A GRONWALL INEQUALITY WITH SINGULARITIES

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ABSTRACT. An inequality of Gronwall type including singularities is derived. Its application to solve uniqueness problems is showed and a connection between this type of inequalities and the Mittag-Leffler functions is also proved.

1. Introduction

Let $t \geq 0$ and $x \in \mathbb{R}^d$. We consider the following non-linear operator

$$\mathcal{M}(v)(t,x) := \int_{\mathbb{R}^d} Z(t, x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)|v(s,y)|^{\gamma-1}v(s,y)dyds$$

where $\gamma > 1$, $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is a given data with $p > 1$. The function v belongs to a Banach space E of continuous functions on the interval $[0,T]$. The space $L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is equipped with the usual norm $\|\cdot\|_1 + \|\cdot\|_p$. The pair (Z, Y) are kernels of convolution. By using the standard notation $f \star g$ for the convolution w.r.t. the spatial variable, we can write

$$\mathcal{M}(v)(t) = Z(t) \star u_0 + \int_0^t Y(t-s) \star |v(s,y)|^{\gamma-1}v(s,y)ds. \tag{1}$$

Depending on E, certain singularities may appear in estimates made on fixed points of $\mathcal{M}$. We recall that a fixed point of a mapping Φ, is an element w of its domain such that $\Phi(w) = w$. Some non-linear Cauchy problems can give rise to this type of operators, which motivates the study of the existence and uniqueness of their fixed points. The aim of this work is to obtain an inequality of Gronwall type and then apply it to obtain a result of fixed point uniqueness.

This paper is organized as follows. Section 2 recalls the classical Gronwall inequalities and compiles some properties of Mittag-Leffler functions. In Section 3 the main result is given in Theorem 3.1 together with Corollary 3.1, which involves standard Mittag-Leffler functions. The last section is devoted to apply the main result.

2. Preliminaries

In what follows we use the notation $f \lesssim g$ in D, which means that there exists a constant $C > 0$ such that $f \leq Cg$ in the set D. The constant may change line by line.

We recall the classical Gronwall inequality (Gronwall's lemma) and some applications (see, e.g. [3, Proposition 9.1.4]). Let $u : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that satisfies the inequality

$$u(t) \leq a + \int_0^t c(s)u(s)ds$$

for all $t \geq 0$, with the constant a and c an integrable function. Then

$$u(t) \leq ae^{\int_0^t c(s)ds}.$$

It is straightforward to see that $u \equiv 0$ whenever $a = 0$. Therefore, we can conclude uniqueness by using this inequality if we suppose two functions u_1, u_2 such that

$$|u_1(t) - u_2(t)| \leq \int_0^t c(s)|u_1(s) - u_2(s)|ds.$$

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Nevertheless, some inequalities may be more complicated in the sense that the function c includes singularities and a is not necessarily constant. For instance, the inequality

$$u(t) \le g(t) + \int_0^t (t-s)^{\alpha-1} u(s) ds$$

with $\alpha \in (0,1)$, is studied in [9] and some results of Gronwall type are proved.

On the other hand, it is well known some analytical properties of the Mittag-Leffler functions (see e.g., [4] and [8]). These functions are so named from the Swedish mathematician Gösta Mittag-Leffler (1846-1927) who introduced them at the beginning of the century XX (1903, 1904, 1905).

The Mittag-Leffler function of two real parameters $\alpha, \vartheta > 0$ ([1, Chapter 18]) is given by

$$E_{\alpha,\vartheta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\vartheta)}, \quad z \in \mathbb{C}.$$

In the literature we can find a generalization of this function with three complex parameters, as well as its relation with the Mellin-Barnes integral and the H-Function (also called Fox's H-function). See e.g., [5, Definition 1.4]. It is also known that $E_{\alpha,\vartheta}$, $\alpha, \vartheta > 0$, is an entire function.

Whenever $\vartheta = 1$,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad z \in \mathbb{C},$$

is the standard Mittag-Leffler function and $E_\alpha(-x)$ is completely monotonic for $x \ge 0$ if $0 < \alpha < 1$. The latter is thanks to the work of the American mathematician Harry Pollard (1919-1985).

In particular, the standard Mittag-Leffler function E_α has become a useful tool to obtain Gronwall type inequalities with singularities, which are crucial for analysing uniqueness of solutions.

