# Revisiting understanding in mathematics

Jesús Gallardo Romero, Universidad de Málaga, gallardoromero@telefonica.net
José Luis González Marí, Universidad de Málaga, gmari@uma.es
Verónica Aurora Quintanilla Batallanos, Universidad de Málaga, veronicaquintanilla@uma.es

**Abstract:** For decades, understanding has been considered as a basic theme of interest and a research object in Mathematics Education. In this theoretical overview paper we present a integrative framework for organizing the diversity of results that emerge from the different studies on mathematical understanding and its interpretation. The proposal is applied onto a representation of relevant literature that has arise in the area over the last two decades. With this overview we seek to provide an useful reference for: (a) advancing towards a better insight of understanding in mathematics, (b) establishing the specific limitations and open questions that demarcate the boundaries of understanding and interpretation in mathematics, and (c) orienting its future study using a shared base of consolidated knowledge.

**Keywords**: Understanding in mathematics; Interpretation; Hermeneutics; overview paper in mathematics education research

## Introduction

One of the main objectives in Mathematics Education is to guarantee that students have comprehensive learning. Over the past few years, increasing specialisation in the study of understanding in mathematics has encouraged the proliferation of different approaches, with specific theoretical frameworks and methods of assessment. These approaches are characterised by a high degree of precision, rigour and prudence in the problems dealt with, in the methods employed and in the results and conclusions obtained. At the same time, the growing specialisation has also generated a considerable diversification between the studies made, it being difficult at present identify consolidated approaches under which to deal, from the same perspective, with the variety of problems derived from the understanding of mathematics.

Furthermore, the available information comes across as heterogeneous and of a different nature. The contributions in the form of theoretical developments and empirical results, which are characteristic of the approaches that contemplate the study from a wide and deep viewpoint, share space with different complementary contributions from works in which the concern for understanding is secondary and its study superficial.

On the other hand, the recognition of teaching and learning with understanding as an fundamental purpose of Mathematics Education (Hiebert et al., 1997; NCTM, 2000) has been motivating the proliferation of initiatives whose main preoccupation lies on the development of the understanding on mathematics classroom. Such initiatives, however, may be affected by important difficulties linked to their foundations and functionality if they do not contemplate the development of learning as a problem included in that of understanding in its fullest extent (Sierpinska, 2000).

In our opinion, all these circumstances justify the pertinence of carrying out efforts in order to organise the field of knowledge around the understanding of mathematics and its interpretation by means of the configuring of concrete references with which:

- To place and relate the different existing issues and approaches (structuring of the current knowledge).

- To characterise those open questions of interest for research (establishment of boundaries and possible lines for progress).

The synthesis carried out in this theoretical paper aims to be a contribution in this sense. More specifically, on the basis of certain notable specific referents we have elaborated an organisational proposal for the advances in the study of understanding in Mathematics Education. The proposal is applied onto a representation of relevant antecedents that have arisen in the area over the last two decades. The work also positions our own contributions to the study on mathematics understanding.

## Dimensions of Mathematical Understanding

The main preoccupation with the development of mathematical understanding in students is part of a larger problem in which other dimensions intervene. In fact, one of the main causes why its study is such a complex task and such a conditioning element for the different research in course lies in its multidimensional character. In general terms, approaches to mathematical understanding admit some of the following dimensions, at least as a provisional reference to act as a starting point of their study:

- Origin and sources.
- Nature and functioning.
- Factors.
- Evolution.
- Effects.

By *origin* we mean the situations and circumstances that are responsible for the appearance of the understanding and by *sources* we are referring to the specific previous events that have generated such situations. For instance, in general constructivist terms, the origin of understanding is to be found in those situations of cognitive imbalance the individuals find themselves involved in during their interaction with the environment. In this context, the sources are to be found in the events that have generated such cognitive imbalances that force the individual to elaborate answers in accordance with the each particular situation (English and Halford, 1995). From this point of view, understanding appears within this space of experiences, cognitive imbalances, adaptive answers and the associated search for stability.

