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Mixed weak-type inequalities in Euclidean spaces and in spaces of the homogeneous type

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Abstract

In this paper, we provide mixed weak-type inequalities generalizing previous results in an earlier work by Caldarelli and the second author and also in the spirit of earlier results by Lorente et al. One of the main novelties is that, besides obtaining estimates in the Euclidean setting, results are provided as well in spaces of the homogeneous type, being the first mixed weak-type estimates that we are aware of in that setting.

KEYWORDS

commutators, Hörmander operators, mixed weighted inequalities

1 | INTRODUCTION

In [28], Muckenhoupt and Wheeden, introduced a new type of the weak-type inequality, that consists in considering a perturbation of the Hardy–Littlewood maximal operator, M, with an A_p weight. The result was the following, if $w \in A_1$ then

$$|\{x \in \mathbb{R}^n : w(x)Mf(x) > \lambda\}| \leq c_w \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x)dx.$$

Inequalities with this kind of perturbation, and some further ones that we describe in the following lines, are known in the literature as mixed weak-type inequalities.

In [34], having as an application an alternative proof to Muckenhoupt's A_p theorem, Sawyer settled the following result. If $u, v \in A_1$ then, in the case $n = 1$,

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{M(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{u,v} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} u(x)v(x)dx. \tag{1.1}$$

The preceding estimate was extended to higher dimensions and for Calderón–Zygmund operators in [7]. In that paper, new sufficient conditions on the weights for Equation (1.1) to hold were introduced as well. To be more precise, there it was shown that is u and v satisfy either $u, v \in A_1$ or $u \in A_1$ and $v \in A_\infty(u)$, then the inequality

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{\mathcal{T}(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{n,u,v} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} u(x)v(x)dx \tag{1.2}$$

holds for every t where $\mathcal{T}$ is either M or a Calderón–Zygmund operator. Note that the assumption $u, v \in A_1$ is weaker, in the sense that the product uv does not necessarily have any regularity, in contrast to what happens with the assumption $u \in A_1$ and $v \in A_\infty(u)$ for which $uv \in A_\infty$. It was also conjectured in [7] that the assumption $u, v \in A_1$ could be further weakened to $u \in A_1$ and $v \in A_\infty$. That conjecture was solved in the positive in [22].

Over the past few years, there have been some contributions on mixed weak-type inequalities such as [3] for the case of fractional integral and related operators, [1] for generalized maximal functions, [5, 29, 30] for related quantitative estimates and [23] for multilinear extensions. Results for commutators of Calderón Zygmund operators and Hörmander-type operators were obtained in [2, 4] (see as well [5] for quantitative estimates).

In this paper, we aim to provide some mixed weak-type estimates both in the Euclidean setting and in spaces of homogeneous type being the results in the latter setting the first ones in such a degree of generality that we are aware of. Also, our results can be regarded as a revisit of certain endpoint estimates in [25], and as a generalization of the estimates settled in [2, 4, 5]. In both cases, our approach will rely upon sparse domination, pushing forward ideas in [5]. We describe our contribution in the following subsections.

1.1 | Results in the Euclidean setting

Our first result is concerned with operators satisfying a bilinear sparse domination result. Given A a Young function (see Section 2.3 for the precise definition), we assume that T admits the following bilinear sparse bound:

$$\int T(f)g \leq c_n \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} \|f\|_{1,Q} \|g\|_{A,Q} |Q|, \tag{1.3}$$

where $\mathcal{S}_j$ are dyadic sparse families.

Theorem 1.1. *Let $1 \leq p, r < \infty$. Let $u \in A_1 \cap RH_q$ with $q = 2r - 1$ and $v \in A_p(u)$. Let A be a Young function such that $A \in B_\rho$ for all $\rho > r$ and T be an operator that satisfies Equation (1.3). Then,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{n,T} C_{u,v} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} u(x)v(x)dx,$$

where

$$C_{uv} = \kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} [u]_{A_1} [uv]_{A_\infty} \log(e + \kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} [u]_{A_1} [uv]_{A_\infty} [v]_{A_p(u)}).$$

In the case of $A(t) = t^r(1 + \log^+ t)^\gamma$ we have $\kappa_u = [u^r]_{A_\infty}^\gamma \leq [u]_{RH_q}^\gamma [u]_{A_1}^\gamma$.

Before presenting our next result, we need some notation. Let $\mathbf{b} = (b_1, b_2, \dots, b_m)$ be a set of symbols with $b_i \in \mathrm{Osc}_{\exp L^{r_i}}$, $i = 1, \dots, m$. Let $\mathbf{b} = \sigma \cup \sigma'$ where σ and σ' are pairwise disjoint sets. We introduce the following notation:

$$(b - \boldsymbol{\lambda})_\sigma = \prod_{i\in\sigma}(b_i(x) - \lambda_i),$$

$$|b - \boldsymbol{\lambda}|_\sigma = \prod_{i\in\sigma}|b_i(x) - \lambda_i|,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$. Let $C_j(b)$ the family of all the subsets σ of b such that $\#\sigma = j$. Observe that the cardinality of $C_j(b)$ is $\binom{m}{j}$. Also, we denote

$$\|\mathbf{b}\| = \prod_{i=1}^{m}\|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}}.$$

We remit the reader to Section 2.3 for the definition of the spaces $\mathrm{Osc}_{\exp L^{r_i}}$.

Having the notation above at our disposal, we define $T_{\mathbf{b}}$ as follows:

$$T_{\mathbf{b}}f(x) = [b_m, \dots, [b_2, [b_1, T]]]f(x).$$

At this point, we are in the position to present our next result. Let A be a Young function. We consider T an operator such that $T_{\mathbf{b}}$ satisfies the following bilinear sparse bound:

$$\int T_{\mathbf{b}}(f)g \le c_n \sum_{j=1}^{3^n}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in\mathcal{S}_j}\|f|b - b_Q|_\sigma\|_{1,Q}\|g|b(x) - b_Q|_{\sigma'}\|_{A,Q}|Q|, \tag{1.4}$$

where $\mathcal{S}_j$ are dyadic sparse families. For such operators, we have the following result.

Theorem 1.2. *Let $1 \le p, s < \infty$. Let $u \in A_1 \cap RH_q$ with $q = 2s - 1$ and $v \in A_p(u)$. Let $m \in \mathbb{N}$, $r_i \ge 1$ for every $1 \le i \le m$, $\frac{1}{r} = \sum_{i=1}^{m}\frac{1}{r_i}$ and $\mathbf{b} = (b_1, \dots, b_m)$, where $b_i \in Osc_{\exp L^{r_i}}$ for $1 \le i \le m$. Let A and B be Young functions such that $B^{-1}(t)\log(t)^{1/r} \lesssim A^{-1}(t)$ for all $t \ge e$ and $B \in B_\rho$ for all $\rho > s$ and T be an operator that satisfies Equation (1.4). Then,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T_{\mathbf{b}}(fv)(x)}{v(x)} > \lambda\right\}\right) \le c_{n,T}C_{u,v}\int_{\mathbb{R}^n}\varphi_{\frac{1}{r}}\left(\frac{|f(x)|\|\mathbf{b}\|}{\lambda}\right)u(x)v(x)dx,$$

where $\varphi_{\frac{1}{r}}(t) = t(1 + \log^+ t)^{\frac{1}{r}}$ and

$$C_{uv} = \sum_{h=0}^{m}\kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[uv]_{A_\infty}^{1+\frac{m-h}{r}}\log(e + [uv]_{A_\infty}^{1+\frac{m-h}{r}}\kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[v]_{A_p(u)})^{1+\frac{1}{r}}.$$

In the case of $A(t) = t^s(1 + \log^+ t)^\gamma$, we have $\kappa_u = [u^s]_{A_\infty}^\gamma \le [u]_{RH_q}^\gamma[u]_{A_1}^\gamma$.

The usual iterated commutator is just a particular case of the result above that consists just in assuming that all the symbols involved coincide. Let A be a Young function, m be a positive integer, and $b \in \mathrm{BMO}$ (bounded mean oscillation). We consider T an operator such that the m-order iterated commutator T_b^m satisfies the following bilinear sparse bound:

$$\int T_b^m(f)g \le c_n\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}\left[\||b - b_Q|^m fv\|_{1,Q}\|gu\|_{A,Q}|Q| + \|fv\|_{1,Q}\|gu|b - b_Q|^m\|_{A,Q}|Q|\right], \tag{1.5}$$

where $\mathcal{S}_j$ are dyadic sparse families. Then, we have the following result.

Theorem 1.3. *Let $1 \leq p, s < \infty$. Let $u \in A_1 \cap RH_q$ with $q = 2s - 1$ and $v \in A_p(u)$. Let $m \in \mathbb{N}$ and $b \in BMO$. Let A and B be Young functions such that $B^{-1}(t)\log(t)^m \lesssim A^{-1}(t)$ for all $t \geq e$ and $B \in B_\rho$ for all $\rho > s$. Let T be an operator that satisfies Equation (1.5). Then,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T_b^m(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{n,T} C_{u,v} \int_{\mathbb{R}^n} \varphi_m\left(\frac{|f(x)|\|b\|_{BMO}^m}{\lambda}\right) u(x)v(x)dx,$$

where $\varphi_m(t) = t(1 + \log^+ t)^m$ and

$$C_{uv} = [u]_{RH_q}^{1+\frac{q}{4s}} \kappa_u [u]_{A_1} [uv]_{A_\infty}^{1+m} \log(e + \kappa_u [u]_{A_1} [uv]_{A_\infty}^{1+m} [v]_{A_p(u)})^{1+m}$$
$$+ [u]_{RH_q}^{1+\frac{q}{4s}} \kappa_u [u]_{A_1} [uv]_{A_\infty} \log(e + \kappa_u [u]_{A_1} [uv]_{A_\infty} [v]_{A_p(u)}).$$

1.2 | Results in spaces of homogeneous type

We recall that (X, d, μ) is a space of homogeneous type if X is a set endowed with a quasi-metric d and a doubling Borel measure μ. d is a quasi-metric if there exists a constant $\kappa_d \geq 1$ such that

$$d(x, y) \leq \kappa_d(d(x, z) + d(z, y)) \qquad x, y, z \in X,$$

namely, if the triangle inequality holds modulo a constant. Since μ satisfies the doubling property, we have that there exists $c_\mu \geq 1$ such that

$$\mu(B(x, 2\rho)) \leq c_\mu \mu(B(x, \rho)) \qquad x \in X, \quad \rho > 0,$$

where $B(x, \rho) := \{y \in X : d(x, y) < \rho\}$. We will assume additionally that all balls B are Borel sets and that $0 < \mu(B) < \infty$.

Since μ is a Borel measure defined on the Borel σ-algebra of the quasi-metric space (X, d) we have that the Lebesgue differentiation theorem holds. This yields that continuous functions with bounded support are dense in $L^p(X)$ for every $1 \leq p < \infty$.

Let A be a Young function. Let T be an operator satisfying the following bilinear sparse bound:

$$\int T(f)g \leq c\frac{1}{1-\varepsilon} \sum_{j=1}^{l} \sum_{Q \in S_j} \|f\|_{1,Q}\|g\|_{A,Q}\mu(Q), \tag{1.6}$$

where S_j are ε-sparse families of dyadic cubes (see Section 2.1 for the precise definition). Observe that in contrast with the results in the Euclidean case, in this setting we need to be precise about the sparseness constant of the families involved since it plays a role in the proof.

Theorem 1.4. *Let $1 \leq p, r < \infty$. Let $u \in A_1 \cap RH_q$ with $q = 2r - 1$ and $v \in A_p(u)$. Let A be a Young function such that $A \in B_\rho$ for all $\rho > r$ and let T be an operator that satisfies Equation (1.6). Then,*

$$uv\left(\left\{x \in X : \frac{T(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{X,T} C_{u,v} \int_X \frac{|f(x)|}{\lambda} u(x)v(x)d\mu(x),$$

where

$$C_{uv} = \kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} [u]_{A_1} [uv]_{A_p} [uv]_{A_\infty} \log(e + c_{n,p}\kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} [u]_{A_1} [v]_{A_p(u)} [uv]_{A_p}^3).$$

In the case of $A(t) = t^r(1 + \log^+ t)^\gamma$ we have $\kappa_u = [u^r]_{A_\infty}^\gamma \leq [u]_{RH_q}^\gamma [u]_{A_1}^\gamma$.

In the case of iterated commutators, we need to assume that

$$\int T_{\mathbf{b}}(f)g \leq c\frac{1}{1-\varepsilon}\sum_{j=1}^{l}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in S_j}\left(\int_Q f|b-b_Q|_{\sigma'}d\mu\right)\|g|b-b_Q|_{\sigma}\|_{A,Q}, \tag{1.7}$$

where S_j are ε-sparse families of dyadic cubes (see Section 2.1 for the precise definition). Under this assumption, we have the following result.

Theorem 1.5. *Let $1 \leq p, s < \infty$. Let $u \in A_1 \cap RH_q$ with $q = 2s-1$ and $v \in A_p(u)$. Let $m \in \mathbb{N}$, $r_i \geq 1$ for every $1 \leq i \leq m$, $\frac{1}{r} = \sum_{i=1}^{m}\frac{1}{r_i}$ and $\mathbf{b} = (b_1, \dots, b_m)$ where $b_i \in Osc_{\exp L^{r_i}}$ for $1 \leq i \leq m$. Let A and B be Young functions such that $B^{-1}(t)\log(t)^{1/r} \lesssim A^{-1}(t)$ for all $t \geq e$ and $B \in B_\rho$ for all $\rho > s$ and T be an operator that satisfies Equation (1.7). Then,*

$$uv\left(\left\{x \in X : \frac{T_{\mathbf{b}}(fv)(x)}{v(x)} > \lambda\right\}\right) \leq c_{n,T}C_{u,v}\int_X \varphi_{\frac{1}{r}}\left(\frac{|f(x)|\|\mathbf{b}\|}{\lambda}\right)u(x)v(x)dx,$$

where $\varphi_{\frac{1}{r}}(t) = t(1+\log^+ t)^{\frac{1}{r}}$ and

$$C_{u,v} = \sum_{h=0}^{m}\tau_{u,v,h}[uv]_{A_\infty}\log(e+\tau_{u,v,h}[uv]_{A_p}^2[v]_{A_p(u)})^{1+\frac{1}{r}},$$

where $\tau_{u,v,h} = \kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}\left([uv]_{A_p}[uv]_{A_\infty}\right)^{\frac{m-h}{r}}[uv]_{A_p}$. In the case $A(t) = t^s(1+\log^+ t)^\gamma$, additionally we have that $\kappa_u = [u^s]_{A_\infty}^\gamma \leq [u]_{RH_q}^\gamma[u]_{A_1}^\gamma$.

Remark 1.6. At this point, we would like to note that the dependencies obtained in the case of spaces of homogeneous type are slightly worse than those in the Euclidean setting. The additional constants appear due to the fact that reverse Hölder inequality is actually a weak reverse Hölder inequality for balls instead of cubes in this setting, and due to the fact that doubling conditions do not behave as good as in the Euclidean setting.

The remainder of the paper is organized as follows. In Section 2, we provide some preliminaries. Section 3 is devoted to the proofs of the main results. Finally, in Section 4 we show how to derive results for A-Hörmander operators and their commutators from the main results and we provide a sparse domination result for $T_{\mathbf{b}}$ in the context of spaces of homogeneous type, generalizing [10, 16].

2 | PRELIMINARIES

2.1 | Dyadic structures on spaces of homogeneous type

We shall follow the presentation and the notation provided in [27]. Let us fix $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$. Assume that for each $k \in \mathbb{Z}$ we have an index set J_k and a pairwise disjoint collection $\mathcal{D}_k = \{Q_j^k\}_{j\in J_k}$ of measurable sets and an associated collection of points $\{z_j^k\}_{j\in J_k}$. We will say that $\mathcal{D} = \bigcup_{k\in\mathbb{Z}}\mathcal{D}_k$ is a dyadic system with parameters c_0, C_0 and δ if the following properties hold.

