HANKEL MATRICES ACTING ON THE HARDY SPACE H^1 AND ON DIRICHLET SPACES

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ABSTRACT. If μ is a finite positive Borel measure on the interval $[0,1)$, we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0,1,2,\dots$, μ_n denotes the moment of order n of μ. This matrix induces formally the operator $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$ on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty}a_k z^k$, in the unit disc $\mathbb{D}$. When μ is the Lebesgue measure on $[0,1)$ the operator $\mathcal{H}_\mu$ is the classical Hilbert operator $\mathcal{H}$ which is bounded on H^p if $1 < p < \infty$, but not on H^1. J. Cima has recently proved that $\mathcal{H}$ is an injective bounded operator from H^1 into the space $\mathscr{C}$ of Cauchy transforms of measures on the unit circle.

The operator $\mathcal{H}_\mu$ is known to be well defined on H^1 if and only if μ is a Carleson measure and in such a case we have that $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$. Furthermore, it is bounded from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

In this paper we prove that when μ is a 1-logarithmic 1-Carleson measure then $\mathcal{H}_\mu$ actually maps H^1 into the space of Dirichlet type $\mathcal{D}_0^1$. We discuss also the range of $\mathcal{H}_\mu$ on H^1 when μ is an α-logarithmic 1-Carleson measure $(0 < \alpha < 1)$. We study also the action of the operators $\mathcal{H}_\mu$ on Bergman spaces and on Dirichlet spaces.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ will be the unit circle. The space of all analytic functions in $\mathbb{D}$ will be denoted by $\mathcal{H}ol(\mathbb{D})$. We also let H^p $(0 < p \leq \infty)$ be the classical Hardy spaces. We refer to [11] for the notation and results regarding Hardy spaces.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}}(1-|z|^2)^\alpha |f(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

Here, dA stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi}\,r\,dr\,d\theta$. The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [12, 18, 29] for the notation and results about Bergman spaces.

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The space of Dirichlet type $\mathcal{D}^p_\alpha$ ($0 < p < \infty$ and $\alpha > -1$) consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A^p_\alpha$. In other words, a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to $\mathcal{D}^p_\alpha$ if and only if

$$\|f\|_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)| + \left((\alpha+1)\int_{\mathbb{D}}(1-|z|^2)^\alpha |f'(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

The Hilbert matrix is the infinite matrix $\mathcal{H} = \left(\frac{1}{k+n+1}\right)_{k,n\geq 0}$. It induces formally an operator, called the Hilbert operator, on spaces of analytic functions as follows:

If $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$, then we set

$$\mathcal{H}f(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \frac{a_k}{n+k+1}\right) z^n, \quad z \in \mathbb{D}, \tag{1}$$

whenever the right-hand side of (1) makes sense for all $z \in \mathbb{D}$ and the resulting function is analytic in $\mathbb{D}$. We define also

$$\mathcal{I}f(z) = \int_0^1 \frac{f(t)}{1-tz}\, dt, \quad z \in \mathbb{D}, \tag{2}$$

if the integrals in the right-hand side of (2) converge for all $z \in \mathbb{D}$ and the resulting function $\mathcal{I}f$ is analytic in $\mathbb{D}$. It is clear that the correspondences $f \mapsto \mathcal{H}f$ and $f \mapsto \mathcal{I}f$ are linear.

If $f \in H^1$, $f(z) = \sum_{n=0}^\infty a_n z^z$, then by the Fejér-Riesz inequality [11, Theorem 3. 13, p. 46] and Hardy's inequality [11, p. 48], we have

$$\int_0^1 |f(t)|\, dt \leq \pi \|f\|_{H^1} \quad \text{and} \quad \sum_{n=0}^\infty \frac{a_n}{n+1} \leq \pi \|f\|_{H^1}.$$

This immediately yields that if $f \in H^1$ then $\mathcal{H}f$ and $\mathcal{I}f$ are well defined analytic functions in $\mathbb{D}$ and that, furthermore, $\mathcal{H}f = \mathcal{I}f$.

Diamantopoulos and Siskakis [9] proved that $\mathcal{H}$ is a bounded operator from H^p into itself if $1 < p < \infty$, but this is not true for $p = 1$. In fact, they proved that $\mathcal{H}(H^1) \nsubseteq H^1$. Cima [6] has recently proved the following result.

Theorem A. (i) *The operator $\mathcal{H}$ maps H^1 into the space $\mathscr{C}$ of Cauchy transforms of measures on the unit circle $\partial\mathbb{D}$.*
(ii) *$\mathcal{H} : H^1 \to \mathscr{C}$ is injective.*

We recall that if σ is a finite complex Borel measure on $\partial\mathbb{D}$, the Cauchy transform $C\sigma$ is defined by

$$C\sigma(z) = \int_{\partial\mathbb{D}} \frac{d\sigma(\xi)}{1 - \overline{\xi}\, z}, \quad z \in \mathbb{D}.$$

We let $\mathscr{M}$ be the space of all finite complex Borel measure on $\partial\mathbb{D}$. It is a Banach space with the total variation norm. The space of Cauchy transforms is $\mathscr{C} = \{C\sigma : \sigma \in \mathscr{M}\}$. It is a Banach space with the norm $\|C\sigma\| \stackrel{\text{def}}{=} \inf\{\|\tau\| : C\tau = C\sigma\}$. We mention [7] as an excellent reference for the main results about Cauchy transforms. We let $\mathcal{A}$ denote the disc algebra, that is, the space of analytic functions in $\mathbb{D}$ with a continuous extension to the closed unit disc, endowed with the $\|\cdot\|_{H^\infty}$-norm. It turns out [7, Chapter 4] that $\mathcal{A}$ can be identified with the pre-dual of $\mathscr{C}$ via the pairing

$$\langle g, C\sigma\rangle \stackrel{\text{def}}{=} \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\overline{C\sigma(re^{i\theta})}\,d\theta. \tag{3}$$

This is the basic ingredient used by Cima to prove the inclusion $\mathcal{H}(H^1)\subset\mathscr{C}$.

Now we turn to consider a class of operators which are natural generalizations of the operators $\mathcal{H}$ and $\mathcal{I}$. If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\dots$, we let μ_n denote the moment of order n of μ , that is, $\mu_n = \int_{[0,1)} t^n\,d\mu(t)$, and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\ge 0}$ with entries $\mu_{n,k}=\mu_{n+k}$. The measure μ induces formally the operators $\mathcal{I}_\mu$ and $\mathcal{H}_\mu$ on spaces of analytic functions as follows:

$$\mathcal{I}_\mu f(z) = \int_{[0,1)}\frac{f(t)}{1-tz}\,d\mu(t),\quad \mathcal{H}_\mu f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_k\mu_{n+k}\right)z^n,\quad z\in\mathbb{D},$$

for $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D})$ being such that the terms on the right-hand sides make sense for all $z\in\mathbb{D}$, and the resulting functions are analytic in $\mathbb{D}$. If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix and the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are simply the operators $\mathcal{H}$ and $\mathcal{I}$.

If $I\subset\partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I)=\{re^{it}:\, e^{it}\in I,\quad 1-\frac{|I|}{2\pi}\le r<1\}$.