The dimensions *nature* and *functioning*, which are closely related, entail having to face complex questions on what understanding is and how it is produced. Since this is a construct that takes place within the individual's internal sphere, and cannot therefore be directly observed, such dimensions are usually studied on the basis of interpretive theoretical proposals of the established relationship between the person's mental states and his or her external conduct. One such proposal, and one with much currency, is to be found in the representational approach which develops a vision of understanding as being linked to internal representations and connections of mathematical knowledge. In this approach, understanding means to create internal representations of mathematics understanding which are connected to mental networks increasingly structured. These internal representations are linked to external representations used to communicate mathematical ideas and which are essentially conceived as objects (generally linguistic) that are in place of others (Goldin, 2002; Hiebert and Carpenter, 1992; Romero, 2000; Rico, 2009). The use of general typologies of understanding (Hiebert and Lefevre, 1986) and that of metaphorical references (Davis, 1992) are other classical strategies to be found in the study of such dimensions.

As for the *factors*, these are to be understood as those aspects conditioning understanding. The specificity of the object of understanding, the individual's general cognitive capacities, the personal assessment this individual carries out about the object itself or the characteristics of the environment are some of the recognised factors whereby understanding is affected (Godino, 2000; Sierpinska, 1994).

The study of the *evolution* is linked to the dynamic facet of understanding and entails recognising that knowledge is not acquired immediately and instantaneously but rather, that it is develops within the individual over time. Understanding is therefore not a static phenomenon, but it emerges, develops and evolves (Carpenter and Lehrer, 1999). Within this context, the Pirie-Kieren's dynamic theory on the growth of mathematical understanding (Kieren, Pirie, and Calvert, 1999; Pirie and Kieren, 1989, 1994) is among the most consolidated and influential within the study of this dimension. The hierarchical models of categories or levels applied with the purpose of capturing the dynamic processes of understanding also constitute another of the widely employed strategies in the research on evolution. One clear example of this latter option is to be found in the two axes process model developed by Koyama (1993, 1997, 2000).

Finally, the *effects* are associated to the results or products derived from the presence of a specific understanding in the individual. Adapted behaviours, the application of knowledge, the solving of problems or description of actions are usually considered to be observable effects. Among the non observable internal effects, mention should be made, as an example, the new cognitive and semantic structures resulting from a change in understanding. This dimension is reflected in approaches such as that of Duffin and Simpson (1997, 2000), which describes some of the internal and external effects (for example, feeling able to reconstruct what has been forgotten or deriving consequences, respectively) associated to the three components of their definition of understanding.

## Understanding and other Cognitive Notions

From a complementary perspective, the study of understanding and its relationship with other cognitive notions of similar complexity also constitutes another approach employed in mathematics Education. From this point of view, understanding shares relevance with other research subjects of interest in the area such as meaning, learning, mathematical thinking or competence, among others. This approach, which recognises understanding as necessarily linked to rest of cognitive configurations, defines an alternative access that extends the position centred on the specific analysis of the different dimensions.

It is possible to appreciate this integral vision of mathematical understanding in works such as those of Byers and Erlwanger (1985), where it is linked with learning and memory, or Bender (1996) when he assumes image and understanding as different but closely related modes of thought. Two recent contributions in this respect comes from Warner, Alcock, Coppolo, and Davis (2003), when studying the contribution of flexible mathematical thinking in the growth of understanding, and from Roth (2004), where a phenomenologically grounded approach to meaning and understanding is proposed in the context of graphs and graphing.

## Research on Understanding and its Contributions to Mathematics Education

Another organisational referent for the approaches to understanding in mathematics, complementary to those described above, is to be obtained attending to the possible consequences derived from them. The approaches to mathematical understanding have consequences in the form of:

- Didactic implications for the teaching of mathematics.
- Influence on other issues of interest for Mathematics Education.

On the one hand, the studies on understanding are usually accompanied by recommendations, proposals and initiatives of different types for promoting learning and understanding among students. On the other hand, the approaches contribute added references with which to improve the present situation of knowledge regarding other research areas of interest for mathematics education, organising, interpreting, explaining, solving or, if applicable, expanding the different existing problems. This is the case of Pirie-Kieren's recursive theory, which helps the educative practise giving an operative approach (mapping) in order to have a detailed record of the process of building/development of understanding, both in a particular student in front of different pieces of mathematical knowledge and several students in front of a specific knowledge (see, for example, Codes, Delgado, González, and Monterrubio, 2013). This approach has consequences in pedagogical realms as the initial training of teachers within the context of learning to teach mathematics. Its application offers a lens for examining growth in prospective teachers' understanding of mathematics and related strategies for teaching mathematics (Cavey and Berenson, 2005).

