A Hankel matrix acting on spaces of analytic functions

Daniel Girela and Noel Merchán

Abstract. If μ is a positive Borel measure on the interval $[0,1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0,1,2,\ldots$, μ_n denotes the moment of order n of μ. This matrix induces formally the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$$

on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. This is a natural generalization of the classical Hilbert operator. In this paper we improve the results obtained in some recent papers concerning the action of the operators H_μ on Hardy spaces and on Möbius invariant spaces.

Mathematics Subject Classification (2010). Primary 47B35; Secondary 30H10.

Keywords. Hankel matrix, Generalized Hilbert operator, Hardy spaces, BMOA, The Bloch space, Conformally invariant spaces, Carleson measures.

1. Introduction and main results

We denote by $\mathbb{D}$ the unit disc in the complex plane $\mathbb{C}$, and by $\mathcal{H}ol(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. We also let H^p $(0 < p \leq \infty)$ be the classical Hardy spaces. We refer to [19] for the notation and results regarding Hardy spaces.

If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\ldots$, we let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n\, d\mu(t)$, and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$.

This research is supported in part by a grant from "El Ministerio de Economía y Competitividad", Spain (MTM2014-52865-P) and by a grant from la Junta de Andalucía (FQM-210). The second author is also supported by a grant from "El Ministerio de de Educación, Cultura y Deporte", Spain (FPU2013/01478).

The matrix $\mathcal{H}_\mu$ can be viewed as an operator on spaces of analytic functions in the following way: if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = \left((n+k+1)^{-1}\right)_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has extensively studied recently (see [1, 13, 14, 17, 27, 28]). Other related generalizations of the Hilbert operator have been considered in [20] and [32].

The question of describing the measures μ for which the operator $\mathcal{H}_\mu$ is well defined and bounded on distinct spaces of analytic functions has been studied in a good number of papers (see [8, 12, 21, 23, 30, 34, 38]). Carleson measures play a basic role in these works.

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$.

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

A 1-Carleson measure will be simply called a Carleson measure.

We recall that Carleson [11] proved that $H^p \subset L^p(d\mu)$ $(0 < p < \infty)$, if and only if μ is a Carleson measure. This result was extended by Duren [18] (see also [19, Theorem 9. 4]) who proved that for $0 < p \leq q < \infty$, $H^p \subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, $0 < q < \infty$, and μ is a positive Borel measure in $\mathbb{D}$, μ is said to be a "*q-Carleson measure for the space X*" or an "*(X,q)-Carleson measure*" if $X \subset L^q(d\mu)$. The q-Carleson measures for the spaces H^p, $0 < p, q < \infty$ are completely characterized. The mentioned results of Carleson and Duren can be stated saying that if $0 < p \leq q < \infty$ then a positive Borel measure μ in $\mathbb{D}$ is a q-Carleson measure for H^p if and only if μ is a q/p-Carleson measure. Luecking [29] and Videnskii [37] solved the remaining case $0 < q < p$. We mention [9] for a complete information on Carleson measures for Hardy spaces.

Following [40], if μ is a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^{\alpha}}{|I|^s} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\left(A \cap [0,1)\right), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right) \leq C(1-t)^s, \quad 0 \leq t < 1.$$

We have a similar statement for α-logarithmic s-Carleson measures.

Widom [38, Theorem 3. 1] (see also [34, Theorem 3] and [33, p. 42, Theorem 7. 2]) proved that $\mathcal{H}_\mu$ is a bounded operator from H^2 into itself if and only μ is a Carleson measure. Galanopoulos and Peláez [21] studied the operators $\mathcal{H}_\mu$ acting on H^1 and Chatzifountas, Girela and Peláez [12] studied the action of $\mathcal{H}_\mu$ on H^p, $0 < p < \infty$.

A key ingredient in [21] and [12] is obtaining an integral representation of $\mathcal{H}_\mu$. If μ is as above, we shall write throughout the paper

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t),$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It turns out that the operators H_μ and I_μ are closely related. Indeed, some of the results obtained in [21] and [12] are the following ones:

Theorem A ([21]). *Let μ be a positive Borel measure on $[0,1)$. Then:*

(i) *The operator I_μ is well defined on H^1 if and only if μ is a Carleson measure.*

(ii) *If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is also well defined on H^1 and, furthermore,*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \textit{for every } f \in H^1.$$

(iii) *The operator I_μ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*

Theorem B ([12]). *Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$. Then:*

(i) *The operator I_μ is well defined on H^p if and only if μ is a 1-Carleson measure for H^p.*

(ii) *If μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is also well defined on H^p and, furthermore,*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \textit{for every } f \in H^p.$$

(iii) *The operator I_μ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.*

Theorem A and Theorem B immediately yield the following.

