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Approaching the square of opposition in terms of the f-indexes of inclusion and contradiction

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ABSTRACT

We continue our research line on the analysis of the properties of the f-indexes of inclusion and contradiction; in this paper, specifically, we show that both notions can be related by means of the, conveniently reformulated, Aristotelian square of opposition. We firstly show that the extreme cases of the f-indexes of inclusion and contradiction coincide with the vertexes of the Aristotelian square of opposition in the crisp case; then, we allocate the rest of f-indexes in the diagonals of the extreme cases and we prove that the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations also hold between the f-indexes of inclusion and contradiction.

1. Introduction

Aristotle's Square of Opposition [21] is a diagrammatic representation of the logical relationships (contradiction, contrariety, subcontrariety, subalternation, superalternation) between statements that assert or deny something about a subject based on its relationship with a predicate. Both the diagram and the theory have been discussed throughout the history of logic, which spans over two millennia. The strength of the theory is that it is at the same time fairly simple and quite rich. Initially, the diagram was employed to present the Aristotelian theory of quantification, but extensions and criticisms of this theory have resulted in various other diagrams [2,10,19].

Logicians such as Boole and De Morgan made contributions to its formalization within algebraic and symbolic logic, more recently the square continues to be an important tool in logic, philosophy, and linguistics, and has found applications in areas such as psychology or computer science. The theory of oppositions has become a topic of intense interest due to the development of a general geometry of opposition (polygons and polyhedra) with many applications [2,3].

The square of opposition admits interpretations in terms of several logic frameworks, from classical logic (both propositional and first-order) to various modal logic approaches, probabilistic logic, and logic of rational agency. Several gradual generalizations have been recently introduced: in [10], different opposition structures (squares and hexagons) are studied both from a classical and a fuzzy standpoint; in [12] the square is in fact a 4-tuple of truth-degrees which should fulfill certain relations among them in order to properly generalize the different notions of opposition; in [19] graded structures of opposition are used in fuzzy natural logic; and in [20] a thorough analysis is done in terms of different intermediate quantifiers.

Our focus is on an interpretation of the square of opposition in terms of the *f-index of inclusion* [17,18] and *f-index of contradiction* [8]. Both notions propose the use of certain *mappings* in the unit interval (called f-indexes) that lead to a functional approach

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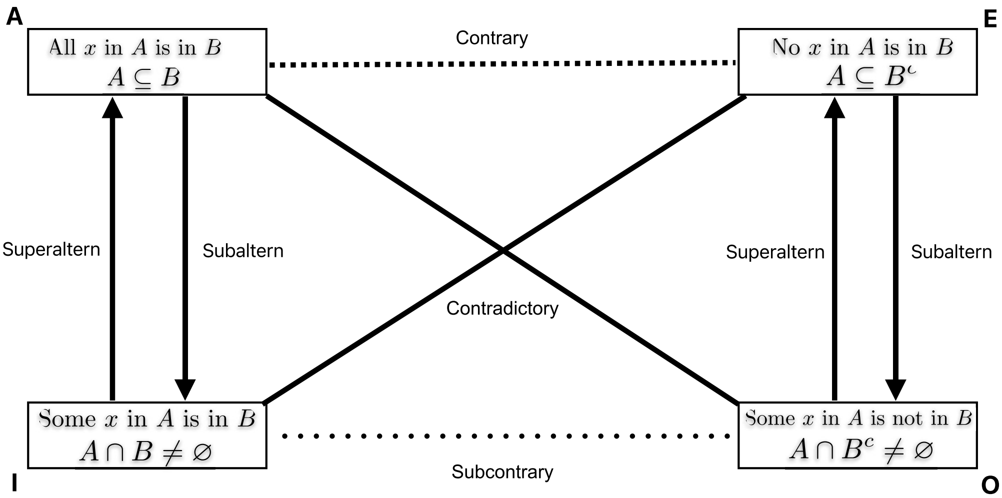

Fig. 1. Classical Aristotelian square of opposition.

to grade inclusion and contradiction between fuzzy sets. Despite of their functional nature, it has been shown that they are in consonance with the standard approaches dealing to measure inclusion and contradiction in the fuzzy framework [16]. Actually, the f-index of inclusion may be considered as a kind of Sinha-Dougherty inclusion [23].

Let us recall the classical Aristotelian square of opposition in Fig. 1. The square of opposition relates those four vertexes by means of five relations, which are represented as arrows in the diagram above and are described as follows:

- *Contradiction*: two statements are contradictory if they cannot be simultaneously *true* nor simultaneously *false*, that is, one must be *true* and the other *false*.
- *Contrariety*: two statements are contrary if they both cannot be *true* simultaneously.
- *Subcontrariety*: two statements are subcontrary if they both cannot be *false* simultaneously.
- *Subalternation*: if a statement is *true* then, its subaltern must be *true* as well.
- *Superalternation*: if a statement is *false* then, its superaltern must be *false* as well.

In this paper, we identify the vertex A as the greatest degree of f-inclusion; the vertex O as the least degree of f-inclusion (there is no inclusion at all); the vertex E as the greatest degree of f-contradiction; and, the vertex I as the least degree of f-contradiction (no contradiction at all). This interpretation collapses to the standard one when the predicates involved are crisp. Then, in the general case, we firstly allocate the rest of f-indexes of inclusion in the segment AO, the rest of f-indexes of contradiction in the segment EI and prove that Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations (conveniently reformulated) also hold between f-indexes of inclusion and f-indexes of contradiction. Consequently, we can use the square of opposition to infer information about one f-index from the knowledge of the other f-index.

The paper is structured as follows: in Section 2 we recall some basic notions of fuzzy set theory, the notion of f-inclusion and the notion of f-weak-contradiction. In Section 3 we introduce some properties of the f-indexes of inclusion and of contradiction and, then, provide a first approach towards an interpretation of the square of opposition. Finally, in Section 6 we present conclusions and future works.

2. Preliminary definitions

Let us recall that a *fuzzy set* A is defined on a referential universe $\mathcal{U}$ (usually omitted) by means of its membership function $A: \mathcal{U} \to [0,1]$. The standard operations *union* and *intersection* between sets can be extended to operations between fuzzy sets as follows: given two fuzzy sets A and B, we define the fuzzy sets $A \cup B$ and $A \cap B$ as $(A \cup B)(u) = \max\{A(u), B(u)\}$ and $(A \cap B)(u) = \min\{A(u), B(u)\}$ for all $u \in \mathcal{U}$, respectively. To define the *complement* of a fuzzy set we need to consider negation operators. Let us recall that a *negation operator* is a decreasing mapping $n: [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$. Given a fixed negation operator n, the complement of a fuzzy set A is defined as $A^c(u) = n(A(u))$ for all $u \in \mathcal{U}$. We say that a negation n is *involutive* if $n(n(x)) = x$ for all $x \in [0,1]$. Note that in the case of considering an involutive negation for the definition of the complement, the double complement law holds; i.e., $(A^c)^c = A$.