Pire-Kieren's theory also suggests finding problematic situations to infer indirectly the students' understanding from the observable actions made by them in their solving attempts. From our approach, we share this proposal and offer an operative procedure for the identification and organisation of useful mathematical situations for teaching (Gallardo and González, 2006). The didactic contribution consists of an establishment of reduced groups of relevant representative situations in order to be used in tasks for diagnosis and assessment, starting from phenomenological and epistemological analysis which can be applied to specific mathematical knowledge.

## Assessment and Understanding

Assessment is present in research of understanding in mathematics. The results stemming from the different routes of access and dimensions contemplated for its study find an important methodological requirement in the assessment. In general terms, approaches in Mathematics Education are usually conscious of this and it is frequent, amidst their theoretical configurations and ideas, to find references and basic assumptions shared about assessment such as the following:

- Its considerable complexity and the existence of limitations that are inherent to its nature.
- The different ways in which we can examine students' understanding in mathematics.
- The suitability of the observable manifestations as a means to obtain information on students' understanding.
- The influence of the specificity of mathematical knowledge in assessment.

Generic referents such as these serve as the base for the different approaches for developing their different assessment proposals in correspondence with those particular aspects of understanding that are at the centre of their interest, thus generating a variety of possibilities on the modes and terms with which to evaluate understanding and on the methods, techniques and instruments to be used. Among the contributions being made in this respect, the most relevant are those proposals that seek to assess understanding according to the representation and internal connections of mathematical knowledge. This approach is performed in terms of external connections which students establish when they face tasks where they have to relate different external representations of the mathematical knowledge (Barmby, Harries, Higgins, and Suggate, 2007; Hiebert and Carpenter, 1992; Romero, 2000). Others alternatives propose to assess understanding taking into account the overcoming of epistemological obstacles (Sierpinska, 1990, 1994) or according to the relations with pre-established institutional meanings (Godino and Batanero, 1994). Also worthy of note are the methods and techniques centred on the elaboration of understanding profiles (Pirie and Kieren, 1994) as well as the strategies and procedures of multifaceted assessment based on the analysis of mathematical knowledge, such as the semantic and structural analyses proposed by Niemi (1996), the analysis of the praxeological meanings of mathematical objects deriving from the onto-semiotic approach to mathematical cognition (Godino, 2002a, 2002b) or, more recently, our epistemological and phenomenological analysis of mathematical knowledge developed and applied in Gallardo and González (2006).

As summary, the Figure 1 synthesises with greater clarity the relation between the different aspects that intervene in the research on mathematical understanding according to the organisation of antecedents carried out.