Theorem C. *Let μ be a positive Borel measure on $[0,1)$.*

(i) *If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*
(ii) *If $1 < p < \infty$ and μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.*

Theorem C does not close completely the question of characterizing the measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into itself. Indeed, in Theorem C we only consider 1-Carleson measures for H^p. In principle, there could exist a measure μ which is not a 1-Carleson measures for H^p but so that the operator $\mathcal{H}_\mu$ is well defined and bounded on H^p. Our first result in this paper asserts that this is not the case.

Theorem 1.1. *Let μ be a positive Borel measure on $[0,1)$.*
(i) *The operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*
(ii) *If $1 < p < \infty$ then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.*

We have the following result for $p = \infty$, a case which was not considered in [12].

Theorem 1.2. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*
(i) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.
(ii) $\sum_{n=0}^{\infty} \mu_n < \infty$.
(iii) *The operator I_μ is a bounded operator from H^∞ into itself.*
(iv) *The operator $\mathcal{H}_\mu$ is a bounded operator from H^∞ into itself.*

In the paper [23] the authors have studied the operators $\mathcal{H}_\mu$ acting on certain conformally invariant spaces such as the Bloch space, BMOA, the analytic Besov spaces B^p $(1 < p < \infty)$, and the Q_s spaces. Let us introduce quickly these spaces.

It is well known that the set of all disc automorphisms (*i.e.*, of all one-to-one analytic maps f of $\mathbb{D}$ onto itself), denoted $\mathrm{Aut}(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself: $\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_a : |a| < 1, |\lambda| = 1\}$, where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$.

A space X of analytic functions in $\mathbb{D}$, defined via a semi-norm ρ, is said to be *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f\circ\varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, $\rho(f\circ\varphi) \le C\rho(f)$ for some positive constant C and all $f \in X$. We mention [3, 15, 42] as references for Möbius invariant spaces.

The *Bloch space* $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ with bounded invariant derivative:

$$f \in \mathcal{B} \;\Leftrightarrow\; \rho_{\mathcal{B}}(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f'(z)| < \infty .$$

A classical reference for the Bloch space is [2]; see also [42]. Rubel and Timoney [35] proved that $\mathcal{B}$ is the biggest "natural" conformally invariant space.

The space $BMOA$ consists of those functions f in H^1 whose boundary values have bounded mean oscillation on the unit circle. Alternatively, $BMOA$ can be characterized in the following way:

If f is an analytic function in $\mathbb{D}$, then $f \in BMOA$ if and only if

$$\|f\|_\star \stackrel{\text{def}}{=} \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^2} < \infty.$$

The seminorm $\|\cdot\|_\star$ is conformally invariant. We mention [22] as a general reference for the space $BMOA$. Let us recall that

$$H^\infty \subsetneq BMOA \subsetneq \bigcap_{0<p<\infty} H^p \quad \text{and } BMOA \subsetneq \mathcal{B}.$$

If $0 \le s < \infty$, we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^s\, dA(z)\right)^{1/2} < \infty.$$

Here, $g(z,a)$ is the Green's function in $\mathbb{D}$, given by $g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right|$, while $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. All Q_s spaces $(0 \le s < \infty)$ are conformally invariant with respect to the semi-norm ρ_{Q_s} (see e.g., [39, p. 1] or [15, p. 47]).

These spaces were introduced by Aulaskari and Lappan in [5] while looking for new characterizations of Bloch functions. They proved that for $s > 1$, Q_s is the Bloch space. Using one of the many characterizations of the space $BMOA$ (see [22, Theorem 6. 2]) we have that $Q_1 = BMOA$. In the limit case $s = 0$, Q_s is the classical Dirichlet space $\mathcal{D}$ of those analytic functions f in $\mathbb{D}$ satisfying $\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty$.

Aulaskari, Xiao and Zhao proved in [7] that

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \qquad 0 < s_1 < s_2 < 1.$$

We mention [39] as an excellent reference for the theory of Q_s-spaces.