2.1. Functional degrees of inclusion and contradiction

The main difference of our approach with respect to other approaches dealing with measures of inclusion between fuzzy sets [15, 26,25,23], is that *the degrees used to express the measure are no longer real values, but certain mappings* from $[0,1]$ to $[0,1]$. These mappings should satisfy the properties of deflation and monotonicity in order to be considered indexes of inclusion.

Definition 1. The set of *f-indexes of inclusion* (denoted by Ω) is the set of mappings $f : [0,1] \to [0,1]$ satisfying the following properties for all $x, y \in [0,1]$:

- $f(x) \le x$;
- if $x \le y$ then $f(x) \le f(y)$

The definition of f-inclusion is given as follows.

Definition 2 *([18]).* Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that *A is f-included in B* (denoted by $A \subseteq_f B$) if and only if the inequality $f(A(u)) \le B(u)$ holds for all $u \in \mathcal{U}$.

It is worth remarking that, different mappings in Ω define different relations of f-inclusion that determine stronger or weaker restrictions.

Ω has a lattice structure with the usual pointwise ordering between mappings. We will use id and $\bot$ to refer to the greatest and lowest mappings in Ω; i.e., $id(x) = x$ and $\bot(x) = 0$ for all $x \in [0,1]$. Note that id and $\bot$ represent the strongest and the weakest degrees of f-inclusion.

Obviously, the greater the mapping $f \in \Omega$ the stronger the restriction imposed by the f-inclusion; since given $A \subseteq_f B$, the value $f(A(u))$ determines a lower bound of the possible values of $B(u)$, for all $u \in \mathcal{U}$. In this way, the degree of inclusion of fuzzy set A into fuzzy set B can be defined by choosing the greatest $f \in \Omega$ such that A is f-included in B.

Definition 3 *(f-index of inclusion [17]).* Let A and B be two fuzzy sets, the *f-index of inclusion* of A in B, denoted by $Inc(A,B)$, is defined by

$$Inc(A,B) = \max\{ f \in \Omega \mid A \subseteq_f B \}\,.$$

For more details on the f-index of inclusion we refer the reader to [17], where the following analytical expression was obtained:

$$Inc(A,B)(x) = \min\{ f_{A,B}(x), x\} \quad \text{where} \quad f_{A,B}(x) = \inf_{u \in \mathcal{U}} \{ B(u) \mid x \le A(u)\}\,. \tag{1}$$

The following result summarizes some properties of the f-index of inclusion defined above that resemble some axiomatic approaches of measures of inclusion given in the literature.

Theorem 1 *([15,23]). Let A, B and C be three fuzzy sets, then*

1. *If A and B are two crisp sets, then either $Inc(A,B) = id$ or $Inc(A,B) = \bot$;*
2. *$Inc(A,B) = id$ if and only if $A(u) \le B(u)$ for all $u \in \mathcal{U}$;*
3. *If $\mathcal{U}$ is finite, $Inc(A,B) = \bot$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$;*
4. *$Inc(B,C) \circ Inc(A,B) \le Inc(A,C)$[1];*
5. *If $B(u) \le C(u)$ for all $u \in \mathcal{U}$ then,*
   (a) *$Inc(A,B) \le Inc(A,C)$ and*
   (b) *$Inc(C,A) \le Inc(B,A)$;*
6. *$Inc(A, B \cap C) = Inc(A,B) \wedge Inc(A,C)$;*
7. *$Inc(A \cup B, C) = Inc(A,C) \wedge Inc(B,C)$.*

2.2. The notion of f-weak contradiction

As with f-inclusion, the underlying idea here is to measure contradiction not with real values, but with certain mappings from $[0,1]$ to $[0,1]$. In this case, those mappings used as degrees of contradiction should satisfy some properties that resemble negation operators.

Definition 4. The set of *f-indexes of contradiction* (denoted by $\overline{\Omega}$) is the set of mappings $f : [0,1] \to [0,1]$ satisfying the following properties for all $x, y \in [0,1]$:

- $f(0) = 1$;
- if $x \le y$ then $f(y) \le f(x)$

The notion of f-weak-contradiction generalizes that of N-contradiction given by Trillas, Alsina and Jacas [24].

[1] The operation $\circ$ is the composition of functions.

Definition 5 *([7,8])*. Let A and B be two fuzzy sets defined over a nonempty universe $\mathcal{U}$ and let $f \in \overline{\Omega}$. We say that A *is f-weak-contradictory w.r.t. B* if and only if $A(u) \leq f(B(u))$ holds for all $u \in \mathcal{U}$.

Note that $\overline{\Omega}$ has a structure of complete lattice with the usual pointwise ordering. We denote by $f_\top$ and $f_\perp$ the greatest and lowest mapping in $\overline{\Omega}$; i.e., the weakest and the strongest degrees of f-weak-contradiction.

Below we summarize some properties of the notion of f-weak-contradiction which motivates the consideration of $\overline{\Omega}$ as a proper set of degrees of contradiction.

Theorem 2 *([8]). Let A, B and C be three fuzzy sets and let $f, g \in \overline{\Omega}$. Then,*

1. *A is $f_\perp$-weak-contradictory w.r.t. B if and only if $B(u) > 0$ implies $A(u) = 0$ for all $u \in \mathcal{U}$;*
2. *If $A \leq B$ and C is f-weak-contradictory w.r.t. B then, C is f-weak-contradictory w.r.t. A;*
3. *If $A \leq C$ and C is f-weak-contradictory w.r.t. B then, A is f-weak-contradictory w.r.t. B;*
4. *If $\mathcal{U}$ is a finite universe, $f_\top$-weak-contradiction is the only f-weak-contradiction of A w.r.t. B if and only if there exists $u \in \mathcal{U}$ such that $A(u) = B(u) = 1$;*

Given two fuzzy sets A and B, a degree of contradiction between A and B can be defined by choosing the least $f \in \overline{\Omega}$ such that A is f-weak-contradictory w.r.t. B.

Definition 6 *([8])*. Let A and B be two fuzzy sets, the *f-index of contradiction of A w.r.t. B*, denoted by $Con(A, B)$, is given by the least mapping $f \in \overline{\Omega}$ verifying that A is f-weak-contradictory w.r.t. B.