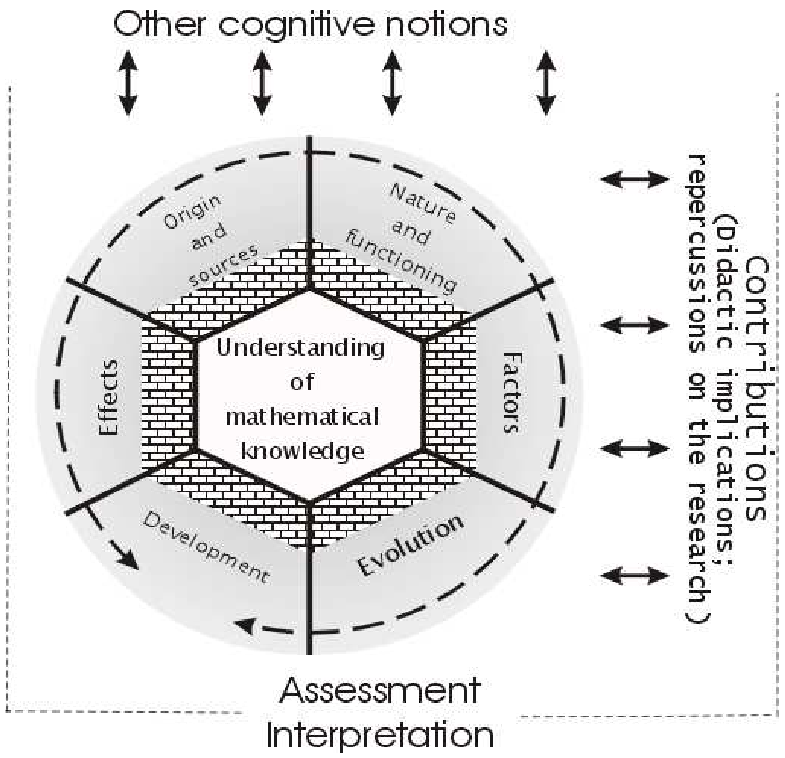


Figure 1. *Organisers for the research on mathematical understanding*

## Interpretation of Understanding in Mathematics

Additionally, the study of understanding is affected by the interpretative nature of assessment. In fact, we can recognize this character of assessment in most of earlier approaches to mathematical understanding. Any observation of students' mathematical activity carried out in order to obtain information on their understanding needs to be interpreted by the observer (Morgan and Watson, 2002). In this way, the basic objective of developing students' understanding is inextricably linked to the interpretation of their mathematical actions in the classroom. This allows us to place interpretation at the heart of the fundamental issues concerning the study of the understanding of mathematical knowledge.

Interpreting mathematical activity presents the constant challenge of finding more and more efficient methods to better grasp students' true understanding. The main operative difficulty lies in how to move from the student's mathematical activities and records to his or her understanding. This problem in turn leads to questions regarding various specific aspects of the interpretation, such as the nature of the mathematical problems and tasks used, the components constituting the scenario in which the interpretation will take place, the traces which reveal understanding on the basis of the recorded mathematical activity and the characterisation of the uses of mathematical knowledge and students' understanding on the basis of these traces.

In Mathematics Education, it is common for the different approaches to understanding to include among their general principles references on how to address interpreting. From a general perspective and with an integrative purpose we can identify three basic approaches in the analysis and processing of interpretation in mathematics.

***Cognitive Approach***

Influenced by the psychological tradition, this approach draws attention to the student's subjectivity and aims primarily to respond to certain internal complexities. It is usually reflected in those approximations which deal with understanding as their main object of study and which decide to address the analysis of some of its recognised dimensions. This approach is characterised by viewing mathematical understanding as a cognitive phenomenon and by recognising the possibility to access and capture it in the students' minds. The interpretation is therefore presented as a transfer towards the student's mental sphere, where mathematical understanding lies, via different manifestations which can be observed during mathematical problem solving. This is recognized by Duffin and Simpson (2000) when they affirmed that:

It suddenly became clear to us that it is only through interpreting the physical manifestations of a learner's use of their understanding that the teacher can make any kind of judgment about the learner's existing understanding. (p. 419)

In essence, in this approach, interpreting entails accessing internal cognitive aspects through the observation of sensitive, objectified realisations. The interpretation's objectivity is supported by the independence accorded by establishing and conserving the external productions in records or representations of various types, verbal and written. Because understanding is an activity which takes place within the individual's internal sphere and is therefore impossible to observe directly, interpreting it from this perspective requires theories on the recognised relation between the individual's mental state and his or her visible external behaviour (Koyama, 1993). The recurrent methodological process used in cognitive interpreting aims at progressively to reduce the distance between the internal and external realities. A clear example of this approach can be found in the aforementioned representational approach. The interpretive access to the mental environment of understanding turns out to be particularly direct in this approach as it presents the assessment according to the mental connections established between the various internal representations of mathematical knowledge (Rico, 2009). Understanding occurs when the student makes a mental model of the essential relations which characterize the mathematical knowledge (English and Halford, 1995), i. e. when the student enriches his/her internal knowledge networks (Romero, 2000).

***Semiotic Approach***

The recognised limits of cognitive interpretation justify presenting the semiotic approach as an alternative way of addressing the interpretation of understanding in mathematics. This option arises from some of the semiotic theories of mathematical knowledge and cognition recently developed in Mathematics Education. The semiotic approach as we derive it from these theories initially assumes a clear distance from the mental aspect of understanding:

Obviously, in this view interiorization or the like does not play a role since a goal of learning is not an internal mental construction but an external, observable activity with diagrams. [...] In a more extreme form: understanding is then not the grasps of abstract objects (based on appropriately constructed mental ones) but the socially accepted expedience with diagrammatic activities. (Dörfler, 2006, p. 109)

As an alternative, it presents understanding as a student's essential ability which is expressed in social practices and which can be publicly interpreted (Font, Godino, and D'Amore, 2007). In this approach, interpretation is circumscribed exclusively to visible mathematical activity and to the use made of the system of mathematical signs within this activity. Basically, interpreting entails transferring oneself into the semiotic environment created by these practices and observable mathematical productions, and even eliminating any reference to the external reality surrounding the semiotic results:

Neither the author nor the reader is the unique source of meaning because meaning is but the sign process itself. The reality of a text is its development, the meaning of a proposition lies in its consequences and the essence of a thing is the essence or meaning of a representation of that thing, and so forth. (Otte, 2006, p. 27)

The objectivity of this approach lies in the internal structure of the semiotic results to which the interpretive task is transferred. The method involved in this interpretation essentially draws on a structural analysis model used in linguistics and aims to capture the complexity of semiotic relations deployed in various mathematical activities observed and recorded in students. Examples can be found in the semiotic analysis included in the onto-semiotic approach to mathematical cognition and instruction (Godino, 2002a; Godino, Batanero, and Font, 2007) and in the peircean view of interpretation as a double semiotic process suggested by Sáenz-Ludlow and Zellweger (2012).

***Hermeneutic Approach***

In this approach the interpretation adopt a more central role in mathematical understanding. By seeing the assessment of mathematics being directed towards the student making sense of his mathematical activity we move in to the realm of interpretations (Brown, 1996). Influenced by moderate hermeneutics, the classroom interaction and processes are contemplated as an exchange of interpretations mediated by the social and cultural context (Ell, 2006). Therefore, the interpretation is considered as a necessary requirement in the identification and characterisation of understanding in mathematical activity instead of limiting or conditioning access to the understanding itself. In this view, the hermeneutic circle is showed as a basic method for interpreting. In essence, in mathematical activity both the teacher and the student are immersed in an open and reiterative process originated to reconcile the own mathematical experience that is happening with ways to describe it and with their prior expectations (Brown, 2001). Moreover, the basic model of the teacher that wants to obtain information on the student involved in a mathematical activity shares the complexity that is characteristic of hermeneutics situations conditioned by language. On this basis, the observable record generated during the mathematical activity and its 'textualization' (mathematical answers written by the student, dialogue transcripts, videotaped actions and so on) is the main depositary source of the visible expression of understanding. However, in the hermeneutic approach although understanding and its interpretation are based on a text, they go beyond that the purely semiotic analysis:

If then the production of any mathematical expression can be seen as an action, the meaning of such an expression is necessarily subject to an interpretation that transcends any meaning in the expression itself. This necessitates looking at how the expression is being used by the individual in a particular context. […] the meaning of any mathematical action goes beyond that which would be found in a purely literal or symbolic investigation. (Brown, 2001, p. 26)

The ability to use mathematical knowledge depends in large part on understanding (one cannot use something one does not possess). This means that the ultimate reference of student's understanding is not only in the written record (sign or text), but in external references as the evident use of mathematical knowledge. An example of this hermeneutic approach can be found in our operative model for interpreting understanding in mathematics (Gallardo, González, and Quispe, 2008a, 2008b; Gallardo, González, and Quintanilla, 2013), which addresses aspects such as those pointed out and will be further described in the last section of this paper.

## Boundaries in Research on Understanding and Interpretation

The results given by the different researches carried out in Mathematics Education have accumulatively created a growing body of confirmed and consolidated knowledge regarding the different aspects linked to mathematical understanding and its interpretation. This progress, however, contrasts with important limitations for which present research has yet to find definitive solutions. More specifically, some boundaries that demarcate the study of understanding and interpretation in mathematics would basically stem from:

*(a) Open questions inherent to each particular dimension of understanding.* Such is the case, among others, of the problem of the existence of limits in the acquisition of understanding or of the encapsulation of its dynamism, present in the study of the evolution. It is also the case of the difficulty entailed by what is impossible to directly observe the internal nature and functioning of the understanding.

*(b) The controversy about the degree of depth and extension that should be demanded from the study of mathematical understanding*. To admit the development of understanding as a purpose of Mathematics Education generates, for the research, the basic issue of clarifying the knowledge that is needed for undertaking this task with guarantees, fulfilling the interests of the area in consensus with the scientific community.

*(c) Limitations of each approach to interpretation of understanding.* For example, the main operative difficulties affecting the cognitive approach are related with the transition from external understanding environments to internal ones along with the mental characteristics of understanding itself; also with the ontological problem of the representations (Font, Godino, and D’Amore, 2007). Moreover, the potential limits of the semiotic approach to interpretation lie in the problematic relation between oral and written signs as well as in the elimination of external references upon which semiotic records are projected. Finally, the hermeneutic approach, searching the mathematical understanding in a reference outside of the language that describes it, is affected by the ontological question of the existence of mathematical objects (Font, Godino, and Gallardo, 2013).