(1) For every $k \in \mathbb{Z}$

$$X = \bigcup_{j\in J_k} Q_j^k.$$

(2) For $k \geq l$ if $P \in \mathcal{D}_k$ and $Q \in \mathcal{D}_l$ then either $Q \cap P = \emptyset$ or $P \subseteq Q$.

(3) For each $k \in \mathbb{Z}$ and $j \in J_k$

$$B(z_j^k, c_0\delta^k) \subseteq Q_j^k \subseteq B(z_j^k, C_0\delta^k).$$

We will call the elements of $\mathcal{D}$ cubes and we will denote

$$\mathcal{D}(Q) := \{P \in \mathcal{D} : P \subseteq Q\}$$

the family of cubes of $\mathcal{D}$ that are contained in Q. We will say, as well, that an estimate depends on $\mathcal{D}$ if it depends on the parameters c_0, C_0, and δ.

The point z_j^k could be regarded as the "center" and δ^k as the "side length" of each cube $Q_j^k \in \mathcal{D}_k$. These need to be with respect a certain $k \in \mathbb{Z}$ since k may not be unique. Consequently, a cube Q also encodes the information of its center z and generation k.

We define the dilations αQ for $\alpha \geq 1$ of $Q \in \mathcal{D}$ as

$$\alpha Q := B(z_j^k, \alpha C_0\delta^k).$$

Abusing of this dilation notation, we denote

$$1Q := B(z_j^k, C_0\delta^k).$$

Note that these dilations are not cubes anymore but balls.

The following proposition, settled in [14], ensures the existence of dyadic systems that provide a convenient replacement for the translations of the usual dyadic systems in Euclidean spaces.

Proposition 2.1. *Let (X, d, μ) be a space of homogeneous type. There exist $0 < c_0 \leq C_0 < \infty$, $\gamma \geq 1$, $0 < \delta < 1$ and $m \in \mathbb{N}$ such that there are dyadic systems $\mathcal{D}_1, \dots, \mathcal{D}_m$ with parameters c_0, C_0, and δ, and with the property that for each $s \in X$ and $\rho > 0$ there is a $j \in \{1, \dots, m\}$ and a $Q \in \mathcal{D}_j$ such that*

$$B(s, \rho) \subseteq Q, \qquad \text{and} \qquad diam(Q) \leq \gamma\rho.$$

We end up this section borrowing from [27] the following covering Lemma.

Lemma 2.2. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D}$ a dyadic system with parameters c_0, C_0, and δ. Suppose that $diam(X) = \infty$, take $\alpha \geq 3c_d^2/\delta$ and let $E \subseteq X$ satisfy $0 < diam(E) < \infty$. Then, there exists a partition $\mathcal{P} \subseteq \mathcal{D}$ of X such that $E \subseteq \alpha Q$ for all $Q \in \mathcal{P}$.*

2.2 | Weights

We recall that given a weight u, $v \in A_p(u)$ if

$$[v]_{A_p(u)} = \sup_Q \frac{1}{u(Q)} \int_Q vu \left(\frac{1}{u(Q)} \int_Q v^{-\frac{1}{p-1}} u \right)^{p-1} < \infty,$$

in the case $1 < p < \infty$ and

$$[v]_{A_1(u)} = \left\| \frac{M_u v}{v} \right\|_\infty < \infty,$$

where $M_u v = \sup_Q \frac{1}{u(Q)} \int_Q vu$. If $u = 1$ we recover the classical Muckenhoupt's condition. We would like also to recall that

$$A_\infty = \bigcup_{p \geq 1} A_p$$

with the constant

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) < \infty.$$

Also we recall the Reverse-Hölder's condition, $w \in \mathrm{RH}_s$, $1 < s < \infty$ if

$$[w]_{\mathrm{RH}_s} = \sup_Q \frac{\left(\frac{1}{|Q|} \int_Q w^s\right)^{\frac{1}{s}}}{\frac{1}{|Q|} w(Q)} < \infty.$$

If $s = 1$ the condition RH_1 is trivial.

An auxiliary lemma that we need is the following.

Lemma 2.3. *Let $1 \leq s \leq q < \infty$. If $u \in A_1 \cap RH_q$ then $u^s \in A_1$ and $[u^s]_{A_1} \leq [u]_{RH_q} [u]_{A_1}$*

In spaces of homogeneous type, we recall that the A_p classes and the A_p and A_∞ constants are defined exactly in the same way as we showed above for the Euclidean setting just replacing cubes by balls. There is an important difference with the Euclidean setting for the reverse Hölder classes. We say that $w \in \mathrm{RH}_q$ if

$$\left(\frac{w^q(B)}{\mu(B)}\right)^{\frac{1}{q}} \leq [w]_{\mathrm{RH}_q} \frac{w(c_d B)}{\mu(c_d B)},$$

where c_d is some constant depending on the space. Another fundamental result for us will be the sharp reverse Hölder inequality that was settled in [15]. There it was established that if $w \in A_\infty$, then

$$\left(\frac{u^t(B)}{\mu(B)}\right)^{\frac{1}{t}} \leq c \frac{u(c_d B)}{\mu(c_d B)},$$

where $1 \leq t \leq 1 + \frac{1}{\tau [u]_{A_\infty}}$ and $c, \tau > 0$ are some constants depending on the space. Note that from this property it follows as well that if $E \subset B$ then

$$w(E) \leq c \left(\frac{\mu(E)}{\mu(c_d B)}\right)^{\frac{1}{c_X [w]_\infty}} w(c_d B)$$

for some constants c, c_X depending on the space X.

Note that if $w \in A_p$, then for every ball B, we have that

$$w(\lambda B) \leq C_{\mu,\lambda} [w]_{A_p} w(B).$$

We continue with the following sum property.

Lemma 2.4. *Let $w \in A_p$ and $\mathcal{S}$ be an η-sparse family of cubes. Then*

$$\sum_{Q\in\mathcal{S}} w(Q) \le c\frac{1}{\eta^p}[w]_{A_p} w\left(\bigcup_{Q\in\mathcal{S}} Q\right).$$

Proof. First, we note that

$$\begin{aligned} 1 = \frac{\mu(Q)}{\mu(Q)} &\le \frac{1}{\eta}\frac{\mu(E_Q)}{\mu(Q)} = \frac{1}{\eta}\frac{1}{\mu(Q)}\int_{E_Q} w^{\frac{1}{p}} w^{-\frac{1}{p}} \\ &\le \frac{1}{\eta}\left(\frac{w(E_Q)}{\mu(Q)}\right)^{\frac{1}{p}}\left(\frac{w^{-\frac{1}{p-1}}(E_Q)}{\mu(Q)}\right)^{\frac{1}{p'}} \\ &\lesssim \frac{1}{\eta}\left(\frac{w(E_Q)}{\mu(Q)}\right)^{\frac{1}{p}}\left(\frac{w^{-\frac{1}{p-1}}(1Q)}{\mu(1Q)}\right)^{\frac{1}{p'}} \\ &\le \frac{1}{\eta}\left(\frac{w(E_Q)}{\mu(Q)}\right)^{\frac{1}{p}}[w]_{A_p}^{\frac{1}{p}}\left(\frac{\mu(1Q)}{w(1Q)}\right)^{\frac{1}{p}} \end{aligned}$$

and hence

$$w(Q) \lesssim \frac{1}{\eta^p}[w]_{A_p} w(E_Q).$$

Taking this into account

$$\sum_{Q\in\mathcal{S}} w(Q) \lesssim \frac{1}{\eta^p}[w]_{A_p}\sum_{Q\in\mathcal{S}} w(E_Q) = \frac{1}{\eta^p}[w]_{A_p} w\left(\bigcup_{Q\in\mathcal{S}} Q\right)$$

and we are done. □

We end this section with the following lemma that readily follows from the definitions both in the Euclidean setting and for spaces of homogeneous type.

Lemma 2.5. *If $u \in A_1$ and $v \in A_p(u)$, then $uv \in A_p$.*

2.3 | Young functions and Orlicz averages

Now, we recall that given a Young function $A : [0, \infty) \to [0, \infty)$, namely a convex, non-decreasing function such that $A(0) = 0$ and $A(t) \to \infty$ when $t \to \infty$ we can define the average on weighted Luxemburg norm

$$\|f\|_{A(u),Q} = \inf\left\{\lambda > 0 : \frac{1}{u(Q)}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)u(x)dx \le 1\right\}.$$

Also, we can define the Luxemburg norm on spaces of homogenous type just replacing the Lebesgue measure, dx, by the corresponding measure, $d\mu$, and cubes by balls.

It is also possible to settle a generalized Hölder inequality for Young functions. If $B^{-1}(t)C^{-1}(t) \leq cA^{-1}(t)$ for $t \geq t_0 > 1$ then

$$\|fg\|_{A(u),Q} \leq c\|f\|_{B(u),Q}\|g\|_{C(u),Q}.$$

We shall drop u in the notation in the case of Lebesgue measure. For each A Young function, we define the associated Young function $\bar{A}$ by $\bar{A}(t) = \sup_{0\leq s<\infty} ts - A(s)$. Note that $A^{-1}(t)\bar{A}^{-1}(t) \leq 2t$.

A Young function A is said to be submultiplicative if there exists $c_A \geq 1$ such that $A(ts) \leq c_A A(t)A(s)$.

Let u be a weight and A be a Young function. We define the maximal operator $M^{\mathcal{F}}_{A(u)}$ by

$$M^{\mathcal{F}}_{A(u)} = \sup_{x\in Q\subset\mathcal{F}} \|f\|_{A(u),Q},$$

where the supremum is taken over all the cubes in the family $\mathcal{F}$.

We say $A \in B_p$ if

$$\int_1^\infty \frac{A(t)}{t^p}\frac{dt}{t} < \infty.$$

Given $1 < p < \infty$, M_A is bounded on L^p if and only if $A \in B_p$, for more details see [31]. Observe that if $A \in B_p$ for all $p > 1$ we get $A(t) \leq c_n\kappa(\varepsilon)t^{1+\varepsilon}$ with $\kappa : (0,\infty) \to (0,\infty)$ for all $\varepsilon > 0$. For example, if $A(t) = t(1+\log^+ t)^\gamma$ then $A(t) \leq (2\gamma)^\gamma\varepsilon^{-\gamma}t^{1+\varepsilon}$ for $t \geq e$.

An important result that connect the average given by a Young function with the class of weights $A_p(u)$ is contained in the following lemma.

Lemma 2.6. *[5] Let u a weight, $v \in A_p(u)$ and Φ a Young function. Then, for every cube Q,*

$$\|f\|_{\Phi(u),Q} \leq \|f\|_{[v]_{A_p(u)}\Phi^p(uv),Q}.$$

Now, we recall if $b \in \mathrm{BMO}$ then

$$\sup_Q \|b-b_Q\|_{\exp L,Q} \leq c_n\|b\|_{\mathrm{BMO}}.$$

It is possible to define classes of symbols with ever better properties of integrability that BMO symbols. Given $r > 1$ we say that $b \in \mathrm{Osc}_{\exp L^r(w)}$ if

$$\|b\|_{\mathrm{Osc}_{\exp L^r(w)}} = \sup_Q \|b-b_Q\|_{\exp L^r(w),Q} < \infty.$$

Note that $\mathrm{Osc}_{\exp L^r} \subsetneq \mathrm{BMO}$ for every $r > 1$. It is not hard to prove that for those classes of functions the following estimate hold

Lemma 2.7. *Let $j > 0$, $w \in A_\infty$ and $b \in Osc_{\exp L^r}$ with $r > 1$. Then*

$$\||b-b_Q|^j\|_{\exp L^{\frac{r}{j}}(w),Q} \leq c[w]_{A_\infty}^{\frac{j}{r}}\|b\|^j_{Osc_{\exp L^r}}.$$

To deal with the case of spaces of the homogeneous type, we will need the following version of the lemma above. Observe that a worse dependence on the constant appears, due to the fact that we need to change balls by cubes to use John–Nirenberg's type inequality.

Lemma 2.8. *Let Q be a cube in a dyadic structure and $b \in Osc_{\exp L^r}$ with $r > 1$. If $w \in A_p$ then*

$$\|b - b_Q\|_{\exp L^r(w),Q} \lesssim \|b\|_{Osc_{\exp L^r}} [w]_{A_p}^{\frac{1}{r}} [w]_{A_\infty}^{\frac{1}{r}}.$$

Proof. First, we note that it is not hard to check that $b \in Osc_{\exp L^r}$, implies that

$$\mu\{x \in B \ : \ |b(x) - b_B| > t\} \le e\mu(B) \mathrm{e}^{-\frac{t^r}{2^r \|b\|^r_{Osc_{\exp L^r}}}}.$$

Bearing that in mind we continue our argument observing that since

$$|b(x) - b_Q| \le |b(x) - b_{1Q}| + c\|b\|_{Osc_{\exp L^r}},$$

we have that

$$\|b - b_Q\|_{\exp L^r(w),Q} \le \|b - b_{1Q}\|_{\exp L^r(w),Q} + c\|b\|_{Osc_{\exp L^r}}$$

and hence it suffices to deal with $\|b - b_{1Q}\|_{\exp L^r(w),Q}$. We argue as follows:

$$\begin{aligned}
&\frac{1}{w(Q)} \int_Q \left(\exp\left(\frac{|b(x) - b_{1Q}|^r}{\lambda^r} \right) - 1 \right) w(x) d\mu(x) \\
&\le \frac{1}{w(Q)} \int_{1Q} \left(\exp\left(\frac{|b(x) - b_{1Q}|^r}{\lambda^r} \right) - 1 \right) w(x) d\mu(x) \\
&\le \frac{1}{w(Q)} \int_0^\infty \mathrm{e}^t w\left(\left\{ x \in 1Q \ : \ \frac{|b(x) - b_{1Q}|^r}{\lambda^r} > t \right\} \right) dt \\
&\le \frac{1}{w(Q)} \int_0^\infty \mathrm{e}^t \left(\frac{\mu\left(\left\{ x \in 1Q \ : \ \frac{|b(x) - b_{1Q}|^r}{\lambda^r} > t \right\} \right)}{\mu(c_d Q)} \right)^{\frac{1}{c_X [w]_{A_\infty}}} w(c_d Q) dt \\
&= \frac{1}{w(Q)} \int_0^\infty \mathrm{e}^t \left(\frac{\mu\left(\left\{ x \in 1Q \ : \ |b(x) - b_{1Q}| > \lambda t^{\frac{1}{r}} \right\} \right)}{\mu(c_d Q)} \right)^{\frac{1}{c_X [w]_{A_\infty}}} w(c_d Q) dt \\
&\le c \frac{1}{w(Q)} \int_0^\infty \mathrm{e}^t \left(\mathrm{e}^{-\left(\frac{t\lambda^r}{2^r \|b\|^r_{Osc_{\exp L^r}}} \right) \frac{1}{c_X [w]_{A_\infty}}} \right) w(c_d Q) dt \\
&\le c \frac{w(c_d Q)}{w(\frac{c_0}{C_0} Q)} \int_0^\infty \mathrm{e}^{t\left[1 - \left(\frac{\lambda^r}{2^r \|b\|^r_{Osc_{\exp L^r}} c_X [w]_{A_\infty}} \right) \right]} dt \\
&\le \tilde{c} [w]_{A_p} \int_0^\infty \mathrm{e}^{t\left[1 - \left(\frac{\lambda^r}{2^r \|b\|^r_{Osc_{\exp L^r}} c_X [w]_{A_\infty}} \right) \right]} dt = (*).
\end{aligned}$$