The proof that the f-index of contradiction is well-defined can be found in [8], together with the following analytic expression for $Con(A, B)(x)$:

$$Con(A,B)(x) = \begin{cases} 1 & \text{if } x = 0 \\ \sup\limits_{u\in\mathcal{U}} \{ A(u) \mid x \leq B(u) \} & \text{otherwise.} \end{cases}$$

The similarities between the f-indexes of inclusion and the f-index of contradiction are now obvious. The following result is a direct consequence of Theorem 2 and characterizes the two extreme degrees of contradiction, namely $f_\perp$ and $f_\top$ that represents the greatest and lowest degree of contradiction, respectively.

Corollary 1. *Let A and B be two fuzzy sets, then,*

1. *$Con(A, B) = f_\perp$ if and only if $B(u) > 0$ implies $A(u) = 0$ for all $u \in \mathcal{U}$.*
2. *If $\mathcal{U}$ is finite, $Con(A, B) = f_\top$ if and only if there is $u \in \mathcal{U}$ such that $A(u) = B(u) = 1$.*

At this point, it is worth noting that the pointwise ordering between the different f-indexes of contradiction classifies the f-indexes inversely to the strength of the contradiction that they represent. That is, the smaller the function $Con(A, B)$, the stronger the contradiction of A w.r.t. B. For more details we refer the reader to [8]

3. f-indexes of inclusion/contradiction in the square of opposition

3.1. Towards a fuzzy interpretation of the square of opposition based on f-indexes

Since we are working in a fuzzy environment, let us rewrite firstly the four vertexes in terms of fuzzy logic. For this purpose, we assume an underlying IMTL algebra, that is, a prelinear residuated lattice $(L, \inf, \sup, *, \to, 1, 0)$ together with an involutive negation n (see [14,13]).

- The vertex **A**, or Universal Affirmative, is given by the statement *"All A is B"*. On the one hand, if A and B are (fuzzy) sets, then such a relationship can be identified with the inclusion. On the other hand, the Universal Affirmative statement in first-order logic is given by the well-formed formula $\forall u(A(u) \Rightarrow B(u))$, whose semantics (in fuzzy logic) is usually given by the standard measure of inclusion defined via the adjoint pair $(*, \to)$; i.e., by $\inf_{u\in\mathcal{U}}\{A(u) \to B(u)\}$ where $\mathcal{U}$ in this later case denotes the set of possible instances for A and B.
- The vertex **E**, or Universal Negative, is represented by the statement *"No A is B"*. As in the case of the Universal Affirmative, we can represent this vertex as the inclusion of A in the complementary of B. Moreover, this statement can be formulated in first order logic by the formula $\forall u(A(u) \Rightarrow \neg B(u))$, whose semantics (in fuzzy logic) is usually given by the standard measure of inclusion defined via the adjoint pair $(*, \to)$; i.e., by $\inf_{u\in\mathcal{U}}\{A(u) \to n(B(u))\}$ where n denotes a negation operator and $\mathcal{U}$ the set of possible instances for A and B.

- The vertex **I**, or Particular Affirmative, is represented by the statement *"Some A is B"*. As the previous vertex, can be formulated in terms of the (nonempty) intersection between two sets $\exists u(A(u) \wedge B(u))$, whose semantics in fuzzy logic is given by: $\sup_{u\in\mathcal{U}}\{A(u) * B(u)\}$ where $\mathcal{U}$ denotes the set of possible instances for A and B.
- The vertex **O**, or Particular Negative, is given by the statement *"Some A is not B"*. As above, it can be represented in set theory as the no inclusion of set A into B and, in first-order logic, as the formula $\exists u(A(u) \wedge \neg B(u))$, whose semantics in fuzzy logic is given by $\sup_{u\in\mathcal{U}}\{A(u) * n(B(u))\}$.

The question now is: how can we relate the square of opposition with the f-index of inclusion and the f-index of contradiction? Following the fuzzy interpretation of the vertexes, we can identify them directly with the greatest and lowest f-indexes of inclusion and contradiction.

3.2. Relationship between vertexes and f-indexes

Underneath, the reader can find in more detail the proposed preliminary identification of vertexes with the extreme f-indexes of inclusion and contradiction, motivated by the standard interpretation of the vertexes in term of fuzzy logic described in the previous section:

- Vertex **A** is identified with the greatest f-index of inclusion id. We recall that if $Inc(A,B) = id$ for two fuzzy sets A and B then, by Theorem 1, we have that $A(u) \leq B(u)$ for all $u \in \mathcal{U}$, which is equivalent to saying that $\forall u(A(u) \Rightarrow B(u))$ has truth degree 1 for any residuated lattice used for its semantics.
- Vertex **E** is identified with the greatest f-index of contradiction $f_\perp$. In this case, Theorem 2 states that a fuzzy set A is $f_\perp$-contradictory with a fuzzy set B if and only if $B(u) > 0$ implies $A(u) = 0$ for all $u \in \mathcal{U}$, which is equivalent to saying that $\forall u(A(u) \Rightarrow \neg B(u))$ has truth degree 1 for any residuated lattice and involutive negation used for its semantics.
- Vertex **I** is identified with the lowest f-index of contradiction $f_\top$. Let us recall that, by Theorem 2, the f-index of contradiction of two fuzzy sets A and B is $f_\top$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = B(u) = 1$, which is equivalent to saying that $\exists u(A(u) \wedge B(u))$ has truth degree 1 for any residuated lattice used for its semantics.
- Vertex **O** is identified with the lowest f-index of inclusion $\perp$. By Theorem 1 we have that if $\mathcal{U}$ is finite, then the f-index of inclusion of a fuzzy set A into another B is $\perp$ if and only if there is an element in the universe $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$, which is equivalent to saying that $\exists u(A(u) \wedge \neg B(u))$ has truth degree 1 for any residuated lattice and involutive negation used for its semantics.

In summary, we have the following identification:

$$\mathbf{A} \equiv id \qquad \mathbf{O} \equiv \perp \qquad \mathbf{E} \equiv f_\perp \qquad \mathbf{I} \equiv f_\top.$$

Moreover, as explained above, we certainly preserve the original meaning of the vertexes with the previous identification. Therefore, it is expected to be able to relate them by the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations.

It is important to recall the requirement of 'existential import' (all categories should have at least one element), which is necessary for all the relations in the crisp square of opposition to hold. In fuzzy set theory, the empty set is identified with the (fuzzy) set that assigns 0 to every $u \in \mathcal{U}$; however, in our framework, this non-emptiness presupposition is rephrased in different terms, namely, under the notion of *normality*.