*(d) The question of the most appropriate interpretation*. Understanding in mathematics gives rise to a limited field of potential interpretations where a confrontation of alternatives and the justified support of certain options to the detriment of others is always a possibility. In this respect, Tahta (1996) recognises the legitimacy and potential of each interpretative approach and he proposes the use of alternatives interpretations, even where they may seem to be contradictory, judging them not for some supposed veracity but in terms of their fruitfulness. For example, one might think that some approaches are preferable to others for their didactic consequences to develop understanding of mathematical knowledge. In order to guarantee their utility and effectiveness in Mathematics Education, it is interesting that such approaches should show a clear descriptive and prescriptive potential (Koyama, 1993).

*(e) The cognitive-semiotic-hermeneutic trichotomy and its methodological dilemma.* When addressing the interpretation of understanding in mathematics, should we assume that the cognitive, semiotic, and hermeneutic approaches (even in their 'weakest' versions) are the poles of a relation of exclusion which imposes upon us a necessary choice between either positions? Or, on the contrary, could we establish dialectical links between them, allowing us to then overcome, or at least reduce, their differences? In connection with the above discussion, to choose the integration of approaches instead of the confrontation and selection of alternatives is another way to face the search of interpretations increasingly more appropriate. For example, among the integrative contributions that provide some light to this dilemma we find the cognitive analysis of mathematical activity proposed by Duval (2006), where it makes it necessary to consider semiotic representations at the level of mind's structure (cognitive-semiotic connection). In our operative model for interpreting understanding in mathematics

(Gallardo, González, and Quispe, 2008a, 2008b; Gallardo, González, and Quintanilla, 2013) the strategy to address the relationship between the three approaches consists of introducing an extended view of interpretation, where the three approaches intervene in different phases of the same interpretive proposal, complement each other and therefore demonstrate solidarity. In concrete, the proposal begins on the cognitive level by recognising that mathematical understanding is a mental phenomenon, then moves onto the semiotic level by analysing the student's mathematical activity diffused throughout the written record, and finally it moves beyond these levels onto a phenomenon-epistemological level which allows us to come back to the student's understanding through his or her uses of mathematical knowledge (cognitive-semiotic-hermeneutic connection).

*(f) The consideration in Mathematics Education given to the attainments and developments about understanding and interpretation achieved in other knowledge areas.* In close connection to point (b), turning to other knowledge areas allows us to assess better the contribution which the knowledge generated in those makes about the specific-research problems covered in Mathematics Education. In fact, some of the results achieved in our area could be observed as indicators of the other research fields influence on the particular aspects studied in Mathematics Education. Thereon, we consider that the field of knowledge on the understanding of mathematics and its interpretation could be extended and consolidated if the links to other areas should be more systematically explored. We observe examples of these connections in the view of interpretation from a peircean perspective of classroom mathematical activity proposed by Sáenz-Ludlow and Zellweger (2012). Also in the contribution of the contemporary hermeneutic philosophy to clarify the cognitive-semiotic-hermeneutic dilemma of the

interpretation of understanding in mathematics (Gallardo, González, and Quispe, 2008b).

## Concluding Remarks

The generic model based on the multifaceted nature of understanding makes it possible to establish a framework of reference with which to organise the diversity of results that emerge from the different studies carried out on understanding in Mathematics Education, while also making it possible to identify, from the components analysed therein, its main purposes when facing the issue of understanding. Likewise, the resulting organisational structure comes across as useful for establishing the specific limitations and issues raised that demarcate the frontiers of the study of mathematical understanding.

The brief exposition developed reveals the complexity facing the researchers in Mathematics Education when dealing with the mathematical understanding. The description made makes it possible to notice a varied panorama in the research with works made according to different approaches, dealing with partial issues of various kinds and establishing non-common objectives on a short-term basis. The variety and extension of the achievements made within this specific area make it recommendable to put integrating efforts into effect and, in this respect, we consider that the elaboration of organisational efforts such as that outlined here opens up a via for facilitating progress towards a better insight of mathematical understanding and for orienting the development of its future study using the starting point of a shared base of consolidated knowledge.

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