If $s>0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right)\le C|I|^s,\quad \text{for any interval } I\subset\partial\mathbb{D}.$$

A 1-Carleson measure will be simply called a Carleson measure. We recall that Carleson [4] proved that $H^p\subset L^p(d\mu)$ $(0<p<\infty)$ if and only if μ is a Carleson measure (see also [11, Chapter 9]).

For $0\le\alpha<\infty$ and $0<s<\infty$ we say that a positive Borel measure μ on $\mathbb{D}$ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log\frac{2\pi}{|I|}\right)^\alpha}{|I|^s}\le C,\quad \text{for any interval } I\subset\partial\mathbb{D}.$$

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\left(A\cap[0,1)\right),\quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right)\le C(1-t)^s,\quad 0\le t<1.$$

We have a similar statement for α-logarithmic s-Carleson measures.

The action of the operators $\mathcal{I}_\mu$ and $\mathcal{H}_\mu$ on distinct spaces of analytic functions have been studied in a number of articles (see, e. g., [2, 5, 14, 15, 16, 22, 25, 27]).

Combining results of [14] and of [16] we can state the following result.

Theorem B. *Let μ be a finite positive Borel measure on $[0,1)$.*

(i) *The operator $\mathcal{I}_\mu$ is well defined on H^1 if and only if μ is a Carleson measure.*
(ii) *If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is also well defined on H^1 and $\mathcal{I}_\mu f = \mathcal{H}_\mu f$ for all $f \in H^1$.*
(iii) *The operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*

Galanopoulos and Peláez [14, Theorem 2. 2] proved the following.

Theorem C. *Let μ be a positive Borel measure on $[0,1)$. If μ is a Carleson measure then $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$.*

This result is stronger than Theorem A(i). In view of these results, the following question arises naturally.

Question 1. Suppose that μ is a 1-logarithmic 1-Carleson measure on $[0,1)$. What can we say about the image $\mathcal{H}_\mu(H^1)$ of H^1 under the action of the operator $\mathcal{H}_\mu$?

To answer Question 1, let us start noticing that it is known that, for $0 < p \le 2$, the space of Dirichlet type $\mathcal{D}^p_{p-1}$ is continuously included in H^p (see [26, Lemma 1. 4]). In particular, the space $\mathcal{D}^1_0$ is continuously included in H^1. In fact, the estimates obtained by Vinogradov in the proof of his lemma easily yield the inequality

$$\|f\|_{H^1} \le 2\|f\|_{\mathcal{D}^1_0}, \quad f \in \mathcal{D}^1_0.$$

We shall prove that if μ is a 1-logarithmic 1-Carleson measure on $[0,1)$ then $\mathcal{H}_\mu(H^1)$ is contained in the space $\mathcal{D}^1_0$. Actually, we have the following stronger result.

Theorem 1. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) *μ is a 1-logarithmic 1-Carleson measure.*
(ii) *$\mathcal{H}_\mu$ is a bounded operator from H^1 into itself.*
(iii) *$\mathcal{H}_\mu$ is a bounded operator from H^1 into $\mathcal{D}^1_0$.*
(iv) *$\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^1_0$ into $\mathcal{D}^1_0$.*

There is a gap between Theorem C and Theorem 1 and so it is natural to discuss the range of H^1 under the action of $\mathcal{H}_\mu$ when μ is an α-logarithmic 1-Carleson measure with $0 < \alpha < 1$. We shall prove the following result.

Theorem 2. *Let μ be a positive Borel measure on $[0,1)$. Suppose that $0 < \alpha < 1$ and that μ is an α-logarithmic 1-Carleson measure. Then $\mathcal{H}_\mu$ maps H^1 into the space $\mathcal{D}^1(\log^{\alpha-1})$ defined as follows:*

$$\mathcal{D}^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)| \left(\log \frac{2}{1-|z|}\right)^{\alpha-1} dA(z) < \infty \right\}.$$

These results will be proved in Section 2. Since the space of Dirichlet type $\mathcal{D}^1_0$ has showed up in a natural way in our work, it seems natural to study the action of the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ on the Bergman spaces A^p_α and the Dirichlet spaces $\mathcal{D}^p_\alpha$ for general values of the parameters p and α. This will be done in Section 3.

Throughout this paper the letter C denotes a positive constant that may change from one step to the next. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2. Proofs of the theorems 1 and 2

Proof of Theorem 1. We already know that (i) and (ii) are equivalent by Theorem B.

To prove that (i) implies (iii) we shall use some results about the Bloch space. We recall that a function $f \in \mathcal{H}ol(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions will be denoted by $\mathcal{B}$. It is a non-separable Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ just defined. A classical source for the theory of Bloch functions is [1]. The closure of the polynomials in the Bloch norm is the *little Bloch space* $\mathcal{B}_0$ which consists of those $f \in \mathcal{H}ol(\mathbb{D})$ with the property that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that (see [1, p. 13])

$$|f(z)| \lesssim \|f\|_{\mathcal{B}} \log\frac{2}{1-|z|}. \tag{4}$$

The basic ingredient to prove that (i) implies (iii) is the fact that the dual $(\mathcal{B}_0)^*$ of the little Bloch space can be identified with the Bergman space A^1 via the integral pairing

$$\langle h, f\rangle = \int_{\mathbb{D}} h(z)\,\overline{f(z)}\,dA(z), \quad h \in \mathcal{B}_0,\ f \in A^1. \tag{5}$$

(See [29, Theorem 5. 15]).

Let us proceed to prove the implication (i) $\Rightarrow$ (iii). Assume that μ is a 1-logarithmic 1-Carleson measure and take $f \in H^1$. We have to show that $\mathcal{I}_\mu f \in \mathcal{D}^1_0$ or, equivalently, that $(\mathcal{I}_\mu f)' \in A^1$. Since $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm, it suffices to show that

$$\left|\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}}\|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{6}$$

So, let h be a polynomial. We have

$$\begin{aligned}
\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z) &= \int_{\mathbb{D}} h(z)\,\overline{\left(\int_{[0,1)}\frac{t\,f(t)}{(1-tz)^2}\,d\mu(t)\right)}\,dA(z)\\
&= \int_{\mathbb{D}} h(z)\int_{[0,1)}\frac{t\,\overline{f(t)}}{(1-t\,\overline{z})^2}\,d\mu(t)\,dA(z)\\
&= \int_{[0,1)} t\,\overline{f(t)}\int_{\mathbb{D}}\frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t).
\end{aligned}$$

Because of the reproducing property of the Bergman kernel [29, Proposition 4. 23], $\int_{\mathbb{D}} \frac{h(z)}{(1-t\,\overline{z})^2}\, dA(z) \,=\, h(t)$. Then it follows that

$$\int_{\mathbb{D}} h(z)\, \overline{(\mathcal{I}_\mu f)'\,(z)}\, dA(z) \,=\, \int_{[0,1)} t\, \overline{f(t)}\, h(t)\, d\mu(t). \tag{7}$$

Since μ is a 1-logarithmic 1-Carleson measure, the measure ν defined by

$$d\nu(t) \,=\, \log \frac{2}{1-t}\, d\mu(t)$$

is a Carleson measure [15, Proposition 2. 5]. This implies that

$$\int_{[0,1)} |f(t)| \log \frac{2}{1-t}\, d\mu(t) \lesssim \|f\|_{H^1}.$$

This and (4) yield

$$\int_{[0,1)} \left| t\, \overline{f(t)}\, h(t) \right|\, d\mu(t) \,\lesssim\, \|h\|_{\mathcal{B}} \|f\|_{H^1}.$$

Using this and (7), (6) follows.