Choosing $\lambda = 2^{\frac{1}{r}} 2\|b\|_{Osc_{\exp L^r}} \tilde{c}^{\frac{1}{r}} c_X^{\frac{1}{r}} [w]_{A_\infty}^{\frac{1}{r}} [w]_{A_p}^{\frac{1}{r}}$ we have that

$$(*) = \tilde{c}[w]_{A_p} \int_0^\infty \mathrm{e}^{t\left[1-\left(\frac{\left(2^{\frac{1}{r}} 2\|b\|_{Osc_{\exp L^r}} \tilde{c}^{\frac{1}{r}} c_X^{\frac{1}{r}} [w]_{A_\infty}^{\frac{1}{r}} [w]_{A_p}^{\frac{1}{r}}\right)^r}{2^r \|b\|_{Osc_{\exp L^r}}^r c_X [w]_{A_\infty}}\right)\right]} dt = \tilde{c}[w]_{A_p} \int_0^\infty \mathrm{e}^{t\left[1-\tilde{c}2[w]_{A_p}\right]} dt$$

$$\leq \tilde{c}[w]_{A_p} \int_0^\infty \mathrm{e}^{\left(1-\tilde{c}[w]_{A_p}2\right)t} dt = \frac{\tilde{c}[w]_{A_p}}{2\tilde{c}[w]_{A_p} - 1} \leq 1$$

and this yields

$$\|b - b_{1Q}\|_{\exp L^r(w),Q} \lesssim \|b\|_{Osc_{\exp L^r}} [w]_{A_\infty}^{\frac{1}{r}} [w]_{A_p}^{\frac{1}{r}}.$$

Gathering the estimates above we are done. □

3 | PROOFS OF MAIN RESULTS

3.1 | Scheme of the proofs of the main results

In this section, we briefly outline the scheme that we are going to follow for each of the proofs of the estimates in the main results. As we mentioned in the introduction, the scheme can be traced back to [5, 9, 21, 24]. Let T be a linear operator and $\mathcal{M}$ maximal type and dyadic, in some sense, operator such that

$$uv(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}) \lesssim \int A\left(\frac{|f|}{t}\right) uv,$$

where A is a submultiplicative Young function. Note that by homogeneity it suffices to show that

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > 1\right\}\right) \lesssim \kappa_{u,v} \int A(|f|)uv,$$

where $\kappa_{u,v} \geq 1$ is the constant given by the dependence on the weights involved. Taking that into account we could proceed as follows:

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > 1\right\}\right) \leq uv\left(\left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > 1, \mathcal{M}f(x) \leq \frac{1}{2}\right\}\right)$$
$$+ uv\left(\left\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \frac{1}{2}\right\}\right).$$

Since the desired estimate holds for the second term it suffices to control the first one. Let us call

$$G = \left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > 1, \mathcal{M}f(x) \leq \frac{1}{2}\right\}.$$

Then, it suffices to prove

$$uv(G) \leq c_{n,T} \kappa_{u,v} \int A(|f|)uv + \frac{1}{2} uv(G),$$

since this yields

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{T(fv)(x)}{v(x)} > 1, \mathcal{M}f(x) \leq \frac{1}{2}\right\}\right) \leq 2c_{n,T}\kappa_{u,v} \int A(|f|)uv.$$

3.2 | Proofs of the results in the Euclidean setting

3.2.1 | Lemmatta

Lemma 3.1. *Let $1 \leq r < \infty$. If $u \in RH_q$ with $q = 2r - 1$ and $u^r \in A_\infty$, then for $s = 1 + \frac{1}{2\tau_n [u^r]_{A_\infty}}$ and any measurable subset $E \subset Q$,*

$$\frac{u^{sr}(E)}{u^{sr}(Q)} \lesssim [u]_{RH_q}^{\frac{q}{4}} \left(\frac{u(E)}{u(Q)}\right)^{\frac{1}{4}}.$$

Proof. First, let see

$$u^r(E) \leq [u]_{\mathrm{RH}_q}^{q/2} \left(\frac{u(E)}{u(Q)}\right)^{1/2} u^r(Q). \tag{3.1}$$

Indeed, since $u \in \mathrm{RH}_q$ we obtain

$$\left(\frac{u^q(Q)}{u(Q)}\right)^{1/2} \leq [u]_{\mathrm{RH}_q}^{q/2} \left(\frac{u(Q)}{|Q|}\right)^{r-1} \leq [u]_{\mathrm{RH}_q}^{q/2} \frac{|Q|}{u(Q)} \left(\frac{u^r(Q)}{|Q|}\right) = [u]_{\mathrm{RH}_q}^{q/2} \frac{u^r(Q)}{u(Q)}.$$

Then

$$u^r(E) \leq u(E)^{1/2} u^q(Q)^{1/2} \leq u(E)^{1/2} [u]_{\mathrm{RH}_q}^{q/2} \frac{u^r(Q)}{u(Q)^{1/2}} \leq [u]_{\mathrm{RH}_q}^{q/2} \left(\frac{u(E)}{u(Q)}\right)^{1/2} u^r(Q).$$

In the other hand, since $s = 1 + \frac{1}{2\tau_n [u^r]_{A_\infty}}$ then by a similar argument as above we get

$$\left(\frac{u^{r(2s-1)}(Q)}{u^r(Q)}\right)^{1/2} \lesssim \frac{u^{rs}(Q)}{u^r(Q)}$$

and

$$\frac{u^{sr}(E)}{u^{sr}(Q)} \lesssim \left(\frac{u^r(E)}{u^r(Q)}\right)^{1/2}. \tag{3.2}$$

Taking into account Equations (3.1) and (3.2), we obtain

$$\frac{u^{sr}(E)}{u^{sr}(Q)} \lesssim \left(\frac{u^r(E)}{u^r(Q)}\right)^{1/2} \leq [u]_{\mathrm{RH}_q}^{q/4} \left(\frac{u(E)}{u(Q)}\right)^{1/4}.$$

□

Lemma 3.2. *Let $1 \le r < \infty$ and A be a Young function such that $A \in B_\rho$ for all $\rho > r$. If $u \in RH_q$ with $q = 2r-1$ and $u^r \in A_\infty$ then for any cube Q and G measurable subset*

$$\|\chi_G u\|_{A,Q} \le c_n \kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} \langle u\rangle_{Q,1} \langle \chi_G\rangle_{Q,s}^u.$$

with $s = 4(1+\frac{1}{2\tau_n [u^r]_{A_\infty}})r$.

Remark 3.3. Observe that if $A(t) = t^r(1+\log^+ t)^\gamma$ then for any $0 < \varepsilon \le 1$ we have $A(t) \le c_{\gamma,r}\varepsilon^{-\gamma} t^{r(1+\varepsilon)}$ and $\kappa_u = C[u^r]_{A_\infty}^\gamma$.

Proof. Since $A \in B_\rho$ for all $\rho > r$ then $A(t) \le c_n \zeta_\varepsilon t^{r(1+\varepsilon)}$ for all $\varepsilon > 0$. Then

$$\|\chi_G u\|_{A,Q} \le c_n \zeta_\varepsilon \|\chi_G u\|_{r(1+\varepsilon),Q} = c_n\zeta_\varepsilon \|\chi_G u^r\|_{1+\varepsilon,Q}^{\frac{1}{r}} = c_n \kappa_u \|\chi_G u^r\|_{1+\varepsilon,Q}^{\frac{1}{r}} \tag{3.3}$$

with $\varepsilon = \frac{1}{2\tau_n [u^r]_{A_\infty}}$ and $\kappa_u = \zeta_{\frac{1}{2\tau_n [u^r]_{A_\infty}}}$. Using Lemma 3.1 we have

$$\left(\frac{u^{r(1+\varepsilon)}(G\cap Q)}{u^{r(1+\varepsilon)}(Q)}\right)^{\frac{1}{1+\varepsilon}} \le c_n [u]_{\mathrm{RH}_q}^{\frac{q}{4(1+\varepsilon)}} \left(\frac{u(G\cap Q)}{u(Q)}\right)^{\frac{1}{4(1+\varepsilon)}}. \tag{3.4}$$

On the other hand, since $1+2\varepsilon = 1 + \frac{1}{\tau_n [u^r]_{A_\infty}}$ and $u \in \mathrm{RH}_q$, with $q = 2r-1 > r$, we get

$$\begin{aligned}
\left(\frac{u^{r(1+\varepsilon)}(Q)}{|Q|}\right)^{\frac{1}{r(1+\varepsilon)}} &\le \left(\frac{u^r(Q)}{|Q|}\right)^{\frac{1}{2r(1+\varepsilon)}} \left(\frac{1}{|Q|}\int_Q u^{r(1+2\varepsilon)}\right)^{\frac{1}{2r(1+\varepsilon)}} \\
&\le 2^{\frac{1}{r}} \left(\frac{u^r(Q)}{|Q|}\right)^{\frac{1}{2r(1+\varepsilon)}} \left(2\frac{u^r(Q)}{|Q|}\right)^{\frac{1+2\varepsilon}{2r(1+\varepsilon)}} = 2^{\frac{1}{r}} \left(\frac{u^r(Q)}{|Q|}\right)^{\frac{1}{r}} \\
&\le 2^{\frac{1}{r}} \left(\frac{u^q(Q)}{|Q|}\right)^{\frac{1}{q}} \le c_n [u]_{RH_q} \frac{u(Q)}{|Q|}.
\end{aligned} \tag{3.5}$$

Taking into account the inequalities above, we obtain

$$\begin{aligned}
\|\chi_G u\|_{A,Q} &\le c_n \kappa_u \|\chi_G u^r\|_{1+\varepsilon,Q}^{\frac{1}{r}} \\
&\le c_n \kappa_u \left(\frac{u^{r(1+\varepsilon)}(Q)}{|Q|}\right)^{\frac{1}{r(1+\varepsilon)}} \left(\frac{u^{r(1+\varepsilon)}(G\cap Q)}{u^{r(1+\varepsilon)}(Q)}\right)^{\frac{1}{r(1+\varepsilon)}} \\
&\le c_n \kappa_u [u]_{RH_q}^{1+\frac{q}{4r(1+\varepsilon)}} \langle u\rangle_{Q,1} \langle \chi_G\rangle_{Q,4(1+\varepsilon)r}^u \\
&\le c_n \kappa_u [u]_{RH_q}^{1+\frac{q}{4r}} \langle u\rangle_{Q,1} \langle \chi_G\rangle_{Q,4(1+\varepsilon)r}^u.
\end{aligned}$$

□

Now, we recall the followings lemmas proved in [5]

Lemma 3.4 [5]. *Let $\gamma_1, \gamma_2 > 1$. For every j, k nonnegative integers let*

$$\alpha_{j,k} = \min \left\{\gamma_1 2^{-k} j^{\rho_1}, \beta\gamma_2 2^{-j} 2^{-k} 2^{\delta k} k^{\rho_2}\right\},$$

where $\rho_1, \rho_2, \delta \geq 0$ and $\beta > 0$. Then

$$\sum_{j,k\geq 0} \alpha_{j,k} \leq c_{\rho_1,\rho_2,\gamma,\delta}\gamma_1 \log_2(e+\gamma_2)^{1+\rho_1} + \frac{1}{2\gamma}\beta,$$

where $\gamma \geq 1$.

Lemma 3.5 [5]. *Let A be a Young function such that $A(xy) \leq c_A A(x)A(y)$ for some $c_A \geq 1$ and $\mathcal{S}$ be a $\frac{c_A 8A(2)}{1+c_A 8A(2)}$-sparse family. Let $f \in C_c^\infty$ and $w \in A_\infty$ and assume that for every $Q \in \mathcal{S}$*

$$2^{-j-1} \leq \|f\|_{A(w),Q} \leq 2^{-j}.$$

Then for every $Q \in \mathcal{S}$ there exist $\tilde{E}_Q \subset Q$ such that

$$\sum_{Q\in\mathcal{S}} \chi_{\tilde{E}_Q} \leq c_n[w]_{A_\infty}$$

and

$$w(Q)\|f\|_{A(w),Q} \leq 4c_A \frac{A(2^{j+1})}{2^{j+1}} \int_{\tilde{E}_Q} A(|f|)w.$$

Lemma 3.6 [5]. *Let $w \in A_\infty$ and $\mathcal{S}$ be a η-sparse family of cubes. Then*

$$\sum_{Q\in\mathcal{S}} w(Q) \leq c_n[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}} Q\right).$$

Lemma 3.7 [5]. *Let A be a Young function such that $A(st) \leq c_A A(s)A(t)$. Let $\mathcal{D}_j$, $j = 1, \dots, k$ be dyadic grids and let w be a weight. Then*

$$w\left(\left\{x \in \mathbb{R}^n : M_{A(w)}^{\mathcal{F}} f(x) > t\right\}\right) \leq c_A c_n \int_{\mathbb{R}^n} A\left(\frac{|f(x)|}{t}\right) w(x)dx,$$

where $\mathcal{F} = \bigcup_{j=1}^k \mathcal{D}_j$.

We end this lemmatta section with the following key Lemma.

Lemma 3.8. *Let $1 < p < \infty$, $\xi \geq 1$ and $\rho \geq 0$. Let $u \in A_1$, $v \in A_p(u)$ and $A(t) = t\log(e+t)^\rho$ and let $\mathcal{S}$ be a sparse family. Then, if $f \in L_c^\infty$ and $g = \chi_G$ where $G \subset \{x : M_{A(uv)}f(x) \leq \frac{1}{2}\}$ is a set of finite measure, we have that for every $\tau_{u,v}, \gamma \geq 1$*

$$\begin{aligned}\tau_{u,v} &\sum_{Q\in\mathcal{S}} \|f\|_{A(uv),Q}\|g\|_{L^\xi(u),Q} uv(Q) \\ &\leq c_{n,p}\tau_{u,v}[uv]_{A_\infty} \log(e+\tau_{u,v}[uv]_{A_\infty}[v]_{A_p(u)})^{1+\rho} \int_{\mathbb{R}^n} A(|f|)uv + \frac{1}{2\gamma} uv(G).\end{aligned}$$

Proof. Taking into account that G is a subset of the set where $M_{A(uv)}(f) \leq \frac{1}{2}$ we can split $\mathcal{S}$ as follows $Q \in \mathcal{S}_{j,k}$, $j, k \geq 0$ if

$$2^{-j-1} < \|f\|_{A(uv),Q} \leq 2^{-j},$$
$$2^{-k-1} < \|g\|_{L^{\xi}(u),Q} \leq 2^{-k}.$$

Let us define

$$s_{j,k} = \sum_{Q \in \mathcal{S}_{j,k}} \|f\|_{A(uv),Q} \|g\|_{L^{\xi}(u),Q} uv(Q).$$

We claim that

$$s_{j,k} \leq \begin{cases} c_n 2^{-k} [uv]_{A_\infty} j^{\rho} \int_{\mathbb{R}} A(|f|)uv \\ c_{n,p} [uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{k(\xi p - 1)} uv(G) \end{cases}$$

For the top estimate, we use Lemma 3.5 with $w = uv$ and $A(t) = t(1 + \log^+ t)^{\rho}$, and we have

$$uv(Q)\|f\|_{A(uv),Q} \leq c j^{\rho} \int_{\tilde{E}_Q} A(|f|)uv$$

with

$$\sum_{Q \in \mathcal{S}_{j,k}} \chi_{\tilde{E}_Q}(x) \leq \lceil c_n [uv]_{A_\infty} \rceil.$$

Then,

$$s_{j,k} \leq c 2^{-k} j^{\rho} \sum_{Q \in \mathcal{S}_{j,k}} \int_{\tilde{E}_Q} A(|f|)uv \leq c_n [uv]_{A_\infty} 2^{-k} j^{\rho} \int_{\mathbb{R}^n} A(|f|)uv.$$