Definition 7. A fuzzy set A is called normal if there exists $u \in \mathcal{U}$ such that $A(u) = 1$.

Note that a crisp set is not empty if and only if it is normal (in the fuzzy sense). Below we show a preliminary result to show how Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations can be applied to those vertexes.

Proposition 1. *Let A and B two normal fuzzy sets, then:*

1. *(Contrariety) If $Inc(A,B) = id$ then $Con(A,B) \neq f_\perp$.*
2. *(Contrariety) If $Con(A,B) = f_\perp$ then $Inc(A,B) \neq id$.*
3. *(Subalternation) If $Inc(A,B) = id$ then $Con(A,B) = f_\top$.*
4. *(Subalternation) If $Con(A,B) = f_\perp$ then $Inc(A,B) = \perp$.*
5. *(Contradiction for crisp sets) If A and B are crisp, then either $Con(A,B) = f_\perp$ or $Con(A,B) = f_\top$*
6. *(Contradiction for crisp sets) If A and B are crisp, then either $Inc(A,B) = id$ or $Inc(A,B) = \perp$.*
7. *(Subcontrariety for crisp sets) If A and B are crisp, $Con(A,B) \neq f_\top$ implies $Inc(A,B) = \perp$.*
8. *(Subcontrariety for crisp sets) If A and B are crisp, $Inc(A,B) \neq \perp$ implies $Con(A,B) = f_\top$.*
9. *(Superalternation for crisp sets) If A and B are crisp, $Con(A,B) \neq f_\top$ implies $Inc(A,B) \neq id$.*
10. *(Superalternation for crisp sets) If A and B are crisp, $Inc(A,B) \neq \perp$ implies $Con(A,B) \neq f_\perp$.*

Proof. Since A and B are normal, we can assume that $u_A, u_B \in \mathcal{U}$ are those such that $A(u_A) = B(u_B) = 1$.

1) By Theorem 1, if $Inc(A, B) = id$, then $A(u) \leq B(u)$ for all $u \in \mathcal{U}$. Then, for u_A we have that $B(u_A) = 1$ since $1 = A(u_A) \leq B(u_A)$. As a consequence of Corollary 1 we have that $Con(A, B) \neq f_\perp$

2) It is the converse of 1)

3) In item 1) we have proved that if $Inc(A, B) = id$ then $B(u_A) = A(u_A) = 1$. Hence, by Corollary 1 we have that $Con(A, B) = f_\top$.

4) Let us assume that $Con(A, B) = f_\perp$. Then, by definition we have that $A(u) \leq f_\perp(B(u))$ for all $u \in \mathcal{U}$. As a consequence, for the specific case of u_B, we have that $A(u_B) \leq f_\perp(B(u_B)) = f_\perp(1) = 0$. In other words, $B(u_B) = 1$ and $A(u_B) = 0$ and then, by Theorem 1 we have $Inc(A, B) = \perp$.

5 and 6)) It is straightforward to check that if A and B are crisp then either $Con(A, B) = f_\perp$ or $Con(A, B) = f_\top$ (resp. either $Inc(A) = id$ or $Inc(A) = \perp$).

7) By 5) we have that $Con(A, B) = f_\perp$, by 2) that $Inc(A, B) \neq id$ and finally by 6) that $Inc(A, B) = \perp$.

8) Similar to 7).

9 and 10) Direct consequences of 7) and 8). □

So far, we have that just the Contrariety and Subalternation relations hold in general whereas Superalternation, Subcontrariety and Contradiction relations only hold for crisp sets. The reason is because in our fuzzy framework we are no longer working with Boolean truth-values, but with degrees. This issue is very clear in the case of the Contradiction relation, since given two fuzzy sets we have infinitely many values to represent the inclusion and contradiction instead of a dichotomy, as in the crisp case. In other words, whereas in crisp set theory, we have either $A \subseteq B$ or $A \nsubseteq B$, in our fuzzy framework we have infinitely many possibilities for the inclusion of A in B. This causes a domino effect in the rest of relations. For instance, given $Inc(A, B) \neq \perp$, we may infer by the Subalternation property that $Con(A, B) \neq f_\perp$, but since there are many different possibilities for the f-index of contradiction, we cannot guarantee that $Con(A, B) = f_\top$ in general (as in the crisp Superalternation).

4. Analyzing the square in terms of f-indexes

4.1. Contradiction: working in the diagonals

In the previous section we have shown that the vertexes of the square of opposition can be identified with the extreme cases of the f-indexes of inclusion and contradiction. At this point, it seems natural to include the rest of f-indexes of contradiction and inclusion in the square of opposition and to reformulate the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations in order to cover them as well. The most natural position to allocate the rest of f-indexes of inclusion and f-indexes of contradiction is in the diagonal joining the respective vertexes with the extreme cases (Fig. 2):

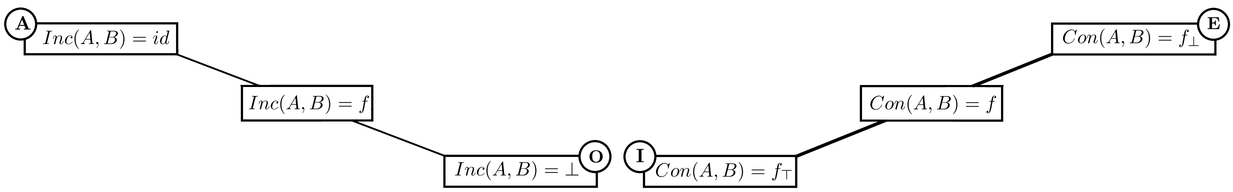

Fig. 2. Location of the f-indexes in the square of opposition. Left, diagonal associated to the f-index of inclusion and right diagonal associated to the f-index of contradiction.

Two remarks about the previous representation. Firstly, note that although in the previous figure the f-indexes are allocated in a straight line, they are not totally ordered but they have the structure of a complete lattice. That representation is only for the sake of presentation. Secondly, the f-indexes of inclusion and contradiction are allocated in order, that is, the stronger the inclusion (resp. contradiction), the closer the respective f-index of inclusion (resp. contradiction) to **A** (resp. to **E**). However, although in the diagonal joining **A** and **O** the f-indexes are ordered by following the natural order between functions in $[0, 1]^{[0,1]}$, in the diagonal joining **E** and **I** the f-indexes are ordered reversely. The reason is because the stronger the contradiction, the lesser the f-index of contradiction. As special cases, note that $\mathbf{E} \equiv f_\perp$ and $\mathbf{I} \equiv f_\top$ represent the strongest and weakest contradictions.