Since $\mathcal{D}^1_0 \subset H^1$, the implication (iii) $\Rightarrow$ (iv) is trivial. To prove that (iv) implies (i) we shall use the following result of Pavlović [23, Theorem 3. 2].

Theorem D. *Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$, and suppose that the sequence $\{a_n\}$ is a decreasing sequence of non-negative real numbers. Then $f \in \mathcal{D}^1_0$ if and only if $\sum_{n=0}^\infty \frac{a_n}{n+1} < \infty$, and we have*

$$\|f\|_{\mathcal{D}^1_0} \asymp \sum_{n=0}^\infty \frac{a_n}{n+1}.$$

Now we turn to prove the implication (iv) $\Rightarrow$ (i). Assume that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^1_0$ into $\mathcal{D}^1_0$. We argue as in the proof of Theorem 1. 1 of [16]. For $\frac{1}{2} < b < 1$ set

$$f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.$$

We have $f_b'(z) = \frac{2b(1-b^2)}{(1-bz)^3}$ $(z \in \mathbb{D})$. Then, using Lemma 3. 10 of [29] with $t = 0$ and $c = 1$, we see that

$$\|f_b\|_{\mathcal{D}^1_0} \asymp \int_{\mathbb{D}} \frac{1-b^2}{|1-bz|^3}\, dA(z) \asymp 1.$$

Since $\mathcal{H}_\mu$ is bounded on $\mathcal{D}^1_0$, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}^1_0}. \tag{8}$$

We also have,

$$f_b(z) \,=\, \sum_{k=0}^\infty a_{k,b} z^k, \quad \text{with } a_{k,b} = (1-b^2)(k+1)b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty}\mu_{n+k}a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem D, (8), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 &\gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}^1_0} \gtrsim \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_{k,b}\right)\\
&= \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=0}^{\infty}a_{k,b}\int_{[0,1)}t^{n+k}\,d\mu(t)\right)\\
&\gtrsim (1-b^2)\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}kb^k\int_{[b,1)}t^{n+k}\,d\mu(t)\right)\\
&\gtrsim (1-b^2)\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}kb^{n+2k}\,\mu\left([b,1)\right)\right)\\
&= (1-b^2)\mu\left([b,1)\right)\sum_{n=1}^{\infty}\frac{b^n}{n}\left(\sum_{k=1}^{\infty}kb^{2k}\right)\\
&= (1-b^2)\mu\left([b,1)\right)\left(\log\frac{1}{1-b}\right)\frac{b^2}{(1-b^2)^2}.
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) = \mathrm{O}\left(\frac{1-b}{\log\frac{1}{1-b}}\right), \quad \text{as } b\to 1.$$

Hence, μ is a 1-logarithmic 1-Carleson measure. □

Before embarking on the proof of Theorem 2 we have to introduce some notation and results. Following [24], for $\alpha\in\mathbb{R}$ the weighted Bergman space $A^1(\log^\alpha)$ consists of those $f\in\mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^1(\log^\alpha)} \stackrel{\text{def}}{=} \int_{\mathbb{D}}|f(z)|\left(\log\frac{2}{1-|z|}\right)^{\alpha}dA(z)<\infty.$$

This is a Banach space with the norm $\|\cdot\|_{A^1(\log^\alpha)}$ just defined and the polynomials are dense in $A^1(\log^\alpha)$. Likewise, we define

$$\mathcal{D}^1(\log^\alpha)=\{f\in\mathcal{H}ol(\mathbb{D}): f'\in A^1(\log^\alpha)\}.$$

We define also the Bloch-type space $\mathcal{B}(\log^\alpha)$ as the space of those $f\in\mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}(\log^\alpha)} \stackrel{\text{def}}{=} |f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)^{-\alpha}|f'(z)|<\infty,$$

and

$$\mathcal{B}_0(\log^\alpha)=\left\{f\in\mathcal{H}ol(\mathbb{D}): |f'(z)|=\mathrm{o}\left(\frac{\left(\log\frac{2}{1-|z|}\right)^{\alpha}}{1-|z|}\right), \text{ as } |z|\to 1\right\}.$$

The space $\mathcal{B}(\log^\alpha)$ is a Banach space and $\mathcal{B}_0(\log^\alpha)$ is the closure of the polynomials in $\mathcal{B}(\log^\alpha)$.

We remark that the spaces $\mathcal{D}^1(\log^\alpha)$, $\mathcal{B}(\log^\alpha)$, and $\mathcal{B}_0(\log^\alpha)$ were called $\mathfrak{B}^1_{\log^\alpha}$, $\mathfrak{B}_{\log^\alpha}$, and $\mathfrak{b}_{\log^\alpha}$ in [24]. Pavlović identified in [24, Theorem 2. 4] the dual of the space $\mathcal{B}_0(\log^\alpha)$.

Theorem E. *Let* $\alpha\in\mathbb{R}$. *Then the dual of* $\mathcal{B}_0(\log^\alpha)$ *is* $A^1(\log^\alpha)$ *via the pairing*

$$\langle h, g\rangle = \int_{\mathbb{D}} f(z)\,\overline{g(z)}\,dA(z), \quad h\in\mathcal{B}_0(\log^\alpha),\ \ g\in A^1(\log^\alpha).$$

Actually, Pavlović formulated the duality theorem in another way but it is a simple exercise to show that his formulation is equivalent to this one which is better suited to our work.

Proof of Theorem 2. Let μ be a positive Borel measure on $[0,1)$ and $0<\alpha<1$. Suppose that μ is an α-logarithmic 1-Carleson measure. Take $f\in H^1$. We have to show that $\mathcal{I}_\mu f\in\mathcal{D}^1(\log^{\alpha-1})$ or, equivalently, that $(\mathcal{I}_\mu f)'\in A^1(\log^{\alpha-1})$. Bearing in mind Theorem E and the fact that $\mathcal{B}_0(\log^{\alpha-1})$ is the closure of the polynomials in $\mathcal{B}(\log^{\alpha-1})$, it suffices to show that

$$\left|\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})}\|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{9}$$

So, let h be a polynomial. Arguing as in the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 1 we obtain

$$\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z) = \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{10}$$

Now, it is clear that

$$|h(z)|\lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})}\left(\log\frac{2}{1-|z|}\right)^\alpha,$$

and then it follows that

$$\int_{[0,1)}\left|t\,\overline{f(t)}\,h(t)\right|\,d\mu(t)\lesssim\|h\|_{\mathcal{B}(\log^{\alpha-1})}\int_{[0,1)}|f(t)|\left(\log\frac{2}{1-t}\right)^\alpha\,d\mu(t).$$

Using the fact that the measure $\left(\log\frac{2}{1-t}\right)^\alpha\,d\mu(t)$ is a Carleson measure [15, Proposition 2. 5], this implies that

$$\int_{[0,1)}\left|t\,\overline{f(t)}\,h(t)\right|\,d\mu(t)\lesssim\|h\|_{\mathcal{B}(\log^{\alpha-1})}\|f\|_{H^1}.$$

This and (10) give (9). □

3. The operators $\mathcal{H}_\mu$ acting on Bergman spaces and on Dirichlet spaces

Jevtić and Karapetrović [20] have recently proved the following result.