For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all functions f analytic in $\mathbb{D}$ such that

$$\rho_p(f) = \left(\int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

All B^p spaces $(1 < p < \infty)$ are conformally invariant with respect to the semi-norm ρ_p (see [3, p. 112] or [15, p. 46]). We have that $\mathcal{D} = B^2$. A lot of information on Besov spaces can be found in [3, 15, 16, 25, 41, 42]. Let us recall that

$$B^p \subsetneq B^q \subsetneq BMOA, \quad 1 < p < q < \infty.$$

Among others, the following results have been proved in [23].

Theorem D. *Let μ be a positive Borel measure on* $[0,1)$.

(i) *For any given* $s > 0$, *the operator* I_μ *is well defined in* Q_s *if and only if*
$$\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty.$$
(ii) *For any given* $s > 0$, *the operator* I_μ *is a bounded operator from* Q_s *into* $BMOA$ *if and only if* μ *is a* 1*-logarithmic* 1*-Carleson measure.*
(iii) *If* μ *is a* 1*-logarithmic* 1*-Carleson measure then* $\mathcal{H}_\mu(f) = I_\mu(f)$, *for all* $f \in \mathcal{B}$.
(iv) *If* μ *is a* 1*-logarithmic* 1*-Carleson measure then* $\mathcal{H}_\mu$ *is a bounded operator from* Q_s *into* $BMOA$ *for any* $s > 0$.

It is natural to look for a characterization of those μ for which I_μ and/or $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{B}$ into itself or, more generally, from Q_s into itself for any $s > 0$. We have the following result.

Theorem 1.3. *Let* μ *be a positive Borel measure on* $[0, 1)$. *Then the following conditions are equivalent.*

(i) *The operator* I_μ *is bounded from* Q_s *into itself for some* $s > 0$.
(ii) *The operator* I_μ *is bounded from* Q_s *into itself for all* $s > 0$.
(iii) *The operator* $\mathcal{H}_\mu$ *is bounded from* Q_s *into itself for some* $s > 0$.
(iv) *The operator* $\mathcal{H}_\mu$ *is bounded from* Q_s *into itself for all* $s > 0$.
(v) *The measure* μ *is a* 1*-logarithmic* 1*-Carleson measure.*

In [23] we also studied the operators $\mathcal{H}_\mu$ acting on Besov spaces. Theorem 3. 8 of [23] asserts that μ being a γ-logarithmic 1-Carleson measure for some $\gamma > 1$ is a sufficient condition for the boundedness of $\mathcal{H}_\mu$ from B^p into itself, for any $p > 1$. On the other hand, Theorem 3. 7 of [23] asserts that if $1 < p < \infty$ and the operator $\mathcal{H}_\mu$ is bounded from B^p to itself then μ is a γ-logarithmic 1-Carleson measure for any $\gamma < 1 - \frac{1}{p}$. We can improve this result as follows.

Theorem 1.4. *Suppose that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$ *such that the operator* $\mathcal{H}_\mu$ *is bounded from* B^p *into itself. Then* μ *is a* $\left(1 - \frac{1}{p}\right)$*-logarithmic* 1*-Carleson measure.*

The paper is organized as follows. The results concerning Hardy spaces will be proved in Section 2; Section 3 will be devoted to prove Theorem 1.3 and Theorem 1.4. We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2. The operator $\mathcal{H}_\mu$ acting on Hardy spaces

This section is devoted to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 *(i).* Suppose that $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself. For $0 < b < 1$, set

$$f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.$$

We have that $f_b \in H^1$ and $\|f_b\|_{H^1} = 1$. Since $\mathcal{H}_\mu$ is bounded on H^1, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^1}. \tag{2.1}$$

We also have,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} = (1-b^2)(k+1)b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem 1. 1 of [31], (2.1), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^1} &\gtrsim \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k}\, d\,\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^k \int_{[b,1)} t^{n+k}\, d\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^{n+2k}\, \mu\left([b,1)\right) \right) \\
&= (1-b^2)\mu\left([b,1)\right) \sum_{n=1}^{\infty} \frac{b^n}{n} \left(\sum_{k=1}^{\infty} k b^{2k} \right) \\
&= (1-b^2)\mu\left([b,1)\right) \left(\log \frac{1}{1-b} \right) \frac{b}{(1-b^2)^2}
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) = \mathrm{O}\left(\frac{1-b}{\log \frac{1}{1-b}} \right), \quad \text{as } b \to 1.$$

Hence, μ is a 1-logarithmic 1-Carleson measure.

