nonlocal continuum theory. Lect Notes Comput Sci Eng. 2013; 89: 45-65.
3. Jirasek M. Nonlocal theories, Encyclopedia of continuum mechanics. Berlin: Springer; 2020. 1869-92.
4. Mahler DH, Rozema L, Fisher K, Vermeyden L, Resch KJ, Wiseman HM, Steinberg A. Experimental nonlocal and surreal Bohmian trajectories. Sci Adv. 2016; 2: e1501466.
5. Tian X, Du Q. Asymptotically compatible schemes for robust discretization of parametrized problems with applications to nonlocal models. SIAM Rev. 2020; 62: 199-227.
6. Sytnyk D, Melnik R. Mathematical models with nonlocal initial conditions: an exemplification from quantum mechanics. Math Comput Appl. 2021; 26: mca26040073.
7. Kerefov AA. Nonlocal boundary value problems for parabolic equations. Differ Equ. 1979; 15: 74-8.
8. Vabishchevich PN. Nonlocal parabolic problems and the inverse problem of heat conduction. Differ Equ. 1981; 17: 1193-9.
9. Charbowski J. On nonlocal problems for parabolic equations. Nagoya Math J. 1984; 93: 109-31.

10. Byszewski L. Theorem about existence and uniqueness of continuous solution of nonlocal problem for nonlinear hyperbolic equation. Appl Anal. 1991; 40: 173-80.
11. Byszewski L. Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with arbitrary functionals. J Math Anal Appl. 1991; 156: 457-70.
12. Byszewski L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J Math Anal Appl. 1991; 162: 494-505.
13. Jackson D. Existence and uniqueness of solutions to semilinear nonlocal parabolic equations. J Math Anal Appl. 1993; 172: 256-65.
14. Boucherif A, Akca H. Nonlocal Cauchy problems for semilinear evolution equation. Dyn Syst Appl. 2002; 11: 415-20.
15. Byszewski L. Uniqueness criterion for solution of abstract nonlocal Cauchy problem. J Appl Math Stoch Anal. 1993; 6: 49-54.
16. Liang J, Liu J, Xiao TJ. Nonlocal Cauchy problems governed by compact operator families. Nonlinear Anal. 2004; 57: 183-9.
17. Byszewski L, Lakshmikantham V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl Anal. 1991; 40: 11-19.
18. Balachandran K, Park JY. Sobolev type integro-differential equation with nonlocal condition in Banach spaces, Taiwan. J Math. 2003; 7: 155-63.
19. Dauer JP, Balachandran K. Existence of solutions for an integro-differential equation with nonlocal condition in Banach spaces. Lib Math. 1996; 16: 133-43.
20. Ntouyas SK. Nonlocal initial and boundary value problems, a survey, handbook of differential equations: ordinary differential equations, 2. Amsterdam: Elsevier; 2005. p. 461-557.
21. Lorenz T. Nonlocal stochastic differential equations: existence and uniqueness of solutions. Newsl Spanish Soc. Appl. Math. 2010; 51: 99-108.
22. Abdou MA, Elborai MM, Youssef MI. On some nonlocal perturbed random fractional integro-differential equations. Life Sci J. 2013; 10: 1601-9.
23. Elborai MM, Youssef MI. On stochastic solutions of nonlocal random functional integral equations. Arab J Math Sci. 2019; 25: 180-8.
24. Varadhan SRS. Asymptotic probabilities and differential equations. Commun Pure Appl Math. 1966; 19: 261-86.
25. Freidlin MI, Wentzell AD. Random perturbations of dynamical systems. New York: Springer; 1984.
26. Mo C, Luo J. Large deviations for stochastic differential delay equations. Nonlinear Anal. 2013; 80: 202-10.
27. Suvinthra M, Balachandran K. Large deviations for nonlinear Ito type stochastic integrodifferential equations. J Appl Nonlinear Dyn. 2017; 6: 1-15.
28. Shruthi G, Suvinthra M. Large deviations for stochastic functional differential equations with infinite delay. Indian J Ind Appl. Math. 2020; 11: 222-33.
29. Budhiraja A, Cheng J, Dupius P. Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stoch Process Their Appl. 2013; 123: 523-60.
30. Li Y. Large deviation principle for the mean reflected stochastic differential equation with jumps. J Inequal Appl. 2018; 295: 1-15.
31. Dupuis P, Ellis RS. A weak convergence approach to the theory of large deviations. New York: Wiley-Interscience; 1997.
32. Fleming WH. A stochastic control approach to some large deviations problems. Lect Notes Math. 1985; 1119: 52-66.

33. Budhiraja A, Dupius P. A variational representation for positive functionals of infinite dimensional Brownian motion. Probab Math Stat. 2008; 20: 39-61.
34. Yamada T, Watanabe S. On the uniqueness of solutions of stochastic differential equations. Kyoto J Math. 1971; 11: 155-67.

Corresponding author
Gopal Shruthi can be contacted at: shruthigopal89@gmail.com