Theorem F. *The Hilbert operator* $\mathcal{H}$ *is a bounded operator from* $\mathcal{D}^p_\alpha$ *into itself if and only if* $\max(-1,p-2)<\alpha<2p-2$.

Now, it is well known that $A^p_\alpha=\mathcal{D}^p_{\alpha+p}$ (see [29, Theorem 4. 28]). Hence, regarding Bergman spaces Theorem F says the following.

Corollary G. *The Hilbert operator* $\mathcal{H}$ *is a bounded operator from* A^p_α *into itself if and only if* $-1 < \alpha < p-2$.

Let us recall that Diamantopoulos [8] had proved before that the Hilbert operator is bounded on A^p for $p > 2$, but not on A^2. The situation on A^2 is even worse. Dostanić, Jevtić, and Vukotić [10] proved that the Hilbert operator is not well defined on A^2. Indeed, they considered the function f defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} z^n, \quad z \in \mathbb{D}, \tag{11}$$

which belongs to A^2. However, the series defining $\mathcal{H}f(0)$ is $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty$ and the integral defining $\mathcal{I}f(0)$ is $\int_0^1 f(t)\,dt = \infty$. Hence neither $\mathcal{H}$ nor $\mathcal{I}$ are defined on A^2.

This result can be extended. We can assert that $\mathcal{H}$ is not well defined on A^p_{p-2} for any $p > 1$. Indeed, let f be the function defined in (11). Notice that the sequence $\{\frac{1}{(n+1)\log(n+1)}\}$ is decreasing and that $\sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^p} < \infty$. Then (see Proposition 1 below) it follows that $f \in A^p_{p-2}$, and we have already seen that $\mathcal{H}f$ and $\mathcal{I}f$ are not defined. Since $\alpha \ge p-2 \Rightarrow A^p_{p-2} \subset A^p_\alpha$, it follows that the Hilbert operator $\mathcal{H}$ is not defined on A^p_α if $\alpha \ge p-2$.

In this section we shall obtain extensions of the mentioned results of Jevtić and Karapetrović considering the generalized Hilbert operators $\mathcal{H}_\mu$.

Theorem 3. *Suppose that* $\max(-1, p-2) < \alpha < 2p-2$ *and let* μ *be a finite positive Borel measure on* $[0,1)$. *If* μ *is a Carleson measure then the operators* $\mathcal{H}_\mu$ *and* $\mathcal{I}_\mu$ *are well defined on* $\mathcal{D}^p_\alpha$. *Furthermore,* $\mathcal{I}_\mu f = \mathcal{H}_\mu f$, *for all* $f \in \mathcal{D}^p_\alpha$.

When dealing with Bergman spaces Theorem 3 reduces to the following.

Corollary 1. *Suppose that* $p > 1$ *and* $-1 < \alpha < p-2$, *and let* μ *be a finite positive Borel measure on* $[0,1)$. *If* μ *is a Carleson measure then the operators* $\mathcal{H}_\mu$ *and* $\mathcal{I}_\mu$ *are well defined on* A^p_α. *Furthermore,* $\mathcal{I}_\mu f = \mathcal{H}_\mu f$, *for all* $f \in A^p_\alpha$.

Proof of Theorem 3. Suppose that μ is a Carleson measure and take $f \in \mathcal{D}^p_\alpha$. Set $\beta = \frac{2+\alpha}{p} - 1$. Observe that $0 < \beta < 1$. Using [29, Theorem 4.14], we see that $|f'(z)| \lesssim \frac{1}{(1-|z|)^{(2+\alpha)/p}}$ and, hence, $|f(z)| \lesssim \frac{1}{(1-|z|)^\beta}$. Then it follows that

$$\int_{[0,1)} |f(t)|\,d\mu(t) \lesssim \int_{[0,1)} \frac{d\mu(t)}{(1-t)^\beta}.$$

Integrating by parts, using that μ is a Carleson measure, and that $0 < \beta < 1$, we obtain

$$\begin{aligned} \int_{[0,1)} \frac{d\mu(t)}{(1-t)^\beta} &= \mu([0,1)) - \lim_{t\to 1} \frac{\mu([t,1))}{(1-t)^\beta} + \beta \int_0^1 \frac{\mu([t,1))}{(1-t)^{\beta+1}}\,dt \\ &= \mu([0,1)) + \beta \int_0^1 \frac{\mu([t,1))}{(1-t)^{\beta+1}}\,dt \\ &\lesssim \mu([0,1)) + \int_0^1 \frac{1}{(1-t)^\beta}\,dt \end{aligned}$$

$$< \infty.$$

Consequently, we obtain that

$$\int_{[0,1)} |f(t)|\,d\mu(t) < \infty. \tag{12}$$

Clearly, this implies that the integral

(13) $\displaystyle\int_{[0,1)} \frac{f(t)\,d\mu(t)}{1-tz}$ converges absolutely and uniformly on compact subsets of $\mathbb{D}$.

This gives that $\mathcal{I}_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that

$$\mathcal{I}_\mu f(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n\, f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. \tag{14}$$

Using [19, Theorem 2. 1] (see also [20, Theorem 2. 1]) we see that for these values of p and α we have that if $f \in A^p_\alpha$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} < \infty$. Now, since μ is a Carleson measure we have that $|\mu_n| \lesssim \frac{1}{n+1}$ ([5, Proposition 1]). Then it follows that

$$\sum_{k=0}^{\infty} |\mu_{n+k} a_k| \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+n+1} \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+1}, \quad \text{for all } n.$$

Clearly, this implies that $\mathcal{H}_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that $\int_{[0,1)} t^n\, f(t)\, d\mu(t) = \sum_{k=0}^{\infty} \mu_{n+k} a_k$ for all n. This and (13) give that $\mathcal{I}_\mu f = \mathcal{H}_\mu f$. □

Our next result is an extension of Corollary G

Theorem 4. *Suppose that $-1 < \alpha < p-2$ and let μ be a finite positive Borel measure on $[0,1)$.*

The operator $\mathcal{H}_\mu$ is well defined on A^p_α and it is a bounded operator from A^p_α to itself if and only if μ is a Carleson measure.

A number of results will be needed to prove this theorem. We start with a characterization of the functions $f \in \mathcal{H}ol(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing which belong to A^p_α.

Proposition 1. *Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ $(z \in \mathbb{D})$. Suppose that $1 < p < \infty$, $\alpha > -1$, and that the sequence $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative real numbers. Then*

$$f \in A^p_\alpha \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty.$$

Furthermore, $\|f\|^p_{A^p_\alpha} \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty$.

This result can be proved with arguments similar to those used in the proofs of [15, Theorem 3.10] and [23, Theorem 3. 1] where the analogous results for the Besov spaces $B^p = \mathcal{D}^p_{p-2}$ $(p > 1)$ and for the spaces $\mathcal{D}^p_{p-1}$ $(p > 1)$ were proved. The case $\alpha = 0$ is proved in [3, Proposition 2. 4]. Consequently, we omit the details.