The converse follows from Theorem C (i). $\square$

Proof of Theorem 1.1 *(ii).* Suppose that $1 < p < \infty$ and that μ is a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself.

For $0 < b < 1$, set

$$f_b(z) = \left(\frac{1-b^2}{(1-bz)^2} \right)^{1/p}, \quad z \in \mathbb{D}.$$

We have that $f_b \in H^p$ and $\|f_b\|_{H^p} = 1$. Since $\mathcal{H}_\mu$ is bounded on H^p, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^p}. \tag{2.2}$$

We also have,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} \approx (1-b^2)^{1/p} k^{\frac{2}{p}-1} b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem A of [31], (2.1), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^p}^p &\gtrsim \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right)^p \\
&= \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k}\, d\mu(t) \right)^p \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^k \int_{[b,1)} t^{n+k}\, d\mu(t) \right)^p \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^{n+2k} \mu\left([b,1)\right) \right)^p \\
&= (1-b^2) \mu\left([b,1)\right)^p \sum_{n=1}^{\infty} n^{p-2} b^{np} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^{2k} \right)^p \\
&\asymp (1-b^2) \mu\left([b,1)\right)^p \frac{1}{(1-b)^2} \sum_{n=1}^{\infty} n^{p-2} b^{np} \\
&\asymp \mu\left([b,1)\right)^p \frac{1}{(1-b)^p}, \quad \text{as } b \to 1.
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) = \mathrm{O}\left(1-b\right), \quad \text{as } b \to 1,$$

and, hence, μ is a Carleson measure.

The other implication follows from Theorem C (ii). $\square$

Proof of Theorem 1.2. The equivalence (i) $\Leftrightarrow$ (ii) is clear because

$$\int_{[0,1)} \frac{d\mu(t)}{1-t} = \int_{[0,1)} \left(\sum_{n=0}^{\infty} t^n \right) d\mu(t) = \sum_{n=0}^{\infty} \int_{[0,1)} t^n d\mu(t) = \sum_{n=0}^{\infty} \mu_n.$$

The implication (i) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Suppose (iii). Let f be the constant function $f(z) = 1$, for all z. Then (iii) implies that there exists a positive constant C such that

$$\left| \int_{[0,1)} \frac{d\mu(t)}{1 - tz} \right| \leq C, \quad z \in \mathbb{D}.$$

Taking $z = 0$ in this inequality, (i) follows.

(iii) $\Rightarrow$ (iv): Suppose (iii). We have seen that then (i) holds, and it is easy to see that (i) implies that μ is a Carleson measure. Using part (ii) of Theorem A, it follows that $\mathcal{H}_\mu$ is well defined in H^∞ and that $\mathcal{H}_\mu(f) = I_\mu(f)$ for all f in H^∞. Then (iii) gives that $\mathcal{H}_\mu$ is bounded from H^∞ into itself.

(iv) $\Rightarrow$ (iii): Suppose that (iv) is true and, as above, let f be the constant function $f(z) = 1$, for all z. Then $\mathcal{H}_\mu(f) \in H^\infty$. But $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^\infty \mu_n z^n$ and then it is clear that

$$\mathcal{H}_\mu(f) \in H^\infty \;\Leftrightarrow\; \sum_{n=0}^\infty \mu_n < \infty.$$

Thus we have seen that (iv) $\Rightarrow$ (ii). Since (ii) $\Leftrightarrow$ (iii), this finishes the proof. $\square$

3. The operator $\mathcal{H}_\mu$ acting on Möbius invariant spaces

A basic ingredient in the proof of Theorem 1.3 will be to have a characterization of the functions $f(z) = \sum_{n=0}^\infty a_n z^n$ whose sequence of Taylor coefficients $\{a_n\}_{n=0}^\infty$ is a decreasing sequence of nonnegative numbers which lie in the Q_s-spaces. This is quite simple for $s > 1$ (recall that $Q_s = \mathcal{B}$ if $s > 1$):

Hwang and Lappan proved in [26, Theorem 1] that if $\{a_n\}$ is a decreasing sequence of nonnegative numbers then $f(z) = \sum_{n=0}^\infty a_n z^n$ is a Bloch function if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.