For the lower estimate, using Lemma 3.6

$$\begin{aligned} s_{j,k} &\leq 2^{-j} 2^{-k} \sum_{Q \in \mathcal{S}_{j,k}} uv(Q) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\cup_{Q \in \mathcal{S}_{j,k}} Q\right) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n : M_u(g)^{\frac{1}{\xi}} > 2^{-k-1} \right\}\right). \end{aligned}$$

Since $v \in A_p(u)$ we have

$$\frac{1}{u(Q)} \int_Q gu \leq \left(\frac{[v]_{A_p(u)}}{uv(Q)} \int_Q guv \right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned} s_{j,k} &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n : \left([v]_{A_p(u)} M_{uv}(g)\right)^{\frac{1}{\xi p}} > 2^{-k-1} \right\}\right) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n : M_{uv}(g) > 2^{-\xi p(k+1)} [v]_{A_p(u)}^{-1} \right\}\right) \\ &\leq c_{n,p} [uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{k(\xi p - 1)} uv(G). \end{aligned}$$

Combining the estimates above

$$\tau_{u,v}\sum_{Q\in\mathcal{S}}\|f\|_{A(uv),Q}\|g\|_{L^{\xi},Q}uv(Q)=\tau_{u,v}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}s_{j,k}$$

$$\leq\sum_{j,k=0}^{\infty}\min\left\{c_n\tau_{u,v}[uv]_{A_\infty}2^{-k}j^{\rho}\int_{\mathbb{R}^n}A(|f|)uv,c_{n,p}\tau_{u,v}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{k(\xi p-1)}uv(G)\right\}.$$

We end the proof using Lemma 3.4 with $\gamma_1=c_n\tau_{u,v}[uv]_{A_\infty}\int_{\mathbb{R}^n}A(|f|)uv$, $\gamma_2=c_{n,p}\tau_{u,v}[uv]_{A_\infty}[v]_{A_p(u)}$, $\beta=uv(G)$, $\delta=\xi p$, $\gamma=3^n$, $\rho_1=\rho$ and $\rho_2=0$. □

3.2.2 | Proof of Theorem 1.1

Let $G=\{x:\frac{T(fv)(x)}{v(x)}>1\}\setminus\{x:M_{uv}f(x)>\frac{1}{2}\}$ and assume that $\|f\|_{L^1(uv)}=1$. If we denote $g=\chi_G$, for the sparse domination we have

$$uv(G)\leq\left|\int T(fv)gudx\right|\lesssim\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}\left(\int_Q fv\right)\|gu\|_{A,Q}.$$

Since $u\in A_1\cap RH_r$ then $u^r\in A_1\subset A_\infty$, then we can take $s=4(1+\frac{1}{2\tau_n[u^r]_{A_\infty}})r$. By Lemma 3.2,

$$\|gu\|_{A,Q}\leq c_n\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}\langle u\rangle_{Q,1}\langle g\rangle_{Q,s}^u.$$

Since $u\in A_1$,

$$uv(G)\lesssim\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}\left(\int_Q fv\right)\|gu\|_{A,Q}$$

$$\leq c_n\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}[u]_{A_1}\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{Q,1}^{uv}uv(Q)\langle g\rangle_{Q,s}^u.$$

Now, we apply Lemma 3.8, with $\xi=s$, $\rho=0$, $\tau_{u,v}=c_n\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}[u]_{A_1}$ and $\gamma=3^n$, then

$$uv(G)\leq c_n\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}[u]_{A_1}\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{Q,1}^{uv}uv(Q)\langle g\rangle_{Q,s}^u$$

$$\leq c_{n,p}\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}[u]_{A_1}[uv]_{A_\infty}\log(e+\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}[u]_{A_1}[uv]_{A_\infty}[v]_{A_p(u)})+\frac{1}{2}uv(G).$$

3.2.3 | Proof of Theorems 1.2 and 1.3

We provide the proof of Theorem 1.2 first and at the end we show how to adjust the argument to settle Theorem 1.3. Let $G=\{x:\frac{T_{\mathbf{b}}(fv)(x)}{v(x)}>1\}\setminus\{x:M_{\varphi_{\frac{1}{r}}(uv)}f(x)>\frac{1}{2}\}$, with $\varphi_{\frac{1}{r}}(t)=t\log(e+t)^{\frac{1}{r}}$ and $g=\chi_G$, for the sparse domination we have

$$uv(G) \le \left|\int T_{\mathbf{b}}(fv)gu\right| \lesssim \sum_{j=1}^{3^n}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in S_j}\left(\int_Q fv|b-b_Q|_{\sigma'}\right)\|gu|b-b_Q|_\sigma\|_{A,Q}.$$

Let $\xi = 4(1+\frac{1}{2\tau_n[u^s]_{A_\infty}})s$. By $u\in A_1$, Theorem 3.2 and Hölder inequality we obtain

$$\begin{aligned}
&\sum_{Q\in S}\left(\int_Q fv|b-b_Q|_{\sigma'}\right)\|gu|b-b_Q|_\sigma\|_{A,Q}\\
&\le \sum_{Q\in S}\left(\int_Q fv|b-b_Q|_{\sigma'}\right)\|gu\|_{B,Q}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}\\
&\lesssim \kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}\sum_{Q\in S}\left(\int_Q fv|b-b_Q|_{\sigma'}\right)\|g\|_{L^\xi(u),Q}\frac{u(Q)}{|Q|}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}\\
&\le \kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}\sum_{Q\in S}\|f|b-b_Q|_{\sigma'}\|_{uv,Q}\|g\|_{L^\xi(u),Q}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}uv(Q)\\
&\lesssim \kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[uv]_{A_\infty}^{\frac{m-h}{r}}\sum_{Q\in S}\prod_{i\in 1}^{m}\|b-b_Q\|_{\exp L^{r_i},Q}\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\|g\|_{L^\xi(u),Q}uv(Q)\\
&= \kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[uv]_{A_\infty}^{\frac{m-h}{r}}\|\mathbf{b}\|\sum_{Q\in S}\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\|g\|_{L^\xi(u),Q}uv(Q).
\end{aligned}$$

Now, we apply Lemma 3.8 with $\rho=\frac{1}{r}$, $A(t)=\varphi_{\frac{1}{r}}(t)$, $\tau_{u,v}=\kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[uv]_{A_\infty}^{\frac{m-h}{r}}$ and $\gamma = 3^n\binom{m}{h}$ (recall that $\binom{m}{h}$ is the cardinality of $C_h(b)$), then

$$\begin{aligned}
&\sum_{\sigma\in C_h(b)}\sum_{Q\in S}\left(\int_Q fv|b-b_Q|_{\sigma'}\right)\|gu|b-b_Q|_\sigma\|_{A,Q}\\
&\qquad\le c_{n,T}C_{u,v}\int_{\mathbb{R}^n}\varphi_{\frac{1}{r}}\left(\frac{|f|\|\mathbf{b}\|}{\lambda}\right)uv+\frac{1}{3^n2}uv(G),
\end{aligned}$$

where

$$C_{uv}=\kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[uv]_{A_\infty}^{1+\frac{m-h}{r}}\log(e+[uv]_{A_\infty}^{1+\frac{m-h}{r}}\kappa_u[u]_{RH_q}^{1+\frac{q}{4s}}[u]_{A_1}[v]_{A_p(u)})^{1+\frac{1}{r}}.$$ □

To settle Theorem 1.3, due to the following lemma, that follows from ideas in [8], it suffices to apply the argument above just to the cases in which $\sigma=\emptyset$ or in which σ contains the m "copies" of b.

Lemma 3.9. *Given $b\in L^m_{loc}$, a sparse family S, a positive integer m and $h\in\{0,\dots,m\}$ we have that*

$$\mathcal{A}^{m-h}_{A,S}(b,f)(x)\le\sum_{Q\in S}|b-b_Q|^m\|f\|_{A,Q}\chi_Q(x)+\sum_{Q\in S}\|f|b-b_Q|^m\|_{A,Q}\chi_Q(x).$$

Proof.

$$
\begin{aligned}
\mathcal{A}^{m-h}_{A,\mathcal{S}}(b,f)(x) &= \sum_{Q\in\mathcal{S}} |b(x)-b_Q|^{m-h}\|f|b-b_Q|^h\|_{A,Q}\chi_Q(x)\\
&= \sum_{Q\in\mathcal{S}} \|f|b(x)-b_Q|^{m-h}|b-b_Q|^h\|_{A,Q}\chi_Q(x)\\
&\leq \sum_{Q\in\mathcal{S}} \|f\max\{|b(x)-b_Q|,|b-b_Q|\}^m\|_{A,Q}\chi_Q(x)\\
&\leq \sum_{Q\in\mathcal{S}} |b(x)-b_Q|^m\|f\|_{A,Q}\chi_Q(x) + \sum_{Q\in\mathcal{S}} \|f|b-b_Q|^m\|_{A,Q}\chi_Q(x).
\end{aligned}
$$

□

3.3 | Proofs of the results in spaces of homogeneous type

3.3.1 | Lemmata

Our first Lemma is the following.

Lemma 3.10. *Let $1\leq r<\infty$. If $u\in RH_q$ with $q=2r-1$ and $u^r\in A_\infty$, then for $s=1+\frac{1}{2\tau_n[u^r]_{A_\infty}}$ and any measurable subset $E\subset Q$,*

$$
\frac{u^{sr}(E)}{u^{sr}(c_dQ)} \lesssim [u]_{RH_q}^{q/4}\left(\frac{u(E)}{u(c_dQ)}\right)^{\frac{1}{4}}.
$$

Proof. First, let us see

$$
u^r(E) \leq [u]_{\mathrm{RH}_q}^{q/2}\left(\frac{u(E)}{u(1Q)}\right)^{1/2} u^r(c_dQ). \tag{3.6}
$$

Indeed, since $u\in \mathrm{RH}_q$ we obtain

$$
\begin{aligned}
\left(\frac{u^q(1Q)}{u(c_dQ)}\right)^{1/2} &= \left(\frac{\frac{u^q(1Q)}{\mu(c_dQ)}}{\frac{u(c_dQ)}{\mu(c_dQ)}}\right)^{1/2} \leq [u]_{\mathrm{RH}_q}^{q/2}\left(\frac{\left(\frac{u(c_dQ)}{\mu(c_dQ)}\right)^q}{\frac{u(c_dQ)}{\mu(c_dQ)}}\right)^{1/2} = [u]_{\mathrm{RH}_q}^{q/2}\left(\frac{u(c_dQ)}{\mu(c_dQ)}\right)^{r-1}\\
&= [u]_{\mathrm{RH}_q}^{q/2}\frac{\mu(c_dQ)}{u(c_dQ)}\left(\frac{u(c_dQ)}{\mu(c_dQ)}\right)^{r} \leq [u]_{\mathrm{RH}_q}^{q/2}\frac{\mu(c_dQ)}{u(c_dQ)}\frac{u^r(c_dQ)}{\mu(c_dQ)}\\
&= [u]_{\mathrm{RH}_q}^{q/2}\frac{u^r(c_dQ)}{u(c_dQ)}.
\end{aligned}
$$

Taking that into account,

$$
\begin{aligned}
u^r(E) &\leq u(E)^{1/2}u^q(1Q)^{1/2} = u(E)^{1/2}u(c_dQ)^{\frac{1}{2}}\left(\frac{u^q(1Q)}{u(c_dQ)}\right)^{1/2}\\
&\leq u(E)^{1/2}[u]_{\mathrm{RH}_q}^{q/2}\frac{u^r(c_dQ)}{u(c_dQ)^{1/2}} = [u]_{\mathrm{RH}_q}^{q/2}\left(\frac{u(E)}{u(c_dQ)}\right)^{1/2}u^r(c_dQ).
\end{aligned}
$$

On the other hand, since $s = 1 + \frac{1}{2\tau_n[u^r]_{A_\infty}}$ we have that

$$\left(\frac{u^{r(2s-1)}(1Q)}{u^r(c_dQ)}\right)^{\frac{1}{2}} = \left(\frac{u^{r(2s-1)}(1Q)}{\mu(1Q)}\frac{\mu(1Q)}{u^r(c_dQ)}\right)^{\frac{1}{2}} \lesssim \left(\frac{u^r(c_dQ)}{\mu(c_dQ)}\right)^{\frac{2s-1}{2}}\left(\frac{\mu(1Q)}{u^r(c_dQ)}\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{u^r(c_dQ)}{\mu(c_dQ)}\right)^{s}\left(\frac{\mu(c_dQ)}{u^r(c_dQ)}\right) \leq \left(\frac{u^{rs}(c_dQ)}{\mu(c_dQ)}\right)\left(\frac{\mu(c_dQ)}{u^r(c_dQ)}\right)$$

$$= \frac{u^{rs}(c_dQ)}{u^r(c_dQ)}.$$

Summarizing

$$\left(\frac{u^{r(2s-1)}(1Q)}{u^r(c_dQ)}\right)^{\frac{1}{2}} \lesssim \frac{u^{rs}(c_dQ)}{u^r(c_dQ)}.$$

Observe that relying upon this estimate

$$\frac{u^{sr}(E)}{u^{sr}(c_dQ)} = \frac{u^{sr-\frac{r}{2}+\frac{r}{2}}(E)}{u^{sr}(c_dQ)} \leq \frac{u^{2r(s-1)}(E)^{\frac{1}{2}}u^r(E)^{\frac{1}{2}}}{u^{sr}(c_dQ)}$$

$$\leq \left(\frac{u^{2r(s-1)}(1Q)}{u^r(c_dQ)}\right)^{\frac{1}{2}}\frac{u^r(c_dQ)^{\frac{1}{2}}u^r(E)^{\frac{1}{2}}}{u^{sr}(c_dQ)}$$

$$\lesssim \frac{u^{rs}(c_dQ)}{u^r(c_dQ)}\frac{u^r(c_dQ)^{\frac{1}{2}}u^r(E)^{\frac{1}{2}}}{u^{sr}(c_dQ)} = \left(\frac{u^r(E)}{u^r(c_dQ)}\right)^{1/2}.$$

Hence,

$$\frac{u^{sr}(E)}{u^{sr}(c_dQ)} \lesssim \left(\frac{u^r(E)}{u^r(c_dQ)}\right)^{1/2}. \tag{3.7}$$

Taking into account Equations (3.6) and (3.7), we obtain

$$\frac{u^{sr}(E)}{u^{sr}(c_dQ)} \lesssim \left(\frac{u^r(E)}{u^r(c_dQ)}\right)^{1/2} \lesssim \left(\frac{[u]_{\mathrm{RH}_q}^{q/2}\left(\frac{u(E)}{u(c_dQ)}\right)^{1/2}u^r(c_dQ)}{u^r(c_dQ)}\right)^{1/2}$$

$$= [u]_{\mathrm{RH}_q}^{q/4}\left(\frac{u(E)}{u(c_dQ)}\right)^{\frac{1}{4}}.$$

□

We continue with the following lemma which is a counterpart of Lemma 3.2.

Lemma 3.11. *Let $1 \leq r < \infty$ and A be a Young function such that $A \in B_\rho$ for all $\rho > r$. If $u \in RH_q$ with $q = 2r - 1$ and $u^r \in A_\infty$ then for any cube Q and G measurable subset*

$$\|\chi_G u\|_{A,1Q} \leq c_X\kappa_u[u]_{RH_q}^{1+\frac{q}{4r}}\langle u\rangle_{c_d^3Q,1}\langle\chi_G\rangle_{c_dQ,s}^u$$

with $s = 4(1 + \frac{1}{2\tau_n[u^r]_{A_\infty}})r$.