The following lemma is a generalization of [13, Lemma 3 (ii)].

Lemma 1. *Let μ be a positive Borel measure on $[0,1)$ which is a Carleson measure. Assume that $0 < p < \infty$ and $\alpha > -1$. Then there exists a positive constant $C = C(p,\alpha,\mu)$ such that for any $f \in A^p_\alpha$*

$$\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha+1}\,d\mu(r) \,\le C\|f\|^p_{A^p_\alpha}.$$

Of course, $M_\infty(r,f) = \sup_{|z|=r}|f(z)|$.

Proof. Take $f \in A^p_\alpha$ and set

$$g(r) = M^p_\infty(r,f)(1-r)^{\alpha+1},\;\; F(r) = \mu([0,r)) - \mu([0,1)) \,=\, -\mu([r,1)),\;\; 0 < r < 1.$$

Integrating by parts, we have

$$\begin{aligned}(15)\qquad &\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha+1}\,d\mu(r) \,=\, \int_{[0,1)} g(r)\,d\mu(r)\\ &= \lim_{r\to 1} g(r)F(r) \,-\, g(0)F(0) \,-\int_0^1 g'(r)F(r)\,dr\\ &= |f(0)|^p\mu([0,1)) \,-\, \lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+1}\mu([r,1)) \,+\, \int_0^1 g'(r)\mu([r,1))\,dr.\end{aligned}$$

Since $f \in A^p_\alpha$ we have that $M^p_\infty(r,f) = \mathrm{o}\left((1-r)^{-2-\alpha}\right)$, as $r \to 1$ (see, e. g., [18, p. 54]). This and the fact that μ is a Carleson measure imply that

$$(16)\qquad \lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+1}\mu([r,1)) \,=\, 0.$$

Using again that μ is a Carleson measure and integrating by parts we see that

$$\begin{aligned}\int_0^1 g'(r)\mu([r,1))\,dr &\lesssim \int_0^1 g'(r)(1-r)\,dr\\ &= \lim_{r\to 1} g(r)(1-r) \,-\, g(0) \,+\, \int_0^1 g(r)\,dr\\ &\le \lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+2} \,+\, \int_0^1 M^p_\infty(r,f)(1-r)^{\alpha+1}\,dr\\ &= \int_0^1 M^p_\infty(r,f)(1-r)^{\alpha+1}\,dr.\end{aligned}$$

Then, using [13, Lemma 3. (ii)], it follows that

$$\int_0^1 g'(r)\mu([r,1))\,dr \,\lesssim\, \|f\|^p_{A^p_\alpha}.$$

Using this and (16) in (15) readily yields $\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha+1}\,d\mu(r) \,\lesssim\, \|f\|^p_{A^p_\alpha}$. $\square$

We shall also need the following characterization of the dual of the spaces A^q_β $(q > 1)$. It is a special case of [21, Theorem 2. 1].

Lemma 2. *If $1 < q < \infty$ and $\beta > -1$, then the dual of A^q_β can be identified with A^p_α where $\frac{1}{p} + \frac{1}{q} = 1$ and α is any number with $\alpha > -1$, under the pairing*

$$(17)\qquad \langle h, f\rangle_{A_{q,\beta,\alpha}} = \int_{\mathbb{D}} h(z)\overline{f(z)}(1-|z|^2)^{\frac{\beta}{q}+\frac{\alpha}{p}}\,dA(z), \quad h \in A^q_\beta, \quad f \in A^p_\alpha.$$

Finally, we recall the following result from [13, (5.2), p. 242] which is a version of the classical Hardy's inequality [17, pp. 244-245].

Lemma 3. *Suppose that* $k>0$, $q>1$, *and* h *is a non-negative function defined in* $(0,1)$, *then*

$$\int_0^1\left(\int_{1-r}^1 h(t)\,dt\right)^q(1-r)^{k-1}\,dr\le\left(\frac{q}{k}\right)^q\int_0^1(h(1-r))^q(1-r)^{q+k-1}\,dr.$$

Proof of Theorem 4. Suppose first that $\mathcal{H}_\mu$ is a bounded operator from A^p_α into itself. For $0<b<1$, set

$$f_b(z)=\frac{(1-b^2)^{1-\frac{\alpha}{p}}}{(1-bz)^{\frac{2}{p}+1}},\quad z\in\mathbb{D}.$$

Recall that $p-\alpha>2$. Then using [29, Lemma 3.10] with $t=\alpha$ and $c=p-\alpha$, we obtain

$$\|f_b\|^p_{A^p_\alpha}\;=\;(1-b^2)^{p-\alpha}\int_{\mathbb{D}}\frac{(1-|z|^2)^\alpha}{|1-bz|^{2+p}}\,dA(z)\;\asymp\;1.$$

Since $\mathcal{H}_\mu$ is bounded on A^p_α, this implies

$$1\gtrsim\|\mathcal{H}\mu(f_b)\|_{A^p_\alpha}. \tag{18}$$

We also have

$$f_b(z)=\sum_{k=0}^\infty a_{k,b}z^k,\ \ (z\in\mathbb{D}),\quad\text{with }\ a_{k,b}\asymp(1-b^2)^{1-\frac{\alpha}{p}}k^{\frac{2}{p}}b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^\infty\mu_{n+k}a_{k,b}\}_{n=0}^\infty$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Proposition 1, (18), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1\gtrsim\|\mathcal{H}_\mu(f_b)\|^p_{A^p_\alpha}&\gtrsim\sum_{n=1}^\infty n^{p-\alpha-3}\left(\sum_{k=1}^\infty\mu_{n+k}a_{k,b}\right)^p\\
&=\sum_{n=1}^\infty n^{p-\alpha-3}\left(\sum_{k=1}^\infty a_{k,b}\int_{[0,1)}t^{n+k}d\mu(t)\right)^p\\
&\gtrsim(1-b^2)^{p-\alpha}\sum_{n=1}^\infty n^{p-\alpha-3}\left(\sum_{k=1}^\infty k^{\frac{2}{p}}b^k\int_{[b,1)}t^{n+k}d\mu(t)\right)^p\\
&\ge(1-b^2)^{p-\alpha}\sum_{n=1}^\infty n^{p-\alpha-3}\left(\sum_{k=1}^\infty k^{\frac{2}{p}}b^{n+2k}\mu([b,1))\right)^p\\
&=(1-b^2)^{p-\alpha}\mu([b,1))^p\sum_{n=1}^\infty n^{p-\alpha-3}b^{np}\left(\sum_{k=1}^\infty k^{\frac{2}{p}}b^{2k}\right)^p\\
&\asymp(1-b^2)^{p-\alpha}\mu([b,1))^p\frac{1}{(1-b^2)^{2+p}}\sum_{n=1}^\infty n^{p-\alpha-3}b^{np}
\end{aligned}$$

$$\asymp (1-b^2)^{p-\alpha}\mu([b,1))^p \frac{1}{(1-b^2)^{2+p}} \cdot \frac{1}{(1-b^2)^{p-\alpha-2}}$$
$$\asymp \mu([b,1))^p \frac{1}{(1-b)^p}.$$

Then it follows that

$$\mu([b,1)) \,=\, \mathrm{O}(1-b), \quad \text{as } b \to 1,$$

and, hence, μ is a Carleson measure.