Another fundamental fact on these functions is because they can be defined for any operator that generates a strongly continuous semigroup in a Banach space, using Zolotarev's formula (or Zolotarev-Pollard formula), in terms of Green functions or strongly continuous semigroups; see [3, Section 8.1]. This representation plays a fundamental role for obtaining estimates of the Green functions in evolution equations with fractional time derivative (see, e.g. [2], [6], [7]).

3. Gronwall-type inequality

Theorem 3.1. *Let $\alpha \in (0,1)$ and $\vartheta \ge 0$ such that $\alpha - \vartheta > 0$. Let $g(t)$ a non-negative function locally bounded on $t \in [0,T)$ with some $T > 0$. Suppose that $f(t)$ is non-negative and locally bounded on $[0,T)$ such that*

$$f(t) \le g(t) + C\int_0^t (t-s)^{\alpha-1} s^{-\vartheta} f(s) ds$$

for all $t \in [0,T)$, with some positive constant C. Then

$$f(t) \le g(t) + \int_0^t \left[\sum_{n=1}^{\infty} a_n (t-s)^{n\alpha-(n-1)\vartheta-1} s^{-\vartheta} g(s) \right] ds, \quad 0 \le t < T,$$

where

$$a_n = C^n \prod_{k=1}^{n-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha - k\vartheta)}.$$

Proof. The case $\vartheta = 0$ is straightforward from [9, Theorem 1]. For the case $\vartheta > 0$ we require some adjustments in its proof. First, we define the operator B given by

$$B\phi(t) := C\int_0^t (t-s)^{\alpha-1} s^{-\vartheta} \phi(s) ds, \quad t \ge 0,$$

for locally bounded functions ϕ. By construction, the operator B is linear and $B\phi_1 \le B\phi_2$ whenever $\phi_1 \le \phi_2$. Therefore, we have that

$$f(t) \le \sum_{k=0}^{n-1} B^k g(t) + B^n f(t), \quad n \ge 1. \tag{2}$$

Next, we prove that

$$B^n f(t) \le C^n \prod_{k=1}^{n-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \int_0^t (t-s)^{n\alpha-(n-1)\vartheta-1} s^{-\vartheta} f(s) ds \tag{3}$$

is true for all $n \in \mathbb{N}$ by induction. The case $n = 1$ follows straightforwardly from the definition of B. Now, we suppose that (3) is true for $N \in \mathbb{N}$ and applying B we obtain

$$\begin{aligned}
&B\left(B^N f\right)(t) \\
&= C\int_0^t (t-s)^{\alpha-1}s^{-\vartheta}B^N f(s)ds \\
&\leq C^{N+1}\prod_{k=1}^{N-1}\frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)}\int_0^t (t-s)^{\alpha-1}s^{-\vartheta}\left[\int_0^s (s-\tau)^{N\alpha-(N-1)\vartheta-1}\tau^{-\vartheta}f(\tau)d\tau\right]ds \\
&= C^{N+1}\prod_{k=1}^{N-1}\frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)}\int_0^t \left[\int_\tau^t (t-s)^{\alpha-1}s^{-\vartheta}(s-\tau)^{N\alpha-(N-1)\vartheta-1}ds\right]\tau^{-\vartheta}f(\tau)d\tau,
\end{aligned}$$

where the last line comes from the Fubini's theorem. Besides, the integral in the square brackets can be estimated with the substitution $s = \tau + z(t-\tau)$ as follows.

$$\begin{aligned}
&\int_\tau^t (t-s)^{\alpha-1}s^{-\vartheta}(s-\tau)^{N\alpha-(N-1)\vartheta-1}ds \\
&= \int_0^1 ((t-\tau)(1-z))^{\alpha-1}(\tau+z(t-\tau))^{-\vartheta}(z(t-\tau))^{N\alpha-(N-1)\vartheta-1}(t-\tau)dz \\
&\leq \int_0^1 ((t-\tau)(1-z))^{\alpha-1}(z(t-\tau))^{-\vartheta}(z(t-\tau))^{N\alpha-(N-1)\vartheta-1}(t-\tau)dz \\
&= (t-\tau)^{(N+1)\alpha-N\vartheta-1}\int_0^1 (1-z)^{\alpha-1}z^{N(\alpha-\vartheta)-1}dz \\
&= (t-\tau)^{(N+1)\alpha-N\vartheta-1}\frac{\Gamma(\alpha)\Gamma(N(\alpha-\vartheta))}{\Gamma((N+1)\alpha-N\vartheta)}.
\end{aligned}$$

Consequently,

$$B\left(B^N f\right)(t) \leq C^{N+1}\prod_{k=1}^{N}\frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)}\int_0^t (t-\tau)^{(N+1)\alpha-N\vartheta-1}\tau^{-\vartheta}f(\tau)d\tau$$

which proves the inductive step in (3).