Proof. Since $A \in B_\rho$ for all $\rho > r$, then $A(t) \leq c\zeta_\varepsilon t^{r(1+\varepsilon)}$ for all $\varepsilon > 0$. Then

$$\|\chi_G u\|_{A,1Q} \leq c\zeta_\varepsilon \|\chi_G u\|_{r(1+\varepsilon),1Q} = c\zeta_\varepsilon \|\chi_G u^r\|_{1+\varepsilon,1Q}^{\frac{1}{r}} = c\kappa_u \|\chi_G u^r\|_{1+\varepsilon,1Q}^{\frac{1}{r}} \tag{3.8}$$

with $\varepsilon = \frac{1}{2\tau_n [u^r]_{A_\infty}}$ and $\kappa_u = \zeta_{\frac{1}{2\tau_n [u^r]_{A_\infty}}}$.

Using Lemma 3.10, we have

$$\left(\frac{u^{r(1+\varepsilon)}(G\cap Q)}{u^{r(1+\varepsilon)}(c_d Q)}\right)^{\frac{1}{1+\varepsilon}} \leq c_X [u]_{\mathrm{RH}_q}^{\frac{q}{4(1+\varepsilon)}} \left(\frac{u(G\cap Q)}{u(c_d Q)}\right)^{\frac{1}{4(1+\varepsilon)}}. \tag{3.9}$$

In the other hand, since $1+2\varepsilon = 1 + \frac{1}{\tau_n [u^r]_{A_\infty}}$ and $u \in \mathrm{RH}_q$, with $q = 2r-1 > r$, we get

$$\begin{aligned}
\left(\frac{u^{r(1+\varepsilon)}(c_d Q)}{\mu(c_d Q)}\right)^{\frac{1}{r(1+\varepsilon)}} &= \left(\frac{u^{\frac{r}{2}+\frac{r}{2}+\varepsilon}(c_d Q)}{\mu(c_d Q)}\right)^{\frac{1}{r(1+\varepsilon)}} \\
&\leq \left(\frac{u^r(c_d Q)}{\mu(c_d Q)}\right)^{\frac{1}{2r(1+\varepsilon)}} \left(\frac{1}{\mu(c_d Q)} \int_{c_d Q} u^{r(1+2\varepsilon)}\right)^{\frac{1}{2r(1+\varepsilon)}} \qquad (3.10) \\
&\leq 2^{\frac{1}{r}} \left(\frac{u^r(1Q)}{\mu(c_d Q)}\right)^{\frac{1}{2r(1+\varepsilon)}} \left(c\frac{u^r(c_d^2 Q)}{\mu(c_d^2 Q)}\right)^{\frac{1+2\varepsilon}{2r(1+\varepsilon)}} \lesssim \left(\frac{u^r(c_d^2 Q)}{\mu(c_d^2 Q)}\right)^{\frac{1}{r}} \\
&\leq \left(\frac{u^q(c_d^2 Q)}{\mu(c_d^2 Q)}\right)^{\frac{1}{q}} \lesssim [u]_{RH_q} \frac{u(c_d^3 Q)}{\mu(c_d^3 Q)}. \qquad (3.11)
\end{aligned}$$

Taking into account the inequalities above we obtain

$$\begin{aligned}
\|\chi_G u\|_{A,1Q} &\leq c\kappa_u \|\chi_G u^r\|_{1+\varepsilon,1Q}^{\frac{1}{r}} \\
&\lesssim \kappa_u \left(\frac{\mu(c_d Q)}{\mu(1Q)}\right)^{\frac{1}{r(1+\varepsilon)}} \left(\frac{u^{r(1+\varepsilon)}(c_d Q)}{\mu(c_d Q)}\right)^{\frac{1}{r(1+\varepsilon)}} \left(\frac{u^{r(1+\varepsilon)}(G\cap 1Q)}{u^{r(1+\varepsilon)}(c_d Q)}\right)^{\frac{1}{r(1+\varepsilon)}} \\
&\leq c_X \kappa_u [u]_{\mathrm{RH}_q}^{1+\frac{q}{4r(1+\varepsilon)}} \langle u\rangle_{c_d^3 Q,1} \langle \chi_G\rangle^u_{c_d Q,4(1+\varepsilon)r} \\
&\leq c_X \kappa_u [u]_{\mathrm{RH}_q}^{1+\frac{q}{4r}} \langle u\rangle_{c_d^3 Q,1} \langle \chi_G\rangle^u_{c_d Q,4(1+\varepsilon)r}.
\end{aligned}$$

□

Lemma 3.12. *Let $w \in A_p$ and let A be a submultiplicative Young function and $\mathcal{S}$ a dyadic $\frac{4c_A A(4)c[w]_{A_p}}{1+4c_A A(4)c[w]_{A_p}}$-sparse family. Let $f \in C_c^\infty$ and assume that for every $Q \in \mathcal{S}$*

$$2^{-j-1} \leq \langle f\rangle_{A(L)(w)Q} \leq 2^{-j}.$$

Then for every $Q \in \mathcal{S}$ there exists $\tilde{E}_Q \subseteq Q$ such that

$$\sum_{Q\in\mathcal{S}} \chi_{\tilde{E_Q}}(x) \le c_X[w]_{A_\infty}$$

and

$$w(Q)\|f\|_{A(w),Q} \le 4\frac{A(2^{j+2})}{2^{j+2}} \int_{\tilde{E_Q}} A(|f|)w.$$

Proof. We split the family $\mathcal{S}$ in the following way:

$$\begin{aligned}
\mathcal{S}^0 &= \{\text{Maximal cubes in } \mathcal{S}\}\\
\mathcal{S}^1 &= \{\text{Maximal cubes in } \mathcal{S}\setminus\mathcal{S}^0\}\\
&\dots\\
\mathcal{S}^i &= \{\text{Maximal cubes in } \mathcal{S}\setminus\cup_{r=0}^{i-1}\mathcal{S}^r\}
\end{aligned}$$

Recall that since $w \in A_\infty$ we have that, for each cube Q and each measurable subset $E \subset Q$,

$$w(E) \le c\left(\frac{\mu(E)}{\mu(1Q)}\right)^{\frac{1}{c_X[w]_{A_\infty}}} w(c_dQ).$$

Now observe that if $Q \in \mathcal{S}^i$ and $J_1 = \bigcup_{P\in\mathcal{S}^{i+1},\, P\subsetneq Q} P$, if we call $\kappa = 4c_A A(4)c[w]_{A_p}$,

$$\begin{aligned}
\mu(J_1) = \mu\left(\bigcup_{P\in\mathcal{S}^{i+1},\, P\subsetneq Q} P\right) &\le \bigcup_{P\in\mathcal{S}^{i+1},\, P\subsetneq Q} \mu(P)\\
&\le \left(\frac{1+\kappa}{\kappa}-1\right)\mu(Q) = \frac{1}{\kappa}\mu(Q).
\end{aligned}$$

Arguing by induction, if we denote $J_\nu = \bigcup_{P\in\mathcal{S}^{i+\nu},\, P\subset Q} P$ then we have that

$$\mu(J_\nu) \le \left(\frac{1}{\kappa}\right)^\nu \mu(Q) \le \left(\frac{1}{\kappa}\right)^\nu \mu(1Q)$$

and hence,

$$w(J_\nu) \le 2\left(\frac{1}{4c_A A(4)c[w]_{A_p}}\right)^{\frac{\nu}{c_X[w]_{A_\infty}}} w(c_dQ).$$

In particular if we choose $\nu = \lfloor c_X[w]_{A_\infty}\rfloor$, then by Lemma 2.4

$$w(J_\nu) \le \frac{2}{\kappa}w(c_dQ) \le \frac{c[w]_{A_p}}{2c_A A(4)c[w]_{A_p}}w(Q) = \frac{1}{2c_A A(4)}w(Q).$$

Let $Q \in \mathcal{S}^i$. and let $\tilde{E_Q} = Q \setminus \bigcup_{P \in \mathcal{S}^{i+\lceil c_X[w]_{A_\infty}\rceil}} P$. Then, we have that

$$
\begin{aligned}
&w(Q)\|f\|_{A(w),Q}\\
&\le w(Q)\left\{2^{-j-2} + \frac{2^{-j-2}}{w(Q)}\int_Q A\left(2^{j+2}|f|\right)w\right\}\\
&\le w(Q)2^{-j-2} + \frac{1}{2^{j+2}}\int_Q A\left(2^{j+2}|f|\right)w\\
&\le w(Q)2^{-j-1} + \frac{1}{2^{j+2}}\int_{\tilde{E_Q}} A\left(2^{j+2}|f|\right)w + \frac{1}{2^{j+2}}\sum_{P\in S_{j,k}^{i+\lceil c_X[w]_{A_\infty}\rceil}}\int_P A\left(2^{j+2}|f|\right)w\\
&\le w(Q)2^{-j-2} + \frac{A(2^{j+2})}{2^{j+2}}\int_{\tilde{E_Q}} A(|f|)w + \frac{1}{2^{j+2}}\sum_{P\in S_{j,k}^{i+\lceil c_X[w]_{A_\infty}\rceil}}\int_P A\left(2^{j+2}|f|\right)w.
\end{aligned}
$$

Observe that we can bound the last term as follows:

$$
\begin{aligned}
&\sum_{P\in S_{j,k}^{i+\lceil c_X[w]_{A_\infty}\rceil}}\int_P A\left(2^{j+2}|f|\right)w \le c_A A(4)\sum_{P\in S_{j,k}^{i+\lceil c_X[w]_{A_\infty}\rceil}} w(P)\frac{1}{w(P)}\int_P A\left(2^j|f|\right)w\\
&\le A(4)\sum_{P\in S_{j,k}^{i+\lceil c_X[w]_{A_\infty}\rceil}} w(P) = c_A A(4) w\left(J_{\lceil c_X[w]_{A_\infty}\rceil}\right)\\
&\le \frac{c_A A(4)}{2}\frac{1}{2c_A A(4)}w(Q) \le \frac{1}{4}w(Q).
\end{aligned}
$$

Hence

$$
\begin{aligned}
w(Q)\|f\|_{A(w),Q} &\le \frac{1}{2^{j+2}}w(Q) + \frac{A(2^{j+2})}{2^{j+2}}\int_{\tilde{E_Q}} A(|f|)w + \frac{1}{2^{j+2}}\frac{1}{4}w(Q)\\
&\le \left(\frac{1}{2}+\frac{1}{4}\right)w(Q)\|f\|_{A(w),Q} + \frac{A(2^{j+2})}{2^{j+2}}\int_{\tilde{E_Q}} A(|f|)w\\
&= \frac{3}{4}w(Q)\|f\|_{A(w),Q} + \frac{A(2^{j+2})}{2^{j+2}}\int_{\tilde{E_Q}} A(|f|)w,
\end{aligned}
$$

from which the desired conclusion readily follows. □

We end this lemmatta with the following key lemma.

Lemma 3.13. *Let $1 < p < \infty$, $s \ge 1$ and $\rho \ge 0$. Let $u \in A_1$, $v \in A_p(u)$, and $A(t) = t\log^\rho(e+t)$. Assume that $\mathcal{S}$ is a $\frac{4c_A A(4)c[w]_{A_p}}{1+4c_A A(4)c[w]_{A_p}}$-sparse family. Then if $f \in L_c^\infty$ and $g = \chi_G$ where $G \subset \left\{x \in X \,:\, M_{A(uv)}f \le \frac{1}{2}\right\}$ is a set of finite measure, we have that for every $\tau_{u,v}, \gamma \ge 1$*

$$
\begin{aligned}
&\tau_{u,v}\sum_{Q\in\mathcal{S}}\|f\|_{A(uv),Q}\langle g\rangle^u_{c_dQ,s}uv(Q)\\
&\le c_\gamma \tau_{u,v}[uv]_{A_\infty}\log(e+\tau_{u,v}[uv]^2_{A_p}[v]_{A_p(u)})^{1+\rho}\int_X A(|f|)d\mu + \frac{1}{2\gamma}uv(G).
\end{aligned}
$$

Proof. Assume that $\|f\|_{A(uv)} = 1$. We split the sparse family $\mathcal{S}$ as follows. We say that $Q \in \mathcal{S}_{j,k}$, $j,k \geq 0$ if

$$2^{-j-1} < \|f\|_{A(uv),Q} \leq 2^{-j},$$
$$2^{-k-1} < \langle g\rangle^{u}_{c_d Q,s} \leq 2^{-k}.$$

Let

$$s_{j,k} = \sum_{Q\in\mathcal{S}_{j,k}} \|f\|_{A(uv),Q}\langle g\rangle^{u}_{c_d Q,s} uv(Q).$$

We claim that

$$s_{j,k} \leq \begin{cases} c_n[uv]_{A_\infty} 2^{-k} j^{\rho} \\ c_{n,p}[uv]^2_{A_p}[v]_{A_p(u)} 2^{-j}2^{k(sp-1)} uv(G) \end{cases}$$

For the top estimate, we argue as follows. Using Lemma 3.12, we have that there exists a set $\tilde{E}_Q \subset Q$ such that

$$\sum_{Q\in\mathcal{S}_{j,k}} \chi_{\tilde{E}_Q}(x) \leq \lceil c_n[uv]_{A_\infty}\rceil$$

and

$$\|f\|_{A(uv),Q} uv(Q) \lesssim \frac{A(2^{j+2})}{2^{j+2}} \int_{\tilde{E}_Q} A(|f|)uv \simeq j^{\rho} \int_{\tilde{E}_Q} A(|f|)uv.$$

Then

$$s_{j,k} \leq 2^{-k} \sum_{Q\in\mathcal{S}_{j,k}} \|f\|_{A(uv),Q} uv(Q) \leq 2^{-k} \sum_{Q\in\mathcal{S}_{j,k}} j^{\rho} \int_{\tilde{E}_Q} A(|f|)uv$$
$$\leq c_n[uv]_{A_\infty} 2^{-k} j^{\rho} \int_X A(|f|)uv = [uv]_{A_\infty} 2^{-k} j^{\rho}.$$

For the lower estimate, by Lemma 2.4

$$s_{j,k} \leq 2^{-j}2^{-k} \sum_{Q\in\mathcal{S}_{j,k}} uv(Q)$$
$$\leq c_n[uv]_{A_p} 2^{-j}2^{-k} uv\left(\cup_{Q\in\mathcal{S}_{j,k}} Q\right)$$
$$\leq c_n[uv]_{A_p} 2^{-j}2^{-k} uv\left(\left\{x \in \mathbb{R}^n : M_u(g)^{\frac{1}{s}} > 2^{-k-1}\right\}\right).$$

Since $v \in A_p(u)$ we have

$$\frac{1}{u(B)} \int_B gu \leq \left(\frac{[v]_{A_p(u)}}{uv(B)} \int_B guv\right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned} s_{j,k} &\le c_n[uv]_{A_p}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n : \left([v]_{A_p(u)}M_{uv}(g)\right)^{\frac{1}{sp}} > 2^{-k-1}\right\}\right) \\ &\le c_n[uv]_{A_p}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n : M_{uv}(g) > 2^{-sp(k+1)}[v]^{-1}_{A_p(u)}\right\}\right) \\ &\le c_{n,p}[uv]^2_{A_p}[v]_{A_p(u)}2^{-j}2^{k(sp-1)}uv(G). \end{aligned}$$

Combining the estimates above

$$\tau_{u,v}\sum_{j,k=0}^{\infty} s_{j,k} \le \sum_{j,k=0}^{\infty}\min\left\{\tau_{u,v}[uv]_{A_\infty}2^{-k}j^{\rho}, c_{n,p}\tau_{u,v}[uv]^2_{A_p}[v]_{A_p(u)}2^{-j}2^{k(sp-1)}uv(G)\right\},$$

where $K_{u,v}=\kappa_u[u]^{1+\frac{q}{4r}}_{RH_q}[u]_{A_1}[uv]_{A_p}$. We end the proof using Lemma 3.4, with $\gamma_1=\tau_{u,v}[uv]_{A_\infty}$, $\gamma_2 = c_{n,p}\tau_{u,v}[uv]^2_{A_p}[v]_{A_p(u)}$, $\beta=uv(G)$, $\delta=sp$, $\rho_1=\rho$, $\rho_2=0$. □