Fefferman gave a characterization of the analytic functions having nonnegative Taylor coefficients which belong to $BMOA$, proofs of this criterium can be found in [10, 22, 24, 36]. Characterizations of the analytic functions having nonnegative Taylor coefficients which belong to Q_s $(0 < s < 1)$ were obtained in [6, Theorem 1. 2] and [4, Theorem 2. 3]. Using the mentioned result in [6, Theorem 1. 2], Xiao proved in [39, Corollary 3. 3. 1, p. 29] the following result.

Theorem E. *Let $s \in (0, \infty)$ and let $f(z) = \sum_{n=0}^\infty a_n z^n$ with $\{a_n\}$ being a decreasing sequence of nonnegative numbers. Then $f \in Q_s$ if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.*

Being based on Theorem 1. 2 of [6], Xiao's proof of this result is complicated. We shall give next an alternative simpler proof. It will simply use the validity of the result for the Bloch space and the simple fact that the mean Lipschitz space $\Lambda^2_{1/2}$ is contained in all the Q_s spaces $(0 < s < \infty)$ (see [4, Remark 4, p. 427] or [39, Theorem 4. 2. 1.]).

We recall [19, Chapter 5] that a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the mean Lipschitz space $\Lambda^2_{1/2}$ if and only if

$$M_2(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1/2}}\right).$$

We have the following simple result for the space $\Lambda^2_{1/2}$.

Lemma 3.1. If $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, then $f \in \Lambda^2_{1/2}$ if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.

Proof. If $a_n = \mathrm{O}\left(\frac{1}{n}\right)$, then

$$M_2(r, f')^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \lesssim \sum_{n=1}^{\infty} r^{2n-2} \lesssim \frac{1}{1-r},$$

and, hence, $f \in \Lambda^2_{1/2}$.

Suppose now that $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers and $f \in \Lambda^2_{1/2}$. Then, for all n

$$\sum_{k=1}^{n} k^2 a_k^2 r^{2k-2} \le \sum_{k=1}^{\infty} k^2 a_k^2 r^{2k-2} = M_2(r, f')^2 \lesssim \frac{1}{1-r}. \tag{3.1}$$

Taking $r = 1 - \frac{1}{n}$ in (3.1), we obtain

$$\sum_{k=1}^{n} k^2 a_k^2 \lesssim n. \tag{3.2}$$

Since $\{a_n\}$ is decreasing, using (3.2) we have

$$a_n^2 \sum_{k=1}^{n} k^2 \lesssim \sum_{k=1}^{n} k^2 a_k^2 \lesssim n$$

and then it follows that $a_n = \mathrm{O}\left(\frac{1}{n}\right)$. $\square$

Now Theorem E follows using the result of Hwang and Lappan for the Bloch space, Lemma 3.1, and the fact that

$$\Lambda^2_{1/2} \subset Q_s \subset \mathcal{B}, \qquad \text{for all } s. \tag{3.3}$$

Using (3.3), it is clear that Theorem 1.3 follows from the following result.

Theorem 3.1. *Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.*

(i) *The operator I_μ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.*
(ii) *The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.*
(iii) *The measure μ is a 1-logarithmic 1-Carleson measure.*
(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

Proof. According to Proposition 2. 5 of [23], μ is a 1-logarithmic 1-Carleson measure if and only if the measure ν defined by $d\nu(t) = \log\frac{1}{1-t}d\mu(t)$ is a Carleson measure and, using Proposition 1 of [12], this is equivalent to (iv). Hence, we have shown that (iii) $\Leftrightarrow$ (iv).

Set $F(z) = \log\frac{1}{1-z}$ $(z\in\mathbb{D})$. We have that $F\in X$.

(i) $\Rightarrow$ (iv): Suppose (i). Then

$$I_\mu(F)(z) = \int_{[0,1)} \frac{\log\frac{1}{1-t}}{1-tz}d\mu(t)$$

is well defined for all $z\in\mathbb{D}$. Taking $z=0$, we see that $\int_{[0,1)}\log\frac{1}{1-t}d\mu(t)<\infty$. Since $F\in X$ we have also that $I_\mu(F)\in\Lambda^2_{1/2}$, but

$$I_\mu(F)(z) = \int_{[0,1)} \frac{\log\frac{1}{1-t}}{1-tz}d\mu(t) = \sum_{n=0}^\infty\left(\int_{[0,1)} t^n\log\frac{1}{1-t}d\mu(t)\right)z^n.$$

Since the sequence $\left\{\int_{[0,1)}t^n\log\frac{1}{1-t}d\mu(t)\right\}_{n=0}^\infty$ is a decreasing sequence of nonnegative numbers, using Lemma 3.1 we see that (iv) holds.