We turn to prove the other implication. So, suppose that μ is a Carleson measure and take $f \in A^p_\alpha$. Let q be defined by the relation $\frac{1}{p}+\frac{1}{q}=1$ and take $\beta = \frac{-\alpha q}{p} = \frac{-\alpha}{p-1}$. Observe that $\beta > -1$ and that with this election of β the weight in the pairing (17) is identically equal to 1. We have to show that $\mathcal{H}_\mu f \in A^p_\alpha$ which is equal to $\left(A^q_\beta\right)^*$ under the pairing $\langle\cdot,\cdot\rangle_{q,\beta,\alpha}$. So take $h \in A^q_\beta$.

$$\begin{aligned}
\langle h, \mathcal{H}_\mu f\rangle_{q,\beta,\alpha} &= \int_{\mathbb{D}} h(z)\,\overline{\mathcal{H}_\mu f(z)}\,dA(z) \\
&= \int_{[0,1)} \overline{f(t)} \left(\int_{\mathbb{D}} \frac{h(z)}{1-t\,\overline{z}}\,dA(z)\right) d\mu(t) \\
&= \int_{[0,1)} \overline{f(t)} \left(\int_0^1 \frac{r}{\pi}\int_0^{2\pi} \frac{h(re^{i\theta})}{1-tre^{-i\theta}}\,d\theta\,dr\right) d\mu(t) \\
&= \int_{[0,1)} \overline{f(t)} \left(\int_0^1 \left(\frac{r}{\pi i}\int_{|\xi|=1} \frac{h(r\xi)}{\xi - tr}\,d\xi\right) dr\right) d\mu(t) \\
&= 2\int_{[0,1)} \overline{f(t)} \left(\int_0^1 rh(r^2t)\,dr\right) d\mu(t).
\end{aligned}$$

Thus,

$$|\langle h, \mathcal{H}_\mu f\rangle_{q,\beta,\alpha}| \le 2\int_0^1 |f(t)|G(t)\,d\mu(t),$$

where $G(t) = \int_0^1 r|h(r^2t)|\,dr$. Using Hölder's inequality we obtain,

$$\begin{aligned}
&\int_{[0,1)} f(t)G(t)\,d\mu(t) = \int_{[0,1)} |f(t)|(1-t)^{\frac{\alpha+1}{p}} G(t)(1-t)^{-\frac{\alpha+1}{p}}\,d\mu(t) \\
&\le \left(\int_{[0,1)} |f(t)|^p(1-t)^{\alpha+1}\,d\mu(t)\right)^{1/p} \cdot \left(\int_{[0,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t)\right)^{1/q}.
\end{aligned}$$

Lemma 1 implies that

$$\left(\int_{[0,1)} |f(t)|^p(1-t)^{\alpha+1}\,d\mu(t)\right)^{1/p} \lesssim \|f\|_{A^p_\alpha}.$$

Next we will show that

$$\int_{[0,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t) \lesssim \|h\|^q_{A^q_\beta}. \tag{19}$$

This will give that

$$|\langle h, \mathcal{H}_\mu f\rangle_{q,\beta,\alpha}| \lesssim \|f\|_{A^p_\alpha} \cdot \|h\|^q_{A^q_\beta}.$$

By the duality theorem, this implies that $\mathcal{H}_\mu f \in A^p_\alpha$.

Let us prove (19). Observe first that if $0 < t < 1/2$ then $|h(r^2t)| \le M_\infty(\frac{1}{2}, h)$ for each $r \in (0,1)$, thus

$$G(t) = \int_0^1 |h(r^2t)| r\, dr \le M_\infty\left(\frac{1}{2}, h\right), \quad 0 < t < 1/2.$$

Clearly, this implies

$$\int_{[0,1/2)} G(t)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \lesssim M^q_\infty\left(\frac{1}{2}, h\right) \lesssim \|h\|^q_{A^q_\beta}. \tag{20}$$

Notice that $-\frac{q(\alpha+1)}{p} = \frac{p-2-\alpha}{p-1} - 1 > -1$. Making the change of variables $r^2t = s$, we obtain $\int_0^1 r|h(r^2t)|\, dr = \frac{1}{2t}\int_0^t |h(s)|\, ds$ and, hence,

$$\begin{aligned}
&\int_{[1/2,1)} G(t)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&= \int_{[1/2,1)} \left(\int_0^1 |h(r^2t)| r\, dr\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&= \int_{[1/2,1)} \frac{1}{(2t)^q} \left(\int_0^t |h(s)|\, ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&\le \int_{[1/2,1)} \left(\int_0^t M_\infty(s,h)\, ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&\le \int_{[0,1)} \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t)
\end{aligned} \tag{21}$$

Let us call $H(t) = \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}$ for $0 \le t < 1$. Integrating by parts we obtain the following

$$\int_{[0,1)} H(t)\, d\mu(t) = H(0)\mu([0,1)) - \lim_{t\to 1^-} H(t)\mu([t,1)) + \int_0^1 \mu([t,1))H'(t)\, dt. \tag{22}$$

The first term is equal to 0. Using the fact that μ is a Carleson measure we have that

$$\begin{aligned}
H(t)\mu([t,1)) &\lesssim (1-t)H(t) \\
&= \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds\right)^q (1-t)^{1-\frac{q(\alpha+1)}{p}} \\
&= \left(\int_0^t M_\infty(s,h)\, ds\right)^q (1-t)^{1-\frac{q(\alpha+1)}{p}}.
\end{aligned}$$

Since $h \in A^q_\beta$ we have $M_\infty(t,h) = o\left((1-t)^{-\frac{\beta+2}{q}}\right)$, as $t \to 1$. Then, bearing in mind that $\frac{\beta+2}{q} > 1$, it follows that

$$H(t)\mu([t,1)) \,=\, \mathrm{o}\left((1-t)^{-\beta-2+q} \cdot (1-t)^{1-\frac{q(\alpha+1)}{p}}\right) = \mathrm{o}(1), \quad \text{as } t \to 1. \tag{23}$$

Actually, we have also proved that

$$(1-t)H(t) \,=\, \mathrm{o}(1), \quad \text{as } t \to 1. \tag{24}$$

Using that μ is a Carleson measure, integrating by parts, and using the definition of H and (24), we obtain

$$\begin{aligned} \int_0^1 \mu([t,1))H'(t)\,dt &\lesssim \int_0^1 (1-t)H'(t)\,dt \\ &= \lim_{t\to 1}(1-t)H(t) \,-\, H(0) \,+\, \int_0^1 H(t)\,dt \\ &= \int_0^1 \left(\int_{1-t}^1 M_\infty(1-s,h)\,ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\,dt. \end{aligned} \tag{25}$$

Now, using Lemma 3 and [13, Lemma 3], we see that

$$\int_0^1 \left(\int_{1-t}^1 M_\infty(1-s,h)\,ds\right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\,dt \lesssim \int_0^1 M^q_\infty(t,h)(1-t)^{\alpha+1}\,dt \lesssim \|h\|^q_{A^q_\beta}.$$

Using this, (25), (23), (22), and (21), it follows that

$$\int_{[1/2,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t) \,\lesssim \|h\|^q_{A^q_\beta}.$$

This and (20) yield (19). □

Our final aim in this article is to find the analogue of Theorem 4 for Dirichlet spaces. In other words, we wish give an answer to the following question.