3.3.2 | Proof of Theorem 1.4

Let $G=\{x : \frac{T(fv)(x)}{v(x)}>1\}\setminus\{x : M_{uv}f(x)>\frac{1}{2}\}$ and assume that $\|f\|_{L^1(uv)}=1$. If we denote $g=\chi_G$, by Equation (1.6)

$$uv(G)\le\left|\int T(fv)gudx\right|\lesssim\frac{1}{1-\varepsilon}\sum_{j=1}^{l}\sum_{Q\in S_j}\left(\int_Q fv\right)\|gu\|_{A,Q}.$$

We shall choose $\varepsilon=\frac{16c[uv]_{A_p}}{1+16c[uv]_{A_p}}$. Hence,

$$uv(G)\lesssim[uv]_{A_p}\sum_{j=1}^{l}\sum_{Q\in S_j}\left(\int_Q fv\right)\|gu\|_{A,Q}.$$

Since $u\in A_1\cap RH_r$, then $u^r\in A_1\subset A_\infty$, and taking $s=4(1+\frac{1}{2\tau_n[u^r]_{A_\infty}})r$, by Lemma 3.11,

$$\|gu\|_{A,Q}\le c_d\kappa_u[u]^{1+\frac{q}{4r}}_{RH_q}\langle u\rangle_{c_d^3Q,1}\langle g\rangle^u_{c_dQ,s}.$$

Taking the estimates above into account and bearing in mind that $u\in A_1$,

$$\begin{aligned} uv(G) &\le c[uv]_{A_p}\sum_{j=1}^{l}\sum_{Q\in S_j}\left(\int_Q fv\right)\|gu\|_{A,Q} \\ &\le c\kappa_u[uv]_{A_p}[u]^{1+\frac{q}{4r}}_{\mathrm{RH}_q}\sum_{j=1}^{l}\sum_{Q\in S_j}\left(\int_Q fv\right)\langle u\rangle_{c_d^3Q,1}\langle g\rangle^u_{c_dQ,s} \\ &\le c\kappa_u[uv]_{A_p}[u]^{1+\frac{q}{4r}}_{\mathrm{RH}_q}[u]_{A_1}\sum_{j=1}^{l}\sum_{Q\in S_j}\langle f\rangle^{uv}_{Q,1}\langle g\rangle^u_{c_dQ,s}uv(Q). \end{aligned}$$

A direct application of Lemma 3.13 with $\tau_{u,v}=c\kappa_u[uv]_{A_p}[u]^{1+\frac{q}{4r}}_{RH_q}[u]_{A_1}$ ends the proof.

3.3.3 | Proof of Theorem 1.5

We shall assume that $\|b\|_{Osc(L^{r_i})}=1$ for every i. Let $G=\{x:\frac{T_{\mathbf{b}}(fv)(x)}{v(x)}>1\}\setminus\{x:M_{\varphi_{\frac{1}{r}}(uv)}f(x)>\frac{1}{2}\}$, with $\varphi_{\frac{1}{r}}(t)=t\log(e+t)^{\frac{1}{r}}$ and $g=\chi_G$, by Equation (1.7), with $\varepsilon=\frac{4c_{\varphi_{\frac{1}{r}}}\varphi_{\frac{1}{r}}(4)c[uv]_{A_p}}{1+4c_{\varphi_{\frac{1}{r}}}\varphi_{\frac{1}{r}}(4)c[uv]_{A_p}}$

$$uv(G)\leq\left|\int T_{\mathbf{b}}(fv)gu\right|\lesssim\frac{1}{1-\varepsilon}\sum_{s=1}^{l}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in\mathcal{S}_s}\left(\int_Q fv|b-b_Q|_{\sigma'}d\mu\right)\|gu|b-b_Q|_{\sigma}\|_{A,Q}$$
$$\lesssim[uv]_{A_p}\sum_{s=1}^{l}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in\mathcal{S}_s}\left(\int_Q fv|b-b_Q|_{\sigma'}d\mu\right)\|gu|b-b_Q|_{\sigma}\|_{A,Q}.$$

Let $\xi=4(1+\frac{1}{2\tau_n[u^s]_{A_\infty}})s$. By $u\in A_1$, Theorem 3.11 and Hölder inequality we obtain

$$\sum_{Q\in\mathcal{S}}\left(\int_Q fv|b-b_Q|_{\sigma'}d\mu\right)\|gu|b-b_Q|_{\sigma}\|_{A,Q}$$
$$\leq\sum_{Q\in\mathcal{S}}\left(\int_Q fv|b-b_Q|_{\sigma'}d\mu\right)\|gu\|_{B,Q}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}$$
$$\leq\kappa_u[u]_{\mathrm{RH}_q}^{1+\frac{q}{4s}}\sum_{Q\in\mathcal{S}}\left(\int_Q fv|b-b_Q|_{\sigma'}d\mu\right)\|g\|_{L^{\xi}(u),c_dQ}\frac{u(Q)}{|Q|}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}$$
$$\leq\kappa_u[u]_{\mathrm{RH}_q}^{1+\frac{q}{4s}}[u]_{A_1}\sum_{Q\in\mathcal{S}}\|f|b-b_Q|_{\sigma'}\|_{uv,Q}\|g\|_{L^{\xi}(u),c_dQ}\prod_{i\in\sigma}\|b-b_Q\|_{\exp L^{r_i},Q}uv(Q)$$
$$\leq\kappa_u[u]_{\mathrm{RH}_q}^{1+\frac{q}{4s}}[u]_{A_1}\left([uv]_{A_p}[uv]_{A_\infty}\right)^{\frac{m-h}{r}}\sum_{Q\in\mathcal{S}}\prod_{i\in 1}^{m}\|b-b_Q\|_{\exp L^{r_i},Q}\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\|g\|_{L^{\xi}(u),c_dQ}uv(Q)$$
$$=\kappa_u[u]_{\mathrm{RH}_q}^{1+\frac{q}{4s}}[u]_{A_1}\left([uv]_{A_p}[uv]_{A_\infty}\right)^{\frac{m-h}{r}}\|\mathbf{b}\|\sum_{Q\in\mathcal{S}}\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\|g\|_{L^{\xi}(u),c_dQ}uv(Q).$$

Taking into account the definition of G, since we removed the set where $M_{u,v}(f)>\frac{1}{2}$ we can split $\mathcal{S}$ as follows. We say that $Q\in\mathcal{S}_{j,k}$, $j,k\geq 0$ if

$$2^{-j-1}<\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\leq 2^{-j},$$
$$2^{-k-1}<\|g\|_{L^{\xi}(u),c_dQ}\leq 2^{-k}.$$

Let us define

$$s_{j,k}=\sum_{Q\in\mathcal{S}_{j,k}}\|f\|_{L\log L^{\frac{1}{r}}(uv),Q}\|g\|_{L^{\xi}(u),c_dQ}uv(Q).$$

We claim that

$$s_{j,k}\leq\begin{cases}c_n2^{-k}[uv]_{A_\infty}j^{\frac{1}{r}}\int_{\mathbb{R}}\varphi_{\frac{1}{r}}(|f|)uv\\ c_{n,p}[uv]_{A_p}^2[v]_{A_p(u)}2^{-j}2^{k(\xi p-1)}uv(G).\end{cases}$$

For the lower estimate we argue as we did in Theorem 1.1. For the top estimate, we use Lemma 3.5 with $w = uv$ and $A(t) = \Phi_{\frac{1}{r}}(t) = t(1 + \log^+ t)^{\frac{1}{r}}$, and we have

$$uv(Q)\|f\|_{L\log L^{\frac{1}{r}}(uv),Q} \leq cj^{\frac{1}{r}} \int_{\tilde{E}_Q} \Phi_{\frac{1}{r}}(|f|)uv$$

with

$$\sum_{Q\in S_{j,k}} \chi_{\tilde{E}_Q}(x) \leq \lceil c_n[uv]_{A_\infty}\rceil.$$

Then,

$$s_{j,k} \leq c2^{-k} j^{\frac{1}{r}} \sum_{Q\in S_{j,k}} \int_{\tilde{E}_Q} \Phi_{\frac{1}{r}}(|f|)uv \leq c_n[uv]_{A_\infty} 2^{-k} j^{\frac{1}{r}} \int_{\mathbb{R}} \Phi_{\frac{1}{r}}(|f|)uv.$$

Combining the estimates above

$$\begin{aligned} uv(G) &\leq lc \sum_{s=1}^{l} \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \kappa_u [u]_{RH_q}^{1+\frac{q}{4s}} [u]_{A_1} \left([uv]_{A_p}[uv]_{A_\infty}\right)^{\frac{m-h}{r}} [uv]_{A_p} \sum_{j,k=0}^{\infty} s_{j,k} \\ &\leq \Gamma_{l,b} \sum_{h=0}^{m} \sum_{j,k=0}^{\infty} \min\left\{ c_n\tau_{u,v,h}[uv]_{A_\infty} 2^{-k} j^{\frac{1}{r}} \int_{\mathbb{R}} \Phi_{\frac{1}{r}}(|f|)uv, c_{n,p}\tau_{u,v,h}[uv]_{A_p}^2 [v]_{A_p(u)} 2^{-j} 2^{k(\xi p-1)} uv(G) \right\}, \end{aligned}$$

where $\tau_{u,v,h} = \kappa_u [u]_{RH_q}^{1+\frac{q}{4s}} [u]_{A_1} \left([uv]_{A_p}[uv]_{A_\infty}\right)^{\frac{m-h}{r}} [uv]_{A_p}$ and $\Gamma_{l,b} = l \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)}$. We end the proof using Lemma 3.4 with $\gamma_1 = c_n\tau_{u,v,h}[uv]_{A_\infty} \int_{\mathbb{R}} \Phi_{\frac{1}{r}}(|f|)uv$, $\gamma_2 = c_{n,p}\tau_{u,v,h}[uv]_{A_p}^2[v]_{A_p(u)}$, $\beta = uv(G)$, $\delta = \xi p$, $\gamma = \Gamma_{l,b}$, $\rho_1 = \frac{1}{r}$ and $\rho_2 = 0$.

4 | SPARSE DOMINATION AND APPLICATIONS OF THE MAIN RESULTS

Note that for each operator for which a bilinear sparse bounds as those presented in the first section of this work hold, the corresponding endpoint estimates in the main results hold as well. We recall that given a Young function A, we say that T is an A-Hörmander operator if

$$\|T\|_{L^2\to L^2} < \infty$$

and T admits the following representation:

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$

with K belonging to the class $\mathcal{H}_A$, namely satisfying that

$$H_{K,A} = \max\left\{H_{K,A,1}, H_{K,A,2}\right\} < \infty,$$

where

$$H_{K,A,1} = \sup_B \sup_{x,z\in\frac{1}{2}B} \sum_{k=1}^{\infty} \mu(2^kB)\left\|(K(x,\cdot) - K(z,\cdot))\chi_{2^kB\setminus 2^{k-1}B}\right\|_{A,2^kB} < \infty,$$

$$H_{K,A,2} = \sup_B \sup_{x,z\in\frac{1}{2}B} \sum_{k=1}^{\infty} \mu(2^kB)\left\|(K(\cdot,x) - K(\cdot,z))\chi_{2^kB\setminus 2^{k-1}B}\right\|_{A,2^kB} < \infty.$$

Observe that the class of L^∞-Hörmander operators contains the class of Calderón–Zygmund operators. A number of applications of A-Hörmander classes of operators are contained in [25, 26]. Among them it is worth mentioning differential transform operators which are $\exp(L^{\frac{1}{1+\varepsilon}})$-Hörmander operators with $\varepsilon > 0$, and multipliers that are $L^r\log(L^r)$-Hörmander operators with $r > 1$.

The following result gathers pointwise sparse domination results for $\bar{A}$-Hörmander operators in the Euclidean setting.

Theorem 4.1. *Let A be a submultiplicative Young function, let T be a $\bar{A}$-Hörmander operator and let $f \in C_c^\infty$. Let $\varepsilon \in (0,1)$. Then, there exist 3^n dyadic lattices $\mathcal{D}_j$ and ε-sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that*

- *[17, 21]*

$$|T_b^m f(x)| \le C_{n,m,T}\frac{1}{1-\varepsilon}\sum_{j=1}^{3^n}\mathcal{A}_{A,\mathcal{S}_j}f(x),$$

where

$$\mathcal{A}_{A,\mathcal{S}}(f)(x) = \sum_{Q\in\mathcal{S}}\|f\|_{A,Q}\chi_Q(x).$$

- *[17] if m is a non-negative integer and $b \in L^m_{loc}(\mathbb{R}^n)$, then*

$$|T_b^m f(x)| \le C_{n,m,T}\frac{1}{1-\varepsilon}\sum_{j=1}^{3^n}\sum_{h=0}^{m}\binom{m}{h}\mathcal{A}^{m,h}_{A,\mathcal{S}_j}(b,f)(x),$$

where

$$\mathcal{A}^{m,h}_{A,\mathcal{S}}(b,f)(x) = \sum_{Q\in\mathcal{S}}|b(x)-b_Q|^{m-h}\|f|b-b_Q|^h\|_{A,Q}\chi_Q(x).$$

- *[33] for $b_1,\dots,b_m \in L^1_{loc}(\mathbb{R}^n)$ such that $\||b|_\sigma\|_{A,Q} < \infty$ for every cube Q and for every $\sigma \in C_j(b)$ where $j \in \{1,\dots,m\}$, then*

$$|T_{\mathbf{b}}f(x)| \le c_{n,m,T}\frac{1}{1-\varepsilon}\sum_{j=1}^{3^n}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\mathcal{A}^\sigma_{A,\mathcal{S}_j}(b,f)(x), \tag{4.1}$$

where

$$\mathcal{A}^\sigma_{A,\mathcal{S}}(b,f)(x) = \sum_{Q\in\mathcal{S}}|b(x)-b_Q|_{\sigma'}\|f|b-b_Q|_\sigma\|_{A,Q}\chi_Q(x).$$

Observe that the required bilinear estimates (1.3)–(1.5) for the main results hold since if G is any of the operators above, then

$$\left|\int_{\mathbb{R}^n} Gfg\right| = \left|\int_{\mathbb{R}^n} fG^*g\right|$$

and the same sparse bounds for G hold as well for G^* and then Equations (1.3)–(1.5) readily follow. Hence, the main results allow us to derive the corresponding estimates for the aforementioned operators.

Remark 4.2. Note that in the case of $\bar{A}$-Hörmander operators the estimates above appeared first in [17] under some additional technical assumption on A. However, such an assumption can be dropped, as it was shown in [20] for T. In the case of commutators (4.1) appeared first in [33] under some additional technical condition on A. Later on in [16, Theorem 3.5], some particular cases of that result were recovered. It is possible as well, to provide the quantitative version in terms of the sparseness constant following ideas in [13]. In the next subsection, we will provide such a result in the realm of spaces of the homogeneous type.

Remark 4.3. In view of the aforementioned sparse domination results, it is worth noting that results obtained here further extend results in [5] and also allow to recover, for instance, results for $T_{\mathbf{b}}$, with T being a Calderón–Zygmund operator, recently obtained in [4].

4.1 | A sparse domination result for $T_{\mathbf{b}}$ commutators in spaces of homogeneous type and applications

In this subsection, we provide a full argument, extending Equation (4.1) to spaces of the homogeneous type, which contains $\mathbf{b} = \emptyset$ as a particular case, and allows us to drop the aforementioned technical condition in [17]. Note that this result also extends the commutator bound from [10]. We remit the reader to the latter mentioned paper and to [27], and references therein for some further insight on sparse domination on spaces of the homogeneous type. We will also show how to apply our main results to A-Hörmander operators and their commutators.