In the Aristotelian square of opposition, the diagonal represents the Contradiction relation. Given two contradictory statements, only one can be true. Under the representation proposed for the f-index of inclusion and contradiction, we reinterpret the Contradiction relation as only one of the f-indexes represented in each diagonal can be true. Since given two fuzzy sets A and B, the f-index of inclusion and the f-index of contradiction are unique, the Contradiction relation between f-index of inclusion and the f-index of contradiction hold straightforwardly.

The rest of relations imposed by the square of opposition (i.e., Contrariety, Subcontrariety, Subalternation and Superalternation) are more intricate. The reason is because we have to reinterpret those relations in the fuzzy framework by incorporating degrees to their definition.

4.2. Reinterpreting the relations in the square

In the crisp Aristotelian square, the relations of Contrariety, Subalternation and Superalternation can be rewritten as follows in terms of an implication:

Contraries	Subalterns	Superalterns
• $\mathbf{A} \Rightarrow \neg\mathbf{E}$	• $\mathbf{A} \Rightarrow \mathbf{I}$	• $\neg\mathbf{I} \Rightarrow \neg\mathbf{A}$
• $\mathbf{E} \Rightarrow \neg\mathbf{A}$	• $\mathbf{E} \Rightarrow \mathbf{O}$	• $\neg\mathbf{O} \Rightarrow \neg\mathbf{E}$

In order to reinterpret those implications in terms of fuzzy logic, we have to assign degrees to the symbols **A**, **E**, **I** and **O** by means of the following natural identification:

- **A**: degree of inclusion
- **E**: degree of contradiction
- **O**: degree of not inclusion
- **I**: degree of not contradiction.

It is worth noting that the previous identifications of the vertexes can be assumed in a sense broader than assuming any underlying algebraic structure (as in Section 3.1 based on residuated lattices), and allowing the use of general measures of inclusion and contradiction as those given by axiomatic definitions [23] and the ones given by the f-indexes. Then, we consider any fuzzy implication $\Rightarrow$ (for instance, that in an IMTL algebra), which is an operator that is decreasing in the antecedent and increasing in the consequent. In this way, given the fuzzy implication $\varphi \Rightarrow \psi$, the greater the degree of φ, the greater the degree of ψ.

Accordingly, $\mathbf{A} \Rightarrow \neg\mathbf{E}$, $\mathbf{A} \Rightarrow \mathbf{I}$, and $\neg\mathbf{O} \Rightarrow \neg\mathbf{E}$ can be roughly reformulated as:

the stronger the inclusion, the weaker the contradiction.

Similarly, $\mathbf{E} \Rightarrow \neg\mathbf{A}$, $\mathbf{E} \Rightarrow \mathbf{O}$, and $\neg\mathbf{I} \Rightarrow \neg\mathbf{A}$, can be reformulated as:

the stronger the contradiction, the weaker the inclusion.

In our framework, that relationship is given by the following proposition, that relates the f-indexes of inclusion and contradiction.

Proposition 2 *(Contrariety and Subcontrariety). Let A and B be two fuzzy sets such that A is normal, then $Inc(A, B) < Con(B, A)$.*

Proof. By definition, given $u_0 \in \mathcal{U}$ it is straightforward to check that

$$Inc(A, B)(A(u_0)) \le B(u_o) \le Con(B, A)(A(u_0)).$$

Now, since A is normal, we can choose $u_0 \in \mathcal{U}$ such that $A(u_0) = 1$ and, then,

$$Inc(A, B)(1) \le Con(B, A)(1),$$

and, by monotonicity of Inc and antitonicity of Con, we have for all $x \in [0, 1]$ that:

$$Inc(A, B)(x) \le Inc(A, B)(1) \le Con(B, A)(1) \le Con(B, A)(x),$$

that is, we have that $Inc(A, B) \le Con(B, A)$.

Finally, since $Inc(A, B)(0) = 0$ and $Con(B, A)(0) = 1$, we have that $Inc(A, B) < Con(B, A)$. □

Three remarks are worth being noticed here: firstly, recall that in the case of contradiction, the stronger the contradiction between two fuzzy sets, the smaller the f-index of contradiction. As a result, by the proposition above we have that, the stronger the inclusion, the greater the f-index of inclusion, the greater the f-index of contradiction and then, the weaker the contradiction, as expected. The second remark is related to the name of the relation derived by Proposition 2. We have named it Contrariety (and Subcontrariety) despite the fact that it can be also linked to the relations of Subalternation and Superalternation. The reason is because those latter relations can be also reformulated in terms of existential and universal quantifiers, as we will show later in this section. The final remark concerns the hypothesis in Proposition 2 that A is normal: that should be expected, since it is similar to the hypothesis in classical Aristotelian square of opposition, where (crisp) sets have to be nonempty. The following example shows that such an hypothesis is necessary.

Example 1. Let $\mathcal{U} = \{u\}$ and $A(u) = 0.5$. Then $Inc(A, A) = id$ and

$$Con(A, A)(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } 0 < x \le 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $Inc(A,A) \not\leq Con(A,A)$, since $Inc(A,A)(1) = 1$ and $Con(A,A)(1) = 0$.

As mentioned, Proposition 2 can be used to establish a relation of Contrariety between the f-indexes of inclusion and contradiction. Specifically, given an f-index of inclusion (resp. an f-index of contradiction) between two fuzzy sets A and B, Proposition 2 determines a lower (resp. upper) bound for all the possible f-indexes of contradiction (resp. inclusion) between A and B. As a result, certain f-indexes of inclusion (resp. contradiction) are contrary to certain f-indexes of contradiction (resp. inclusion); in the sense that both are incompatible. The following corollaries specify exactly which are those bounds.

Corollary 2. *Let A and B be two fuzzy sets such that $Inc(A,B)(x) = f(x)$ and A is normal, then*

$$Con(B,A)(x) \geq \begin{cases} 1 & \text{if } x = 0 \\ f(1) & \text{otherwise.} \end{cases}$$

Corollary 3. *Let A and B be two fuzzy sets such that $Con(B,A)(x) = f(x)$ and A is normal, then*

$$Inc(A,B)(x) \leq \begin{cases} x & \text{if } x = f(1) \\ f(1) & \text{otherwise.} \end{cases}$$

Let us describe how the previous results determine a Contrariety relation between the f-indexes of inclusion and contradiction. Given the f-index of inclusion $Inc(A,B)$ (resp. contradiction $Con(B,A)$), Corollary 2 (resp. Corollary 3) determines a lower bound (resp. upper bound) on the set of possible f-indexes of contradiction for $Con(B,A)$ (resp. inclusion for $Inc(A,B)$). As a result, all the f-indexes that do not satisfy those bounds are contrary, since they are incompatible with the given f-index of inclusion/contradiction. Note that in our approach, a given f-index of inclusion (resp. contradiction) does not have one contrary f-index of contradiction (resp. inclusion) but a set of them.