Question 2. If $\max(-1,p-2) < \alpha < 2p-2$, is it true that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^p_\alpha$ into itself if and only if μ is a Carleson measure?

Since $p-1 < \alpha < 2p-2$ implies that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$, Theorem 4 answers the question affirmatively for these values of p and α. It remains to consider the case $\max(-1,p-2) < \alpha \le p-1$. We shall prove the following result which gives a positive answer to Question 2 in the case $p > 1$.

Theorem 5. *Suppose that $p > 1$ and $p-2 < \alpha \le p-1$, and let μ be a finite positive Borel measure on $[0,1)$.*

The operator $\mathcal{H}_\mu$ is well defined on $\mathcal{D}^p_\alpha$ and it is a bounded operator from $\mathcal{D}^p_\alpha$ into itself if and only if μ is a Carleson measure.

The following two lemmas will be needed in the proof of Theorem 5. The first one follows trivially from Proposition 1.

Lemma 4. *Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{D})$. Suppose that $1 < p < \infty$ and $p-2 < \alpha \le p-1$, and that the sequence $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative real numbers. Then*

$$f \in \mathcal{D}^p_\alpha \ \Leftrightarrow \ \sum_{n=0}^{\infty} (n+1)^{2p-\alpha-3} a_n^p < \infty.$$

The following lemma is a generalization of [13, Lemma 4].

Lemma 5. *Let μ be a positive Borel measure on $[0,1)$ which is a Carleson measure. Assume that $0 < p < \infty$ and $\alpha > -1$. Then there exists a positive constant $C = C(p,\alpha,\mu)$ such that for any $f \in \mathcal{D}^p_\alpha$*

$$\int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\, d\mu(r) \le C\|f\|^p_{\mathcal{D}^p_\alpha}.$$

Proof. We argue as in the proof of Lemma 1. Take $f \in \mathcal{D}^p_\alpha$ and set

$$g(r) = M_\infty^p(r,f)(1-r)^{\alpha-p+1}, \ F(r) = \mu([0,r)) - \mu([0,1)) = -\mu([r,1)), \ 0 < r < 1.$$

Integrating by parts, we have

$$\begin{aligned}(26)\qquad \int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\, d\mu(r) &= \int_{[0,1)} g(r)\, d\mu(r)\\ &= \lim_{r\to1} g(r)F(r) - g(0)F(0) - \int_0^1 g'(r)F(r)\, dr\\ &= |f(0)|^p \mu([0,1)) - \lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\mu([r,1)) + \int_0^1 g'(r)\mu([r,1))\, dr.\end{aligned}$$

Since $f \in \mathcal{D}^p_\alpha$ we have that $M_\infty^p(r,f') = \mathrm{o}\left((1-r)^{-2-\alpha}\right)$, as $r \to 1$. Hence, $M_\infty^p(r,f) = \mathrm{o}\left((1-r)^{-2-\alpha+p}\right)$, as $r \to 1$. This and the fact that μ is a Carleson measure imply that

$$\lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\mu([r,1)) = 0. \tag{27}$$

Using again that μ is a Carleson measure and integrating by parts we see that

$$\begin{aligned}\int_0^1 g'(r)\mu([r,1))\, dr &\lesssim \int_0^1 g'(r)(1-r)\, dr\\ &= \lim_{r\to1} g(r)(1-r) - g(0) + \int_0^1 g(r)\, dr\\ &\le \lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+2} + \int_0^1 M_\infty^p(r,f)(1-r)^{\alpha-p+1}\, dr\\ &= \int_0^1 M_\infty^p(r,f)(1-r)^{\alpha-p+1}\, dr.\end{aligned}$$

Then, using [13, Lemma 3], it follows that

$$\int_0^1 g'(r)\mu([r,1))\, dr \lesssim \|f\|^p_{\mathcal{D}^p_\alpha}.$$

Using this and (27) in (26) readily yields $\int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\,d\mu(r) \lesssim \|f\|_{\mathcal{D}_\alpha^p}^p$.
□

Proof of Theorem 5. Suppose first that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}_\alpha^p$ into itself. For $1/2 < b < 1$ we set

$$f_b(z) = \frac{(1-b^2)^{1-\frac{\alpha}{p}}}{(1-bz)^{2/p}}, \quad z \in \mathbb{D}.$$

We have $\|f_b\|_{\mathcal{D}_\alpha^p} \asymp 1$. Then arguing as in the proof of the correspondent implication in Theorem 4 we obtain that μ is a Carleson measure. We omit the details.

To prove the other implication, suppose that μ is a Carleson measure and take $f \in \mathcal{D}_\alpha^p$. Since $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on $\mathcal{D}_\alpha^p$, we have to prove that $\mathcal{I}_\mu f \in \mathcal{D}_\alpha^p$ and that $\|\mathcal{I}_\mu f\|_{\mathcal{D}_\alpha^p} \lesssim \|f\|_{\mathcal{D}_\alpha^p}$ or, equivalently, that $(\mathcal{I}_\mu f)' \in A_\alpha^p$ and

$$\|\,(\mathcal{I}_\mu f)'\,\|_{A_\alpha^p} \lesssim \|f\|_{A_\alpha^p}. \tag{28}$$

We shall distinguish two cases.

First case: $\boldsymbol{\alpha < p-1}$. Let q be defined by the relation $\frac{1}{p}+\frac{1}{q}=1$ and take $\beta = \frac{-\alpha q}{p}$. In view of Lemma 2, (28) is equivalent to

$$\left|\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| \lesssim \|f\|_{\mathcal{D}_\alpha^p}\|h\|_{A_\beta^q}, \quad h \in A_\beta^q. \tag{29}$$

So, take $h \in A_\beta^q$. Just as in the proof of Theorem 1, we have

$$\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z) \;=\; \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{30}$$

Set $s = -1+\frac{\alpha+1}{p}$. Observe that $ps = \alpha-p+1$ and $-qs = \beta+1$. Then, using (30), Hölder's inequality, Lemma 1, and Lemma 5, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{D}} h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| &\le \int_{[0,1)} |f(t)|(1-t)^s\,|h(t)|(1-t)^{-s}\,d\mu(t)\\
&\le \left(\int_{\mathbb{D}} |f(t)|^p(1-t)^{\alpha-p+1}\,d\mu(t)\right)^{1/p}\left(\int_{[0,1)} |h(t)|^q(1-t)^{\beta+1}\,d\mu(t)\right)^{1/q}\\
&\le \left(\int_{\mathbb{D}} M_\infty^p(t,f)(1-t)^{\alpha-p+1}\,d\mu(t)\right)^{1/p}\left(\int_{[0,1)} M_\infty^q(t,h)(1-t)^{\beta+1}\,d\mu(t)\right)^{1/q}\\
&\le \|f\|_{\mathcal{D}_\alpha^p}\|h\|_{A_\beta^q}.
\end{aligned}$$

Thus, (29) holds.