We begin recalling that given C a submultiplicative Young function

$$\mu(\{x \in X \, : \, M_C f(x) > t\}) \leq \|M_C\| \int_X C\left(\frac{|f|}{t}\right) d\mu.$$

The proof of this fact can be obtained relying upon covering by dyadic structures and then using the same argument as in, for instance, [21, Lemma 2.6].

We recall as well that given $\alpha > 0$ we can define the operator $\mathcal{M}^{\#}_{T,\alpha}$, that was introduced in [20] by

$$\mathcal{M}^{\#}_{T,\alpha} f(x) = \sup_{B \ni x} \operatorname*{esssup}_{y,z \in B} |T(f\chi_{X\setminus \alpha B})(y) - T(f\chi_{X\setminus\alpha B})(z)|.$$

The statement of the sparse domination result we intend to prove is the following.

Theorem 4.4. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D}$ a dyadic system with parameters c_0, C_0, and δ. Let us fix $\alpha \geq \frac{3c_d^2}{\delta}$ and let $f : X \to \mathbb{R}$ be a boundedly supported function such that $f \in L^\infty(X)$. Assume as well that $b_1, \dots, b_m \in L^\infty$. Let A and B be submultiplicative Young functions and $C = \max(A, B)$, and assume that there exist non-increasing functions ψ and ϕ such that for every $Q \in \mathcal{D}$ and any supported function $g \in L^\infty(X)$*

$$\mu\big(\{x \in Q : |T(g\chi_Q)(x)| > \psi(\rho)\|g\|_{A,Q}\}\big) \leq \rho\mu(Q) \quad (0 < \rho < 1)$$

and

$$\mu\Big(\{x \in Q : \mathcal{M}^{\#}_{T,\alpha}(g\chi_Q)(x) > \phi(\rho)\|g\|_{B,Q}\}\Big) \leq \rho\mu(Q) \quad (0 < \rho < 1).$$

Then, given $\varepsilon \in (0, 1)$, there exists a $(1 - \varepsilon)$-sparse family $\mathcal{S} \subset \mathcal{D}$ such that

$$|T_{\mathbf{b}} f(x)| \lesssim \kappa_{\xi,\varepsilon,C} \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{Q \in \mathcal{S}} \| |b - b_{\alpha Q}|_\sigma f\|_{C,\alpha Q} |b - b_{\alpha Q}|_{\sigma'} \chi_Q(x),$$

where

$$\kappa_{\xi,\varepsilon,C} = \left(2\xi\left(\frac{\varepsilon}{3c_1c_2}\right) + \xi\left(\frac{1}{3c_1c_2}\right)\frac{\varepsilon}{3c_1c_2}\|M_C\|\right)$$

and $\xi(\rho) = \psi(\rho) + \phi(\rho)$ with c_1 and c_2 being constants depending on the parameters defining $\mathcal{D}$. Furthermore, there exist $0 < c_0 \le C_0 < \infty$, $0 < \delta < 1$, $\gamma \ge 1$ and $k \in \mathbb{N}$ such that there are dyadic systems $\mathcal{D}_1, \dots, \mathcal{D}_k$ with parameters c_0, C_0, and δ and k $(1-\varepsilon)$-sparse families $\mathcal{S}_i \subset \mathcal{D}_i$ such that

$$|T_{\mathbf{b}} f(x)| \lesssim \kappa_{\xi,\rho,C} \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{Q \in \mathcal{S}} \sum_{j=1}^{k} \sum_{Q \in \mathcal{S}_j} \| |b - b_Q|_{\sigma} f \|_{C,Q} |b - b_Q|_{\sigma'} \chi_Q(x).$$

Before settling this result let us show how to provide applications from it. Observe that if an operator G satisfies the bound

$$\mu(\{x \in Q : |Gf(x)| > \lambda\}) \le C_G \int_X A\left(\frac{|f|}{\lambda}\right)$$

then we have that if $\varphi(\rho) \ge 1$ then

$$\begin{aligned}
\mu\big(\{x \in Q : |G(g\chi_Q)(x)| > \varphi(\rho)\|g\|_{A,Q}\}\big) &\le C_G \int_X A\left(\frac{|g\chi_Q|}{\varphi(\rho)\|g\|_{A,Q}}\right) \\
&\le C_G \frac{1}{\varphi(\rho)} \int_Q A\left(\frac{|g|}{\|g\|_{A,Q}}\right) \\
&\le C_G \frac{1}{\varphi(\rho)} \mu(Q).
\end{aligned}$$

Hence, we have that choosing, $\varphi(t) = \frac{C_G}{t}$, since $\rho \in (0,1)$, then

$$\mu\big(\{x \in Q : |G(g\chi_Q)(x)| > \varphi(\rho)\|f\|_{A,Q}\}\big) \le \rho\mu(Q).$$

Observe that this would be a suitable choice for φ in order to apply Theorem 4.4. Furthermore, note that for this choice of φ, then

$$\begin{aligned}
2\varphi\left(\frac{\varepsilon}{3c_1c_2}\right) + \varphi\left(\frac{1}{3c_1}\right)\frac{\varepsilon}{3c_1c_2}\|M_C\| &= 2\frac{C_G}{\frac{\varepsilon}{3c_1c_2}} + \frac{C_G}{\frac{1}{3c_1}}\frac{\varepsilon}{3c_1c_2}\|M_C\| \\
&\le 4\frac{c_1c_2C_G\|M_C\|}{\varepsilon}.
\end{aligned}$$

Taking these ideas into account we have the following corollary:

Corollary 4.5. *Let (X, d, μ) be a space of homogeneous type and $\mathcal{D}$ a dyadic system with parameters c_0, C_0, and δ. Let us fix $\alpha \ge \frac{3c_d^2}{\delta}$ and let $f : X \to \mathbb{R}$ be a boundedly supported function such that $f \in L^\infty(X)$. Assume as well that $b_1, \dots, b_m \in L^\infty$. Let A and B be Young functions and $C = \max(A, B)$, and assume that there exist non-increasing functions ψ and ϕ such that for every $Q \in \mathcal{D}$ and any boundedly supported function $g \in L^\infty(X)$*

$$\mu(\{x \in Q : |Tg(x)| > \lambda\}) \le \|T\| \int_X A\left(\frac{|f|}{\lambda}\right) \tag{4.2}$$

and

$$\mu\Big(\{x \in Q : \mathcal{M}^{\#}_{T,\alpha}(g) > \lambda\}\Big) \le \|\mathcal{M}^{\#}_{T,\alpha}\| \int_X B\left(\frac{|f|}{\lambda}\right). \tag{4.3}$$

Then, given $\varepsilon \in (0,1)$, there exists a ε-sparse family $\mathcal{S} \subset \mathcal{D}$ such that

$$|T_{\mathbf{b}}f(x)| \lesssim \kappa \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{Q \in \mathcal{S}} \| |b - b_{\alpha Q}|_\sigma f \|_{C,\alpha Q} |b - b_{\alpha Q}|_{\sigma'} \chi_Q(x),$$

where

$$\kappa = \frac{c_1 c_2 \left(\|T\| + \|\mathcal{M}^{\#}_{T,\alpha}\| \right) \|M_C\|}{1-\varepsilon}$$

and $\xi(\rho) = \psi(\rho) + \phi(\rho)$ with c_1 and c_2 being constants depending on the parameters defining $\mathcal{D}$. Furthermore, there exist $0 < c_0 \leq C_0 < \infty$, $0 < \delta < 1$, $\gamma \geq 1$ and $k \in \mathbb{N}$ such that there are dyadic systems $\mathcal{D}_1, \dots, \mathcal{D}_k$ with parameters c_0, C_0, and δ and k $(1-\varepsilon)$-sparse families $\mathcal{S}_i \subset \mathcal{D}_i$ such that

$$|T_{\mathbf{b}}f(x)| \lesssim \kappa \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{Q \in \mathcal{S}} \sum_{j=1}^{k} \sum_{Q \in \mathcal{S}_j} \| |b - b_Q|_\sigma f \|_{C,Q} |b - b_Q|_{\sigma'} \chi_Q(x).$$

If T is an $\bar{\Psi}$-Hörmander operator, then we have as well that it is not hard to check that Equation (4.2) holds for $A(t) := t$ and Equation (4.3) holds for the Young function $B(t) := \Psi(t)$. Hence, the sparse control in the corollary above holds. Note that since

$$\left| \int_X T_{\mathbf{b}} f(x) g(x) d\mu(x) \right| = \left| \int_X T^*_{\mathbf{b}} g(x) f(x) d\mu(x) \right|$$

and T^* is as well an $\bar{\Psi}$-Hörmander operator, then

$$\left| \int_X T_{\mathbf{b}} f(x) g(x) d\mu(x) \right| \lesssim \frac{1}{1-\varepsilon} \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{j=1}^{k} \sum_{Q \in \mathcal{S}_j} \| |b - b_Q|_\sigma g \|_{\Psi,Q} \int_Q |f| |b - b_Q|_{\sigma'}.$$

This yields that if T is an $\bar{\Psi}$-Hörmander operator, then T itself and its commutators $T_{\mathbf{b}}$ satisfy the sparse domination conditions (1.6) and (1.7), respectively, and hence the estimates obtained in Theorems 1.4 and 1.5 hold as well for T and $T_{\mathbf{b}}$.

Remark 4.6. The results that we have just mentioned in the preceding lines extend some results from [5] and the estimate for $T_{\mathbf{b}}$, with T being a Calderón–Zygmund operator, recently obtained in [4] to spaces of the homogeneous type.

4.1.1 | Proof of Theorem 4.4

To settle Theorem 4.4, we need a few Lemmas. The first of them is an adaption of ideas in [19] and was settled in [10] and relates sparse and Carleson families.

Lemma 4.7. *Let $\mathcal{D}$ be a dyadic system. If $\mathcal{S} \subset \mathcal{D}$ is a Λ-Carleson with $\Lambda > 1$, then it is a $\frac{1}{\Lambda}$-sparse family. Conversely, if $\mathcal{S}$ is a ε-sparse family with $\varepsilon \in (0,1)$, then $\mathcal{S}$ is a $\frac{1}{\varepsilon}$-Carleson family.*

The second of them is contained in [19, Section 6.3]. Although there it is stated in the Euclidean setting the same proof works as well for dyadic systems in spaces of the homogeneous type.

Lemma 4.8. *Let $\mathcal{D}$ be a dyadic system. If $\mathcal{S} \subset \mathcal{D}$ is a Λ-Carleson family and $t \geq 2$ then $\mathcal{S} = \cup_{i=1}^{t} \mathcal{S}_i$ each $\mathcal{S}_i$ is a $1 + \frac{\Lambda-1}{t}$-Carleson family.*

The third and last lemma we need is the following one, which contains the key estimate to perform the iterative process that will allow us to settle Theorem 4.4.

Lemma 4.9. *Let (X, d, μ) be a space of the homogeneous type and $\mathcal{D}$ a dyadic system with parameters c_0, C_0, and δ. Let us fix $\alpha \geq \frac{3c_d^2}{\delta}$ and let f be a boundedly supported function such that $f \in L^\infty(X)$. Let A and B be Young functions and $C = \max(A, B)$, and assume that there exist non-increasing functions ψ and ϕ such that for every $Q \in \mathcal{D}$, and every boundedly supported function $g \in L^\infty(X)$*

$$\mu\big(\{x \in Q : |T(g\chi_Q)(x)| > \psi(\rho)\|g\|_{A,Q}\}\big) \leq \rho\mu(Q) \quad (0 < \rho < 1)$$

and

$$\mu\Big(\{x \in Q : \mathcal{M}^{\#}_{T,\alpha}(g\chi_Q)(x) > \phi(\rho)\|g\|_{B,Q}\}\Big) \leq \rho\mu(Q) \quad (0 < \rho < 1).$$

Then, given $\varepsilon \in (0, 1)$ there exist disjoint subcubes $Q_j \in \mathcal{D}(Q)$ such that

$$\sum_j \mu(Q_j) \leq \varepsilon\mu(Q)$$

and for every $\sigma \in C_h(\mathbf{b})$ and $h = 0, \dots, m$,

$$\left|T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_Q - \sum_j T_{\mathbf{b}}(f\chi_{\alpha P_j})(x)\chi_{P_j}(x)\right| \leq \kappa_{\xi,\varepsilon,C}\sum_{h=0}^{m}\sum_{\sigma\in C_h(\mathbf{b})} |b(x) - c_Q|_\sigma \|f|b - c_Q|_{\sigma'}\|_{C,\alpha Q}\chi_Q(x),$$

where

$$\kappa_{\xi,\varepsilon,C} = 2\xi\left(\frac{\varepsilon}{3c_1c_2}\right) + \xi\left(\frac{1}{3c_1}\right)\frac{\varepsilon}{3c_1c_2}\|M_C\|$$

and $\xi(\rho) = \psi(\rho) + \phi(\rho)$ with c_1 and c_2 being constants depending on the parameters defining $\mathcal{D}$.

Proof. We shall use the following identity that was obtained in [32, p. 684]

$$T_{\mathbf{b}}f(x) = \sum_{h=0}^{m}\sum_{\sigma\in C(\mathbf{b})} (-1)^{m-h}(b(x) - \boldsymbol{\lambda})_{\sigma'}T\big((b - \boldsymbol{\lambda})_\sigma f\big)(x).$$

By the doubling condition of the measure, there exists c_1 such that $\mu(\alpha P) \leq c_1\mu(P)$ for any cube P. Now, we observe that for any disjoint family $\{P_j\} \subset \mathcal{D}(Q)$

$$T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_Q(x) = T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_{Q\setminus\cup P_j}(x) + \sum_j T_{\mathbf{b}}(f\chi_{\alpha Q\setminus\alpha P_j})(x)\chi_{P_j}(x) + \sum_j T_{\mathbf{b}}(f\chi_{\alpha P_j})(x)\chi_{P_j}(x)$$

and we have that

$$\begin{aligned}
T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_Q(x) - \sum_j T_{\mathbf{b}}(f\chi_{\alpha P_j})(x)\chi_{P_j}(x) &= T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_{Q\setminus\cup P_j}(x) + \sum_j T_{\mathbf{b}}(f\chi_{\alpha Q\setminus\alpha P_j})(x)\chi_{P_j}(x)\\
&= \sum_{h=0}^{m}\sum_{\sigma\in C(\mathbf{b})} (-1)^{m-h}\big(b(x) - c_Q\big)_\sigma T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q}\Big)(x)\chi_{Q\setminus\cup P_j}(x)\\
&\quad + \sum_{h=0}^{m}\sum_{\sigma\in C(\mathbf{b})}\sum_j (-1)^{m-h}\big(b(x) - c_Q\big)_\sigma T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x)\chi_{P_j}(x).
\end{aligned}$$

We claim that there exists a disjoint family $\{P_j\} \subset \mathcal{D}(Q)$ with

$$\sum_j \mu(P_j) \leq \varepsilon \mu(Q)$$

and

$$|T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_{Q\setminus \cup P_j} + \sum_j T_{\mathbf{b}}(f\chi_{\alpha Q\setminus \alpha P_j})(x)\chi_{P_j}|$$

$$\leq \kappa_{n,\rho,s} \sum_{h=0}^{m} \sum_{\sigma\in C_h(\mathbf{b})} |b(x) - c_Q|_\sigma \| f|b - c_Q|_{\sigma'} \|_{C,\alpha Q} \chi_Q(x).$$