Finally, since $\frac{\Gamma((k+1)(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \leq 1$ for k large enough, we have that

$$\lim_{n\to\infty} B^n f(t) = 0$$

and the expression (2) can be written as

$$f(t) \leq \sum_{n=0}^{\infty} B^n g(t).$$

The proof is complete. □

Under hypothesis of the previous theorem, if $\vartheta = 0$ and M is an upper bound of g, we see that

$$f(t) \leq M + M\int_0^t \left[\sum_{n=1}^{\infty} a_n(t-s)^{n\alpha-1}\right]ds, \quad 0 \leq t < T,$$

where

$$a_n = \frac{C^n\Gamma(\alpha)^n}{\Gamma(n\alpha)}, \quad n \geq 1.$$

It follows that

$$\begin{aligned} f(t) &\leq M + M\sum_{n=1}^{\infty} a_n \left[\int_0^t (t-s)^{n\alpha-1} ds\right] \\ &= M + M\sum_{n=1}^{\infty} a_n \frac{t^{n\alpha}}{n\alpha} \\ &= M + M\sum_{n=1}^{\infty} \frac{(C\Gamma(\alpha)t^{\alpha})^n}{\Gamma(n\alpha+1)} \\ &= M\sum_{n=0}^{\infty} \frac{(C\Gamma(\alpha)t^{\alpha})^n}{\Gamma(n\alpha+1)}. \end{aligned}$$

Therefore, we have the following estimate in terms of the standard Mittag-Leffler function.

Corollary 3.1. *Under hypothesis of Theorem 3.1, if $\vartheta = 0$ and M is an upper bound of g, it holds*

$$f(t) \leq M E_{\alpha}\left(C\Gamma(\alpha)t^{\alpha}\right).$$

In particular, $g \equiv 0$ implies that $f \equiv 0$.

4. An application for uniqueness of fixed points

In this section we use previous inequalities for showing uniqueness of fixed points of some non-linear operators in Banach spaces. For instance, considering $0 < \alpha < 1$ and $0 < \beta < 2$, we want to conclude that the non-linear operator (1),

$$\mathcal{M}(v)(t) = Z(t) \star u_0 + \int_0^t Y(t-s) \star |v(s,y)|^{\gamma-1} v(s,y) ds,$$

can only have at most one fixed point (solution) in the Banach space

$$E := C([0,T]; L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)) \cap C((0,T]; L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_E := \sup_{t\in[0,T]} \left(\|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1\right) + \sup_{t\in(0,T]} t^{\frac{\alpha d}{\beta p}} \|v(t,\cdot)\|_\infty,$$

whenever $\frac{d(\gamma-1)}{\beta p} < 1$. In this case, we suppose that there are two fixed points, u_1 and u_2, of $\mathcal{M}$. Using the property

$$|\,|a|^c a - |b|^c b\,| \lesssim |\,a-b\,|(|a|^c + |b|^c) \lesssim |\,a-b\,|(\,|a| + |b|\,)^c, \quad a, b \in \mathbb{R}, c > 0,$$

together with Young's inequality for convolutions, it follows that

$$\begin{aligned} \|u_1(t) - u_2(t)\|_1 &\leq \int_0^t \|Y(t-s,\cdot)\|_1 \||u_1(s,\cdot)|^{\gamma-1}u_1(s,\cdot) - |u_2(s,\cdot)|^{\gamma-1}u_2(s,\cdot)\|_1 ds \\ &\lesssim (\|u_1\|_E + \|u_2\|_E)^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \|u_1(s) - u_2(s)\|_1 ds. \end{aligned}$$

By applying Theorem 3.1, with $g = 0$ and $\vartheta = \frac{\alpha d(\gamma-1)}{\beta p}$, we see that $u_1 = u_2$.

Further details on $\mathcal{M}$ are available in [6, Section 3].

Availability of data and material

Not applicable. No datasets were generated or analysed during the current work.

Code availability

Not applicable.

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