It is worth noting also that Proposition 2, Corollary 2 and Corollary 3 can be applied to any given f-index of inclusion and contradiction, in particular to the extreme cases $\perp$ and $f_\top$, which somehow contradicts the application of the contrariety in the crisp square of opposition. However, for both extreme cases the reader can check that Corollary 2 and Corollary 3 do not determine any strict bound but the naive ones; that is $Con(B,A) \geq f_\perp$ and $Inc(A,B) \leq id$. As a result, there is no f-index of inclusion (resp. contradiction) contrary to $f_\top$ (resp. $\perp$).

The following examples show how to apply the Contrariety relation with specific values.

Example 2. Let us assume that the f-index of inclusion between two fuzzy sets A and B is $Inc(A,B)(x) = x^2$. By Proposition 2 we have that $x^2 \leq Inc(A,B)(x) < Con(B,A)(x)$. Note that, since $Con(B,A)$ is decreasing and $Inc(A,B)(1) = 1^2 = 1 \leq Con(x)$ for all $x \in [0,1]$, then $Con(B,A)(x)) = f_\top$; that is, the contradiction between A and B is the weakest f-index of contradiction (or no contradiction). Then, despite $Inc(A,B)$ is not the greatest f-index of inclusion, we have that there is no contradiction between A and B at all. As a result, we can conclude that $Inc(A,B) = x^2$ is contrary to all the f-indexes of contradiction but $Con(B,A) = f_\top$.

Example 3. Let us assume that the f-index of inclusion between two fuzzy sets A and B is

$$Inc(A,B)(x) = \begin{cases} x & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise.} \end{cases}$$

By Proposition 2 we have that $Inc(A,B)(x) < Con(B,A)(x)$. As in the previous example, since $Con(B,A)$ is decreasing, we have that $Con(B,A)(x) \geq Inc(1) = 0.5$ and then, the strongest possible f-index of contradiction is

$$Con(B,A)(x) \geq f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{otherwise.} \end{cases}$$

Therefore, there are many possible f-indexes of contradiction between A and B, but they can never be greater than $f(x)$. As a result, we can conclude that the f-index of inclusion $Inc(A,B)$ is contrary to all the f-indexes of contradiction such that they are not greater than $f(x)$; since those f-indexes are incompatible with $Inc(A,B)$.

The following pair of examples are related to apply the Contrariety to given f-indexes of contradiction in order to infer some information about the f-indexes of inclusion.

Example 4. Let us assume that the f-index of contradiction between two fuzzy sets A and B is $Con(B,A) = 1 - x$. Then, by Proposition 2 we have that $Inc(A,B)(x) < Con(B,A) = 1 - x$ for all $x \in [0,1]$. By monotonicity of $Inc(A,B)$, we have that

$$Inc(A,B)(x) \leq Inc(A,B)(1) \leq Con(B,A)(1) = 0$$

for all $x \in [0,1]$. In other words, despite $Con(B,A) = 1 - x$ is not the strongest f-index of contradiction the only possible f-index of inclusion is $Inc(A,B) = \perp$; i.e., there is no inclusion of A in B. In other words, we can conclude that $Con(B,A) = 1 - x$ is contrary to all the f-indexes of inclusion but $Inc(A,B)(x) = \perp$.

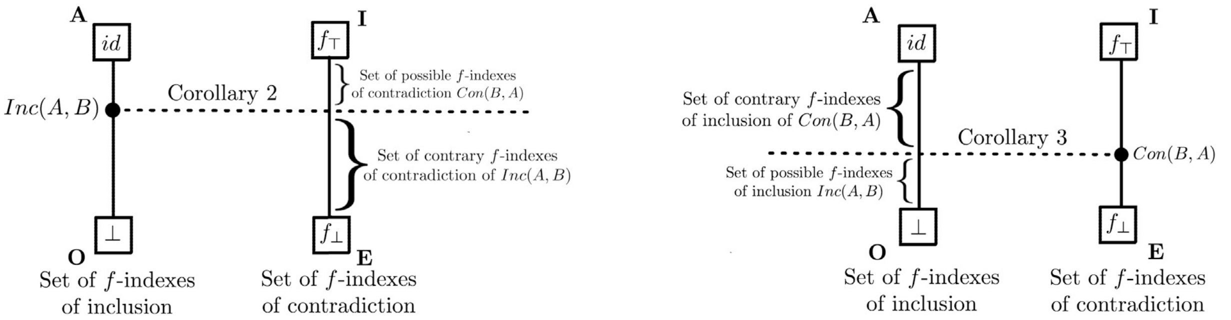

Fig. 3. Graphical representation of the effect of applying the Contrariety relation on the set of the f-index of contradiction given an f-index of inclusion (left) and on the set of the f-index of inclusion given an f-index of contradiction (right).

Example 5. Let us assume that the f-index of contradiction between two fuzzy sets A and B is

$$Con(B,A)(x) = \begin{cases} 1 & \text{if } x \leq 0.4 \\ 0.6 & \text{otherwise.} \end{cases}$$

By Proposition 2, we have that $Inc(A,B)(x) \leq Con(B,A)(x)$ and by monotonicity of $Inc(A,B)$, for all $x \in [0,1]$, we have that

$$Inc(A,B)(x) \leq Inc(A,B)(1) \leq Con(B,A)(1) = 0.6$$

By using now that $Inc(A,B)(x) \leq x$ for all $x \in [0,1]$ as well, we can finally find an upper bound the f-index of inclusion by:

$$Inc(A,B)(x) \leq f(x) = \begin{cases} x & \text{if } x \leq 0.6 \\ 0.6 & \text{otherwise.} \end{cases}$$

Following the reasoning used in the previous examples, we can conclude that the previous f-index of contradiction is contrary to all the f-indexes of inclusion such that are not lower than $f(x)$, since those f-indexes are incompatible with $Con(B,A)$.