Second case: $\boldsymbol{\alpha = p-1}$. We let again q be defined by the relation $\frac{1}{p}+\frac{1}{q}=1$ and take $\beta = q-1$. Using Lemma 2 and arguing as in the preceding case, we have to show that

$$\left|\int_{\mathbb{D}} (1-|z|^2)\,h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| \lesssim \|f\|_{\mathcal{D}_{p-1}^p}\|h\|_{A_{q-1}^q}, \quad h \in A_{q-1}^q. \tag{31}$$

We have

$$\int_{\mathbb{D}} (1-|z|^2)\,h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z) \;=\; \int_{[0,1)} t\,\overline{f(t)}\int_{\mathbb{D}} \frac{(1-|z|^2)h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t). \tag{32}$$

Now, $\int_{\mathbb{D}} \frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z) \,=\, h(t)$ and

$$\begin{aligned}\int_{\mathbb{D}} \frac{|z|^2\,h(z)}{(1-t\,\overline{z})^2}\,dA(z) \,&=\, \int_0^1 \frac{r^3}{\pi}\int_0^{2\pi} \frac{h(re^{i\theta})\,d\theta}{(1-tre^{-i\theta})^2}\,dr \\ = \int_0^1 \frac{2r^3}{2\pi i}\int_0^{2\pi} \frac{e^{i\theta}h(re^{i\theta})ie^{i\theta}\,d\theta}{(e^{i\theta}-tr)^2}\,dr \,&=\, \int_0^1 \frac{2r^3}{2\pi i}\int_{|z|=1} \frac{zh(rz)}{(z-tr)^2}\,dz\,dr \\ = \int_0^1 2r^3\left[h(r^2t)\,+\,r^2th'(r^2t)\right]\,dr.\end{aligned}$$

Then it is clear that $\left|\int_{\mathbb{D}} \frac{(1-|z|^2)\,h(z)}{(1-t\,\overline{z})^2}\,dA(z)\right| \,\lesssim\, M_\infty(t,h)$. Using this, (32), Hölder's inequality, Lemma 1, and Lemma 5, we obtain

$$\begin{aligned}\left|\int_{\mathbb{D}} (1-|z|^2)\,h(z)\,\overline{(\mathcal{I}_\mu f)'\,(z)}\,dA(z)\right| \,&\lesssim\, \int_{[0,1)} M_\infty(t,f)\,M_\infty(t,h)\,d\mu(t) \\ \le \left(\int_{[0,1)} M_\infty^p(t,f)\,d\mu(t)\right)^{1/p} &\left(\int_{[0,1)} M_\infty^q(t,h)\,d\mu(t)\right)^{1/q} \,\le\, \|f\|_{\mathcal{D}^p_{p-1}}\|h\|_{A^q_{q-1}}.\end{aligned}$$

This is (31). □

We shall close the article with some comments about the case $p=1$ in Question 2. We have the following result.

Theorem 6. *Let μ be a finite positive Borel measure on $[0,1)$ and $-1<\alpha<0$. If μ is a Carleson measure then the operator $\mathcal{H}_\mu$ is a bounded operator form $\mathcal{D}^1_\alpha$ to itself.*

Proof. Using [29, Theorem 5. 15, p. 113], we see that A^1_α can be identified as the dual of the little Bloch space under the pairing

$$\langle h,g\rangle \,=\, \int_{\mathbb{D}} (1-|z|^2)^\alpha\,h(z)\,\overline{g(z)}\,dA(z),\quad h\in\mathcal{B}_0,\;\; g\in A^1_\alpha. \tag{33}$$

Suppose that μ is a Carleson measure. Using this duality relation and the fact that $\mathcal{H}_\mu=\mathcal{I}_\mu$ on $\mathcal{D}^1_\alpha$, showing that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^1_\alpha$ to itself is equivalent to showing that

$$\left|\int_{\mathbb{D}} (1-|z|^2)^\alpha\,h(z)\,\overline{(\mathcal{I}_\mu f)'\,(z)}\,dA(z)\right| \,\lesssim\, \|h\|_{\mathcal{B}}\cdot\|f\|_{\mathcal{D}^1_\alpha},\quad h\in\mathcal{B}_0,\;\; f\in\mathcal{D}^1_\alpha. \tag{34}$$

Let us prove (34). Take $h\in\mathcal{B}_0$ and $f\in\mathcal{D}^1_\alpha$. We have

$$\int_{\mathbb{D}} (1-|z|^2)^\alpha\,h(z)\,\overline{(\mathcal{I}_\mu f)'\,(z)}\,dA(z) \,=\, \int_{[0,1)} t\,\overline{f(t)}\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t). \tag{35}$$

Using [29, Lemma 5. 14, pp. 113-114] we have that the operator T defined by

$$T\phi(\xi) \,=\, (1-|\xi|^2)^{-\alpha}\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha\phi(z)}{(1-\xi\,\overline{z})^2}\,dA(z)$$

is a bounded operator from $\mathcal{B}$ into $L^\infty(\mathbb{D})$. Then it follows that

$$\left|\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha h(z)}{(1-t\,\overline{z})^2}\,dA(z)\right| \,\lesssim\, \|h\|_{\mathcal{B}}(1-t^2)^\alpha,\quad t\in[0,1).$$

Using this in (35), we obtain

$$\left|\int_{\mathbb{D}}(1-|z|^2)^\alpha\, h(z)\,\overline{(\mathcal{I}_\mu f)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}}\int_{\mathbb{D}}(1-t)^\alpha|f(t)|\,d\mu(t). \tag{36}$$

The fact that μ is a Carleson measure readily implies that the measure ν defined by $d\nu(t)=(1-t)^\alpha\,d\mu(t)$ is a $(1-\alpha)$-Carleson measure. Using Theorem 1 of [28] we see that then ν is a Carleson measure for $\mathcal{D}^1_\alpha$, that is,

$$\int_{[0,1)}(1-t)^\alpha|g(t)|\,d\mu(t)\lesssim\|g\|_{\mathcal{D}^1_\alpha},\quad g\in\mathcal{D}^1_\alpha.$$

Using this in (36), (34) follows. □

We do not know whether the converse of Theorem 6 is true. This is due to the fact that we do not know whether Lemma 4 remains true for $p=1$. The inequality

$$\sum_{n=0}^{\infty}|a_n|(n+1)^{-(1+\alpha)}\lesssim\|f\|_{\mathcal{D}^1_\alpha}. \tag{37}$$

is certainly true with no assumption on the sequence $\{a_n\}$. Indeed, by Hardy's inequality [11, p. 48], $\sum_{n=1}^\infty|a_n|r^{n-1}\lesssim\int_0^{2\pi}|f'(re^{i\theta})|d\theta$. Hence

$$\begin{aligned}\|f\|_{\mathcal{D}^1_\alpha}&\asymp\int_0^1(1-r)^\alpha\int_0^{2\pi}|f'(re^{i\theta})|d\theta dr\\ &\gtrsim\sum_{n=1}^\infty|a_n|\int_0^1(1-r)^\alpha r^{n-1}\,dr=\sum_{n=1}^\infty|a_n|B(\alpha+1,n),\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. Stirling's formula gives $B(\alpha+1,n)\asymp n^{-(\alpha+1)}$ and then (37) follows.

However, the proof of Theorem D in [23] does not seen to work to prove the opposite inequality when $\{a_n\}$ is decreasing.

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