(iv) $\Rightarrow$ (i): Suppose (iv) and take $f\in X$. Since $X\subset\mathcal{B}$, it is well known that $|f(z)|\lesssim\log\frac{2}{1-|z|}$, see [2, p. 13]. This and (iv) give

$$\int_{[0,1)} t^n|f(t)|d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right). \tag{3.4}$$

Then it follows easily that $I_\mu(f)$ is well defined and that

$$I_\mu(f)(z) = \sum_{n=0}^\infty\left(\int_{[0,1)} t^n f(t)d\mu(t)\right)z^n.$$

Now (3.4) implies that $\int_{[0,1)}t^nf(t)d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$ and then it follows that $I_\mu(f)\in\Lambda^2_{1/2}$.

The implication (iv) $\Rightarrow$ (ii) follows using Theorem 2. 3 of [23] and the already proved equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

It remains to prove that (ii) $\Rightarrow$ (iv). Suppose (ii) then $\mathcal{H}_\mu(F)\in\Lambda^2_{1/2}$. Now

$$\mathcal{H}_\mu(F)(z) = \sum_{n=0}^\infty\left(\sum_{k=1}^\infty\frac{\mu_{n+k}}{k}\right)z^n.$$

Notice that the sequence $\{\sum_{k=1}^\infty\frac{\mu_{n+k}}{k}\}_{n=0}^\infty$ is a decreasing sequence of nonnegative numbers. Then, using Lemma 3.1 and the fact that $\mathcal{H}_\mu(F)\in\Lambda^2_{1/2}$, we deduce that

$$\sum_{k=1}^\infty\frac{\mu_{n+k}}{k} = \mathrm{O}\left(\frac{1}{n}\right). \tag{3.5}$$

Now

$$\sum_{k=1}^\infty\frac{\mu_{n+k}}{k} = \int_{[0,1)}\sum_{k=1}^\infty\frac{t^{n+k}}{k}d\mu(t) = \int_{[0,1)} t^n\log\frac{1}{1-t}d\mu(t).$$

Then (iv) follows using (3.5). □

Remark 3.1. It is clear that Theorem 3.1 actually implies the following result.

Theorem 3.2. *Let μ be a positive Borel measure on $[0,1)$ and let $0 < s_1, s_2 < \infty$. Then following conditions are equivalent.*

(i) *The operator I_μ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.*
(ii) *The operator $\mathcal{H}_\mu$ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.*
(iii) *The measure μ is a 1-logarithmic 1-Carleson measure.*

Proof of Theorem 1.4. Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. For $\frac{1}{2} < b < 1$, set

$$g_b(z) = \left(\log\frac{1}{1-b^2}\right)^{-1/p}\log\frac{1}{1-bz}, \quad z\in\mathbb{D}.$$

We have,

$$g_b'(z) = \left(\log\frac{1}{1-b^2}\right)^{-1/p}\frac{b}{1-bz}, \quad z\in\mathbb{D}$$

and then, using Lemma 3. 10 of [42] with $t = p-2$ and $c = 0$, we have

$$\int_{\mathbb{D}}(1-|z|^2)^{p-2}|g_b'(z)|^p\,dA(z) \asymp \left(\log\frac{1}{1-b^2}\right)^{-1}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2}}{|1-bz|^p}\,dA(z) \asymp 1.$$

In other words, we have that

$$g_b\in B^p \quad\text{and}\quad \|g_b\|_{B^p}\asymp 1.$$

Since $\mathcal{H}_\mu$ is a bounded operator from B^p into itself, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(g_b)\|_{B^p}^p. \tag{3.6}$$

We have

$$g_b(z) = \sum_{k=0}^{\infty} a_{k,b}z^k, \quad\text{with}\quad a_{k,b} = \left(\log\frac{1}{1-b^2}\right)^{-1/p}\frac{b^k}{k}.$$