Note that to settle the claim, we are actually going to show that

$$\Bigg| \sum_{h=0}^{m} \sum_{\sigma\in C(\mathbf{b})} (-1)^{m-h} (b(x) - c_Q)_\sigma T\Big((b - c_Q)_{\sigma'} f\chi_{\alpha Q}\Big)(x)\chi_{Q\setminus\cup P_j}(x)$$

$$+ \sum_{h=0}^{m} \sum_{\sigma\in C(\mathbf{b})} \sum_j (-1)^{m-h} (b(x) - c_Q)_\sigma T\Big((b - c_Q)_{\sigma'} f\chi_{\alpha Q\setminus \alpha P_j}\Big)(x)\chi_{P_j}(x)\Bigg|$$

$$\leq \kappa_{n,\rho,s} \sum_{h=0}^{m} \sum_{\sigma\in C_h(\mathbf{b})} |b(x) - c_Q|_\sigma \| f|b - c_Q|_{\sigma'} \|_{C,\alpha Q} \chi_Q(x).$$

For $\rho \in (0,1)$ to be chosen let

$$\tilde{\mathcal{M}}_T(f) = \max_{\sigma\in C_h(\mathbf{b}),\, h=0,\dots,m} \left\{ \frac{|T(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})|}{\xi(\rho)\||b - c_Q|_{\sigma'} f\|_{A,\alpha Q}}, \frac{\mathcal{M}^{\#}_{T,\alpha}(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})}{\xi(\rho)\||b - c_Q|_{\sigma'} f\|_{B,\alpha Q}} \right\}$$

and let us define the sets

$$\Omega = \left\{ x \in Q \,:\, \max\left\{ \frac{M_C(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})(x)}{\rho\|M_C\|\||b - c_Q|_{\sigma'} f\|_{C,\alpha Q}}, \tilde{\mathcal{M}}_T(f) \right\} > 1 \right\}.$$

Observe that

$$\mu(\Omega) \leq \sum_{\sigma\in C_h(\mathbf{b}),\, h=0,\dots,m} \mu\left(\left\{ x \in Q \,:\, \frac{M_C(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})(x)}{\rho\|M_C\|\||b - c_Q|_{\sigma'} f\|_{C,\alpha Q}} > 1 \right\}\right)$$

$$+ \sum_{\sigma\in C_h(\mathbf{b}),\, h=0,\dots,m} \mu\left(\left\{ x \in Q \,:\, \frac{|T(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})|}{\xi(\rho)\||b - c_Q|_{\sigma'} f\|_{A,\alpha Q}} > 1 \right\}\right)$$

$$+ \sum_{\sigma\in C_h(\mathbf{b}),\, h=0,\dots,m} \mu\left(\left\{ x \in Q \,:\, \frac{\mathcal{M}^{\#}_{T,\alpha}(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})}{\xi(\rho)\||b - c_Q|_{\sigma'} f\|_{B,\alpha Q}} > 1 \right\}\right)$$

$$\leq 3\Gamma_m \rho\mu(\alpha Q) \leq 3\Gamma_m \rho c_1 \mu(Q),$$

where $\Gamma_k = \sum_{\sigma\in C_h(\mathbf{b}),\, h=0,\dots,m}$. Now, we take the local Calderón–Zygmund decomposition (see [11, Lemma 4.5]) of

$$\Omega_{c_2} = \left\{ s \in Q \,:\, M^{\mathcal{D}(Q)}(\chi_\Omega) > \frac{1}{c_2} \right\} \qquad c_2 \geq 2.$$

For a suitable choice of c_2 we have that Ω_{c_2} is a proper subset of Q and that there exists a family $\{P_j\} \subset \mathcal{D}(Q)$ such that $\Omega_{c_2} = \bigcup P_j$ and

$$\frac{1}{c_2} \leq \frac{\mu(P_j \cap \Omega)}{\mu(P_j)} \leq \frac{1}{2}.$$

Then, we have that

$$\sum_j \mu(P_j) \leq c_2 \sum_j \mu(P_j \cap \Omega) \leq c_2\mu(\Omega) \leq 3\rho c_1 c_2 \mu(Q),$$

and choosing $\rho = \frac{\varepsilon}{3\Gamma_m c_1 c_2}$,

$$\sum_j \mu(P_j) \leq \varepsilon\mu(Q).$$

Note that by the Lebesgue differentiation theorem there exists some set N of measure zero such that

$$\Omega \setminus N \subset \Omega_{c_2} = \bigcup_j P_j. \tag{4.4}$$

Now, we show that this family $\{P_j\}$ is suitable for the claim above to hold. Taking Equation (4.4) into account if $x \in Q \setminus \cup P_j$ then the inequalities in Ω hold reversed a.e. in particular,

$$\left|T(|b - c_Q|_{\sigma'} f\chi_{\alpha Q})\right| \leq \xi(\rho)\||b - c_Q|_{\sigma'} f\|_{A,\alpha Q}.$$

Now, we deal with each term $T(f\chi_{\alpha Q\setminus\alpha P_j})(x)\chi_{P_j}(x)$. First, we note that $\mu(P_j \setminus \Omega) \neq 0$. Indeed

$$\mu(P_j) = \mu(P_j \cap \Omega) + \mu(P_j \setminus \Omega) \leq \frac{1}{2}\mu(P_j) + \mu(P_j \setminus \Omega).$$

Then for each $x' \in P_j \setminus \Omega$

$$\begin{aligned}\left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x)\right| &= \left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x) - T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x')\right| \\ &\quad + \left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x')\right|.\end{aligned}$$

For I we observe that since $x' \in P_j \setminus \Omega$

$$\begin{aligned}&\left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x) - T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x')\right| \\ &\leq \mathcal{M}^{\#}_{T,\alpha}(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q})(x') \leq \xi(\rho)\||b - c_Q|_{\sigma'} f\|_{A,\alpha Q}.\end{aligned}$$

Then

$$\left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x)\right| \leq \xi(\rho)\||b - c_Q|_{\sigma'} f\|_{C,\alpha Q} + \inf_{x'\in P_j\setminus\Omega}\left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x')\right|.$$

For the remaining term, we note that

$$\left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j}\Big)(x')\right| \leq \left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha Q}\Big)(x')\right| + \left|T\Big(\big(b - c_Q\big)_{\sigma'} f\chi_{\alpha P_j}\Big)(x')\right|.$$

For the first term, as above

$$\left|T\left(\left(b-c_Q\right)_{\sigma'} f\chi_{\alpha Q}\right)(x')\right| \le \xi(\rho)\||b-c_Q|_{\sigma'} f\|_{A,\alpha Q}.$$

Then

$$\inf_{x'\in P_j\setminus\Omega}\left|T(\left(b-c_Q\right)_{\sigma'} f\chi_{\alpha Q\setminus\alpha P_j})(x')\right| \le \xi(\rho)\||b-c_Q|_{\sigma'} f\|_{A,\alpha Q} + \inf_{x'\in P_j\setminus\Omega}\left|T\left(\left(b-c_Q\right)_{\sigma'} f\chi_{\alpha P_j}\right)(x')\right|.$$

Now, we note that for $t=\frac{1}{3c_1}$

$$\begin{aligned}&\mu\left(\left\{x\in P_j\ :\ \left|T\left((b-c_Q)_{\sigma'} f\chi_{\alpha P_j}\right)(x)\right| > \xi(t)\|(b-c_Q)_{\sigma'} f\|_{A,\alpha P_j}\right\}\right)\\ &\le t\mu(\alpha P_j)\le \frac{c_1}{3c_1}\mu(P_j)=\frac{1}{3}\mu(P_j).\end{aligned}$$

Then, necessarily

$$\begin{aligned}\inf_{x'\in P_j\setminus\Omega}\left|T\left(\left(b-c_Q\right)_{\sigma'} f\chi_{\alpha P_j}\right)(x')\right| &\le \xi(t)\|(b-c_Q)_{\sigma'} f\|_{A,\alpha P_j}\\ &\le \xi(t)\inf_{x'\in P_j\setminus\Omega} M_C(\left(b-c_Q\right)_{\sigma'} f\chi_{\alpha Q})(x')\\ &\le \xi(t)\rho\|M_C\|\|\left(b-c_Q\right)_{\sigma'} f\|_{C,\alpha Q}.\end{aligned}$$

Indeed, first we recall that since $\frac{\mu(P_j\cap\Omega)}{\mu(P_j)}\le\frac{1}{2}$, then

$$\mu(P_j)=\mu(P_j\cap\Omega)+\mu(P_j\setminus\Omega)\le\frac{1}{2}\mu(P_j)+\mu(P_j\setminus\Omega)$$

and we have that

$$\frac{1}{2}\mu(P_j)\le\mu(P_j\setminus\Omega).$$

Now, we observe that if we had

$$P_j\setminus\Omega\subset\left\{x\in P_j\ :\ \left|T\left((b-c_Q)_{\sigma'} f\chi_{\alpha P_j}\right)(x)\right| > \xi(\rho)\|\left(b-c_Q\right)_{\sigma'} f\|_{A,\alpha P_j}\right\}$$

then

$$\begin{aligned}&\frac{1}{2}\mu(P_j)\le\mu(P_j\setminus\Omega)\\ &\quad\le\mu\left(\left\{x\in P_j\ :\ \left|T\left((b-c_Q)_{\sigma'} f\chi_{\alpha P_j}\right)(x)\right| > \xi(\rho)\|\left(b-c_Q\right)_{\sigma'} f\|_{A,\alpha P_j}\right\}\right)\\ &\quad\le\frac{1}{3}\mu(P_j)\end{aligned}$$

which would be a contradiction. Gathering the estimates above the claim holds and hence we are done. □

Finally armed with the lemma above we are in the position to settle Theorem 4.4.

Proof of Theorem 4.4. Let Q be a cube. We iterate Lemma 4.9 and we get $\{Q_j^l\}$ families of cubes with

$$\sum_{Q_j^{l+1}\subset Q_i^l} \mu(Q_j^{l+1}) \leq \varepsilon\mu(Q_i^l)$$

such that

$$|T_{\mathbf{b}}(f\chi_{\alpha Q})(x)|\chi_Q(x) \leq \kappa_{n,\rho,s}\sum_{h=0}^{m}\sum_{\sigma\in C_h(\mathbf{b})}\sum_{l=0}^{L-1}\sum_j |b(x)-c_{Q_j^l}|_\sigma \| f|b-c_{Q_j^l}|_{\sigma'}\|_{C,\alpha Q_j^l}\chi_{Q_j^l}(x) + \sum_j T_{\mathbf{b}}(f\chi_{\alpha Q_j^L})(x)\chi_{Q_j^L}(x).$$

Note that

$$\sum_j \mu(Q_j^L) \leq \varepsilon^L\mu(Q).$$

Hence, letting $L\to\infty$

$$|T_{\mathbf{b}}(f\chi_{\alpha Q})(x)|\chi_Q(x) \leq \kappa_{n,\rho,s}\sum_{h=0}^{m}\sum_{\sigma\in C_h(\mathbf{b})}\sum_{l=0}^{\infty}\sum_j |b(x)-c_{Q_j^l}|_\sigma \| f|b-c_{Q_j^l}|_{\sigma'}\|_{C,\alpha Q_j^l}\chi_{Q_j^l}(x)$$

and clearly

$$\mathcal{S}_Q = \bigcup_{j,l}\{Q_j^l\}$$

is a $(1-\varepsilon)$-sparse family. Now, we use Lemma 2.2 with $E = \operatorname{supp}(f)$, there exists a partition of X, $\mathcal{P}\subset\mathcal{D}$ such that $E\subseteq\alpha Q$ for every $Q\in\mathcal{P}$. Then

$$T_{\mathbf{b}}f(x) = \sum_{Q\in\mathcal{D}} T_{\mathbf{b}}(f\chi_{\alpha Q})(x)\chi_Q(x)$$

and it suffices to apply the estimate above to each term and we are done just choosing in this case $c_{Q_j^k} = b_{Q_j^k}$. Note that since $\mathcal{D}$ is a partition $\mathcal{S} = \cup_{Q\in\mathcal{D}}\mathcal{S}_Q$ is a $(1-\varepsilon)$-sparse family and hence we are done.

To prove the furthermore part, we fix the parameters in Proposition 2.1. Then, there exist $\mathcal{D}_1,\dots,\mathcal{D}_{t_0}$ dyadic systems associated to those parameters. We repeat the argument above for $\mathcal{D}_1$ and its parameters. Then, there exists a $(1-\varepsilon)$-sparse family $\mathcal{S}\subset\mathcal{D}_1$ such that

$$|T_{\mathbf{b}}f(x)| \lesssim \kappa_{\xi,\varepsilon,C}\sum_{h=0}^{m}\sum_{\sigma\in C_h(\mathbf{b})}\sum_{Q\in\mathcal{S}} \||b-c_Q|_{\sigma'}f\|_{C,\alpha Q}|b-c_Q|_\sigma\chi_Q(x).$$

Now by Proposition 2.1, we have that for any $Q\in\mathcal{S}$ with center z and side length δ^k we can find $Q'\in\mathcal{D}_j$ for some $1\leq j\leq m_0$ such that

$$\alpha Q = B(z,\alpha C_0\delta^k)\subseteq Q' \qquad \operatorname{diam}(Q')\leq\gamma\alpha C_0\delta^k$$

then there exists $c>0$ depending on X and α such that

$$\mu(Q') \leq \mu(B(z,\alpha\gamma C_0\delta^k)) \leq c\mu B(z,C_0\delta^k) \leq c\mu(Q).$$

Taking $E_{Q'} = E_Q$ we have that

$$\mu(Q') \le c\mu(P) \le \frac{c}{1-\varepsilon}\mu(E_Q) = \frac{c}{1-\varepsilon}\mu(E_{Q'})$$

and hence

$$\mu(Q')\frac{1-\varepsilon}{c} \le \mu(E_{Q'})$$

and hence the collections of cubes

$$\tilde{\mathcal{S}}_j = \left\{Q' \in \mathcal{D}_j \ : \ Q \in \mathcal{S}\right\}$$

are $\frac{1-\varepsilon}{c}$-sparse. Gathering the facts above, at this point we have, choosing $c_Q = b_{Q'}$, the following estimate:

$$|T_{\mathbf{b}} f(x)| \lesssim c\kappa_{\xi,\varepsilon,C} \sum_{h=0}^{k} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{j=1}^{t_0} \sum_{Q \in \tilde{\mathcal{S}}_j} \| |b - b_Q|_{\sigma'} f \|_{C,\alpha Q} |b(x) - b_Q|_{\sigma} \chi_Q(x)$$

and all we are left to do is to show that we can actually choose $(1-\varepsilon)$-sparse families. For that purpose, observe that our families $\tilde{\mathcal{S}}_j$ are $\frac{c}{1-\varepsilon}$-Carleson. Let $u > c$ an integer. In virtue of Lemma 4.8 we have that we can write $\tilde{\mathcal{S}}_j = \cup_{i=1}^{u} \mathcal{S}_j^i$, where each $\mathcal{S}_j^i$ is $1 + \frac{\frac{c}{(1-\varepsilon)} - 1}{u}$-Carleson. Note that

$$1 + \frac{\frac{c}{1-\varepsilon} - 1}{u} = 1 + \frac{c - 1 + \varepsilon}{u(1-\varepsilon)} \le 1 + \frac{\varepsilon}{1-\varepsilon} = \frac{1}{1-\varepsilon}.$$

This yields that each $\mathcal{S}_j^i$ is $\frac{1}{1-\varepsilon}$-Carleson and consequently, $(1-\varepsilon)$-sparse. Then, we have that

$$|T_{\mathbf{b}} f(x)| \lesssim c\kappa_{\xi,\varepsilon,C} \sum_{h=0}^{m} \sum_{\sigma \in C_h(\mathbf{b})} \sum_{j=1}^{t_0} \sum_{i=1}^{u} \sum_{Q \in \mathcal{S}_i^j} \| |b - b_Q|_{\sigma'} f \|_{C,\alpha Q} |b(x) - b_Q|_{\sigma} \chi_Q(x)$$

and reindexing, we are done. □

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