The notion of Contrariety between the f-indexes of inclusion and contradiction can be also represented graphically in the square of opposition. However, for the sake of a better representation, it is convenient to reorder the four vertexes locating them as follows:

- **A**: left top corner
- **O**: left bottom corner
- **I**: right top corner
- **E**: right bottom corner

The new position of those vertexes can be seen in Fig. 3. The first consequence of this reorder is that now, the set of f-indexes of inclusion can be identified with the left side of the square and the f-indexes of contradiction with the right side. The second consequence is that the f-indexes can be allocated in order from the top to the bottom element according to the standard order between mappings. In this way, we keep the identification of vertexes with the extreme cases. Note that, for the sake of presentation, the set of f-indexes are represented by a straight line despite the ordering is not total. The third consequence of this representation is that the bounds determined by Corollary 2 and Corollary 3 can be easily represented by a dotted line that splits the set of f-indexes in two: those compatible f-indexes and those contrary f-indexes.

To finish the discussion about the Contrariety relation, the following proposition shows that if the f-indexes of inclusion and contradiction do not satisfy the order described in Proposition 2 (i.e., they are not contrary), then one of the fuzzy sets should be *partially empty*, i.e. considering degrees of emptiness in the following sense: the closer to the fuzzy set $\emptyset$, defined by $\emptyset(u) = 0$ for all $u \in \mathcal{U}$, the emptier.

Proposition 3. *Let A and B be two fuzzy sets such that there exists $\alpha \in (0,1]$ satisfying the inequality $Inc(A,B)(\alpha) > Con(B,A)(\alpha)$. Then:*

1. *$A(u) < \alpha$ for all $u \in \mathcal{U}$;*
2. *$Inc(A,B)(x) = x$ for all $x \geq \alpha$;*
3. *$Con(B,A)(x) = 0$ for all $x \geq \alpha$.*

Proof. 1. Let us assume, by reductio ad absurdum, that there exists $u_0 \in \mathcal{U}$ such that $A(u_0) \geq \alpha$. Then, by definition, we have that

$$Inc(A,B)(A(u_0)) \leq B(u_0) \leq Con(B,A)(A(u_0)).$$

By monotonicity of Inc and antitonicity of Con, we would have

$$Inc(A,B)(\alpha) \le Inc(A,B)(A(u_0)) \le Con(B,A)(A(u_0)) \le Con(B,A)(\alpha),$$

which contradicts the hypothesis $Inc(A,B)(\alpha) > Con(B,A)(\alpha)$. Therefore, $A(u) < \alpha$ for all $u \in \mathcal{U}$.

2. By the analytical expressions of $Inc(A,B)$, for all $x \ge \alpha$ we have:

$$Inc(A,B)(x) = \inf_{u\in\mathcal{U}}\{B(u) \mid x \le A(u)\} \wedge x = \inf_{u\in\mathcal{U}}\{\emptyset\} \wedge x = 1 \wedge x = x.$$

3. By the analytical expressions of $Con(B,A)$, for all $x \ge \alpha$ we have:

$$Con(B,A)(x) = \sup_{u\in\mathcal{U}}\{B(u) \mid x \le A(u)\} = \sup_{u\in\mathcal{U}}\{\emptyset\} = 0. \quad \square$$

4.3. Analysis of subalternation and superalternation relations

Now, let us reconsider again the Subalternation and Superalternation relations, and let us reformulate them in terms of f-indexes of inclusion and contradiction. In Fig. 1 we can observe that Subalternation relation is always pointing to the existential quantifier. Specifically, the Subalternation relation between **A** and **I** can be reformulated in terms of the f-index of inclusion as follows: the greater the f-index of inclusion (truth degree of **A**) between fuzzy sets A and B, the more different is $A \cap B$ from the emptyset (truth degree of **I**); i.e., there exists an element with highest truth degree in $A \cap B$. This relation is formally given by the following proposition.

Proposition 4 *(Subaltern f-index of inclusion). Let A and B be two fuzzy sets such that A is normal, then there exists $u \in \mathcal{U}$ such that $(A \cap B)(u) \ge Inc(A,B)(1)$.*

Proof. Since A is normal there exists $u_0 \in \mathcal{U}$ such that $A(u_0) = 1$. Moreover, by definition of Inc, we have that $Inc(A,B)(A(u_0)) \le B(u_0)$. As a result

$$(A \cap B)(u_0) = \min\{A(u_0), B(u_0)\} = B(u_0) \ge Inc(A,B)(A(u_0)) = Inc(A,B)(1) \quad \square$$

It is not difficult to check that the statement fails whenever $A \subseteq B$ and A is not normal.

Example 6. Consider A as in Example 1, and $B = A$. Then, $(A \cap B)(u) = 0.5$ but $Inc(A,B)(1) = 1$.

Note that the previous result determines both the Subalternation and the Superalternation relation concerning the f-index of inclusion. On the one hand, as written, we can determine a lower bound for the least truth degree of the intersection of two fuzzy sets (positive version of the statement **I**) by the f-index of inclusion (positive version of the statement **A**). On the other hand, the following corollary shows that the closer the intersection of two fuzzy sets to the empty set (negated version of the statement **I**), the lesser the f-index of inclusion (negated version of the statement **A**).

Corollary 4 *(Superaltern f-index of inclusion). Let A and B be two fuzzy sets such that A is normal and let $\alpha \in [0,1]$. If $(A \cap B)(u) \le \alpha$ for all $u \in \mathcal{U}$, then $Inc(A,B)(x) \le \alpha$ for all $x \in [0,1]$ and then:*

$$Inc(A,B)(x) \le \begin{cases} x & \textit{if } x \le \alpha \\ \alpha & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 4 we know that there exists $u \in \mathcal{U}$ such that $(A \cap B)(u) \ge Inc(A,B)(1)$. Since $(A \cap B)(u) \le \alpha$ for all $u \in \mathcal{U}$, then, $Inc(A,B)(1) \le \alpha$. By monotonicity of Inc we have that $Inc(A,B)(x) \le Inc(1) \le \alpha$ for all $x \in [0,1]$. The additional inequality is because of monotonicity of Inc and that $Inc(x) \le x$. $\square$

Note that if $(A \cap B)$ is normal, the previous corollary does not impose any restriction. The following three examples show how can we use the Subalternation and Superalternation relations related to the f-index of inclusion to infer information about the fuzzy set $(A \cap B)$, and vice versa.

Example 7. Let us reconsider the f-index of inclusion given in Example 2; i.e., $Inc(A,B)(x) = x^2$. Then, by Proposition 4 we have that there exists $u \in \mathcal{U}$ such that $(A \cup B)(u) \ge Inc(1) = 1$. Note that we can obtain that $A \cup B$ is normal (the least degree of emptiness) despite of the inclusion of A in B not being the identity function id.