Since the $a_{k,b}$'s are positive it follows that the sequence $\{\sum_{k=0}^{\infty}\mu_{n+k}a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(g_b)$ is a decreasing sequence of non-negative real numbers. Using this, [23, Theorem 3. 10], and (3.6) we see that

$$
\begin{aligned}
1 \gtrsim \|\mathcal{H}_\mu(g_b)\|_{B^p}^p &\gtrsim \sum_{n=1}^{\infty} n^{p-1}\left(\sum_{k=1}^{\infty}\mu_{n+k}a_{k,b}\right)^p \\
&= \left(\log\frac{1}{1-b^2}\right)^{-1}\sum_{n=1}^{\infty} n^{p-1}\left(\sum_{k=1}^{\infty}\frac{b^k}{k}\int_{[0,1)} t^{n+k}d\mu(t)\right)^p \\
&\geq \left(\log\frac{1}{1-b^2}\right)^{-1}\sum_{n=1}^{\infty} n^{p-1}\left(\sum_{k=1}^{\infty}\frac{b^k}{k}\int_{[b,1)} t^{n+k}d\mu(t)\right)^p \\
&\geq \left(\log\frac{1}{1-b^2}\right)^{-1}\sum_{n=1}^{\infty} n^{p-1}\left(\sum_{k=1}^{\infty}\frac{b^{n+2k}}{k}\right)^p \mu\left([b,1)\right)^p \\
&= \left(\log\frac{1}{1-b^2}\right)^{-1}\sum_{n=1}^{\infty} n^{p-1}b^{np}\left(\sum_{k=1}^{\infty}\frac{b^{2k}}{k}\right)^p \mu\left([b,1)\right)^p \\
&= \left(\log\frac{1}{1-b^2}\right)^{p-1}\frac{1}{(1-b^p)^p}\mu\left([b,1)\right)^p \\
&\asymp \left(\log\frac{1}{1-b^2}\right)^{p-1}\frac{1}{(1-b)^p}\mu\left([b,1)\right)^p .
\end{aligned}
$$

Then it follows that $\mu\left([b,1)\right) \lesssim \frac{1-b}{\left(\log\frac{1}{1-b}\right)^{1-\frac{1}{p}}}$. This finishes the proof. □

References

[1] A. Aleman, A. Montes-Rodríguez and A. Sarafoleanu, *The eigenfunctions of the Hilbert matrix*, Const. Approx. **36** n. 3, (2012), 353–374.

[2] J. M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.

[3] J. Arazy, S. D. Fisher and J. Peetre, *Möbius invariant function spaces*, J. Reine Angew. Math. **363** (1985), 110–145.

[4] R. Aulaskari, D. Girela and H. Wulan, *Taylor coefficients and mean growth of the derivative of Q_p functions*, J. Math. Anal. Appl. **258** (2001), no. 2, 415-428.

[5] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex Analysis and its Applications (Harlow), Pitman Research Notes in Math, vol. 305, Longman Scientific and Technical, 1994, 136–146.

[6] R. Aulaskari, D. A. Stegenga and J. Xiao, *Some subclasses of BMOA and their characterization in terms of Carleson measures* Rocky Mountain J. Math. **26** (1996), no. 2, 485-506.

[7] R. Aulaskari, J. Xiao and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), 101–121.

[8] G. Bao and H. Wulan, *Hankel matrices acting on Dirichlet spaces*, J. Math. Anal. Appl. **409** (2014), no. 1, 228-235.

[9] O. Blasco and H. Jarchow, *A note on Carleson measures for Hardy spaces*, Acta Sci. Math. (Szeged) **71** (2005), no. 1–2, 371–389.

[10] F. F. Bonsall, *Boundedness of Hankel matrices*, J. London Math. Soc. (2) **29** (1984), no. 2, 289-300.

[11] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547-559.

[12] Ch. Chatzifountas, D. Girela and J. A. Peláez, *A generalized Hilbert matrix acting on Hardy spaces*, J. Math. Anal. Appl. **413** (2014), no. 1, 154-168.

[13] E. Diamantopoulos, *Hilbert matrix on Bergman spaces*, Illinois J. of Math. **48**, n. 3, Fall (2004), 1067–1078.

[14] E. Diamantopoulos and A. G. Siskakis, *Composition operators and the Hilbert matrix*, Studia Math. **140** (2000), 191–198.

[15] J. J. Donaire, D. Girela and D. Vukotić, *On univalent functions in some Möbius invariant spaces*, J. Reine Angew. Math. **553** (2002), 43–72.

[16] J. J. Donaire, D. Girela and D. Vukotić, *On the growth and range of functions in Möbius invariant spaces*, J. Anal. Math. **112** (2010), 237-260.

[17] M. Dostanić, M. Jevtić and D. Vukotić, *Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type*, J. Funct. Anal. **254** (2008), 2800–2815.

[18] P. L. Duren, *Extension of a Theorem of Carleson*, Bull. Amer. Math. Soc. **75** (1969), 143–146.

[19] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York-London 1970. Reprint: Dover, Mineola, New York 2000.

[20] P. Galanopoulos, D. Girela, J. A. Peláez and A. Siskakis, *Generalized Hilbert operators*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 1, 231-258.

[21] P. Galanopoulos and J. A. Peláez, *A Hankel matrix acting on Hardy and Bergman spaces*, Studia Math. **200**, 3, (2010), 201–220.

[22] D. Girela, *Analytic functions of bounded mean oscillation.* In: Complex Function Spaces, Mekrijärvi 1999 Editor: R. Aulaskari. Univ. Joensuu Dept. Math. Rep. Ser. 4, Univ. Joensuu, Joensuu (2001) pp. 61–170.

[23] D. Girela and N. Merchán, *A generalized Hilbert operator acting on conformally invariant spaces*, Banach J. Math. Anal., advance publication, 8 September 2017. doi:10.1215/17358787-2017-0023. https://projecteuclid.org/euclid.bjma/1504857614

[24] F. Holland and D. Walsh, *Boundedness criteria for Hankel operators*, Proc. Roy. Irish Acad. Sect. A **84** (1984), no. 2, 141-154.

[25] F. Holland and D. Walsh, *Growth estimates for functions in the Besov spaces A_p*, Proc. Roy. Irish Acad. Sect. A **88** (1988), 1–18.

[26] J. S. Hwang and P. Lappan, *Coefficients of Bloch functions and normal functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), no. 1, 69-75.

[27] M. Jevtić and B. Karapetrović, *Hilbert matrix on spaces of Bergman-type*, J. Math. Anal. Appl. **453** (2017), no. 1, 241–254.

[28] B. Lanucha, M. Nowak and M. Pavlović, *Hilbert matrix operator on spaces of analytic functions*, Ann. Acad. Sci. Fenn. Math. **37** (2012), 161–174.

[29] D. H. Luecking, *Embedding derivatives of Hardy spaces into Lebesgue spaces*, Proc. London Math. Soc. **63** (1991), no. 3, 595–619.

[30] M. Papadimitrakis and J. A. Virtanen, *Hankel and Toeplitz transforms on Hq: continuity, compactness and Fredholm properties*, Integral Equations Operator Theory **61** (2008), no. 4, 573-591.

[31] M. Pavlović, *Analytic functions with decreasing coefficients and Hardy and Bloch spaces*, Proc. Edinburgh Math. Soc. Ser. 2 **56**, 2, (2013), 623–635.

[32] J. A. Peláez and J. Rättyä, *Generalized Hilbert operators on weighted Bergman spaces*, Adv. Math. **240** (2013), 227–267.

[33] V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[34] S. C. Power, *Vanishing Carleson measures*, Bull. London Math. Soc. **12**, (1980), 207–210.

[35] L. E. Rubel and R. M. Timoney, *An extremal property of the Bloch space*, Proc. Amer. Math. Soc. **75** (1979), no. 1, 45–49.

[36] W. T. Sledd and D. A. Stegenga, *An H^1 multiplier theorem.* Ark. Mat. **19** (1981), no. 2, 265-270.

[37] I. V. Videnskii, *An analogue of Carleson measures*, Dokl. Akad. Nauk SSSR **298**, no. 5, (1988) 1042–1047 (Russian); translation in Soviet Math. Dokl. **37** no. 1, (1988), 186–190.

[38] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121**, (1966), 1–35.

[39] J. Xiao, *Holomorphic Q classes*, Lecture Notes in Mathematics **1767**, Springer-Verlag, 2001.

[40] R. Zhao, *On logarithmic Carleson measures*, Acta Sci. Math. (Szeged) **69** (2003), no. 3-4, 605–618.

[41] K. Zhu, *Analytic Besov spaces*, J. Math. Anal. Appl. **157** (1991), 318–336.

[42] K. Zhu, *Operator Theory in Function Spaces, Second Edition*, Math. Surveys and Monographs, **138** (2007).

Daniel Girela
Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
e-mail: `girela@uma.es`

Noel Merchán
Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
e-mail: `noel@uma.es`