Example 8. Let us reconsider the f-index of inclusion given in Example 2; i.e.,

$$Inc(A,B)(x) = \begin{cases} x & \text{if } x \le 0.5 \\ 0.5 & \text{otherwise.} \end{cases}$$

Then, by Proposition 4 we have that there exists $u \in \mathcal{U}$ such that $(A \cup B)(u) \ge Inc(1) = 0.5$.

Example 9. Let us that A and B are two fuzzy sets such that $(A \cup B)(u) \le 0.4$ for all $u \in \mathcal{U}$. Then, by Corollary 4 we have that $Inc(A, B)(x) \le 0.5$ and, moreover, that

$$Inc(A, B)(x) \le \begin{cases} x & \text{if } x \le 0.4 \\ 0.4 & \text{otherwise.} \end{cases}$$

The Subalternation relation between **E** and **O** can be reformulated in terms of the f-index of contradiction as follows: the lesser the f-index of contradiction (truth degree of **E**) between two fuzzy sets A and B, the more different is $A \cap B^c$ from the emptyset (truth degree of **O**). This relation is formally given by the following proposition.

Proposition 5 *(Subaltern f-index of contradiction). Let A and B be two fuzzy sets such that A is normal and let n be an involutive negation. Then there exists $u \in \mathcal{U}$ such that $(A \cap B^c)(u) \ge n(Con(B, A)(1))$.*

Proof. Since A is normal there exists $u_0 \in \mathcal{U}$ such that $A(u_0) = 1$. Moreover, by definition of Con, we have that $B(u_0) \le Con(B, A)(A(u_0))$, or equivalently $n(Con(B, A)(A(u_0))) \le n(B(u_0))$. As a result

$$(A \cap B^c)(u_0) = \min\{A(u_0), n(B(u_0))\} = n(B(u_0)) \ge n(Con(B, A)(A(u_0))) = n(Con(B, A)(1)) \quad \square$$

Once again, Example 1 provides a counterexample for the case of A not being a normal fuzzy set. Moreover, as in the case for the f-index of inclusion, Proposition 5 also determines the Superalternation relation for the f-index of contradiction.

Corollary 5 *(Superaltern f-index of contradiction). Let A and B be two fuzzy sets such that A is normal, let n be an involutive negation and let $\alpha \in [0, 1]$. If $(A \cap B^c)(u) \le \alpha$ for all $u \in \mathcal{U}$, then $Con(B, A)(x) \ge n(\alpha)$ for all $x \in [0, 1]$.*

Proof. By Proposition 5 we know that there exists $u \in \mathcal{U}$ such that $(A \cap B^c)(u) \ge n(Con(B, A)(1))$. Since $(A \cap B^c)(u) \le \alpha$ for all $u \in \mathcal{U}$, then, $n(Con(B, A)(1)) \le \alpha$, which is equivalent to say that $n(\alpha) \le Con(B, A)(1)$. By monotonicity of Con we have that $Con(B, A)(x) \ge Con(B, A)(1) \ge n(\alpha)$ for all $x \in [0, 1]$. $\square$

Following the statement of the previous corollary, the closer $(A \cap B^c)$ to the empty set, the smaller α and then, the greater the least bound of the f-index of contradiction (weaker the contradiction). The following proposition presents an alternative formulation of the Subalternation relation for the f-index of contradiction. Among the pros of this alternative approach we have that it does not need to consider negation operators in the statement; among the cons, the statement is far from the formula $A \cap B^c$ used in the classical Aristotelian square of opposition.

Proposition 6. *Let A and B be two fuzzy sets such that A is normal, then there exists $u \in \mathcal{U}$ such that $(A \cap B)(u) \le Con(B, A)(1)$.*

Proof. Since A is normal there exists $u_0 \in \mathcal{U}$ such that $A(u_0) = 1$. Moreover, by definition of Con, we have that $B(u_0) \le Con(B, A)(A(u_0))$. As a result

$$(A \cap B)(u_0) = A(u_0) \wedge B(u_0) = B(u_0) \le Con(B, A)(A(u_0)) = Con(B, A)(1) \quad \square$$

The following examples illustrate how the previous results can be used to infer information about the emptiness of $A \cap B^c$ from the f-index of contradiction (Subalternation) and vice versa (Superalternation).

Example 10. Let us reconsider the Example 4, i.e., $Con(B, A) = 1 - x$. Then, by Proposition 5 and for any involutive negation n, we have that there exists $u \in \mathcal{U}$ such that $(A \cap B^c)(u) \ge n(Con(B, A)(1)) = n(0) = 1$. In other words, for any involutive negation n, the set $A \cap B^c$ is always normal.

Example 11. Let us reconsider the Example 5, that is

$$Con(B, A)(x) = \begin{cases} 1 & \text{if } x \le 0.4 \\ 0.6 & \text{otherwise.} \end{cases}$$

By Proposition 5 we have that there exists $u \in \mathcal{U}$ such that $(A \cap B^c)(u) \ge n(Con(B, A)(1)) = n(0.6)$. For example, if we consider the standard negation $n(x) = 1 - x$ to model the complement, we can conclude that $A \cap B^c \ge n(0.6) = 0.4$.

Example 12. Let us assume that the complement of fuzzy sets is given by the standard negation $n(x) = 1 - x$ and that A and B are two normal fuzzy sets such that $(A \cap B^c)(u) \le 0.4$ for all $u \in \mathcal{U}$. Then, by Corollary 5, we have that $Con(B, A)(x) \ge n(0.4) = 1 - 0.4 = 0.6$. By monotonicity of the f-index of contradiction, we can find a lower bound of it by

$$Con(B, A)(x) \ge \begin{cases} 1 & \text{if } x = 0 \\ 0.6 & \text{otherwise.} \end{cases}$$

5. Other approaches with alternative interpretations of the square

There are various alternative interpretations of the vertexes of Aristotle's square and its possible extensions (such as the hexagon) in the literature, and we will focus on just two of them. Firstly, we survey recent papers with a probabilistic analysis of the relations of opposition; secondly, another line of alternative interpretations is related to formal concept analysis.

In [22], the authors present a probabilistic analysis of the square (and the hexagon) of opposition under coherence and study the semantics of basic key relations in order present an application of the square to the study of generalized quantifiers. Continuing with the probabilistic approach, [4] considers hexagons built from a probabilistic rough set and studies relations of oppositions among vertexes derived from different pairs of thresholds; [1] introduces a combination of three-way decisions and probabilistic rough sets to generate hexagons of opposition, which are then used to detect influential news in online communities.

From a different perspective, [5] introduces fuzzy concept-forming operators based on intermediate quantifiers, which are then employed to define new fuzzy concept lattices. Additionally, the generalized concept-forming operators are used to build graded polygons (hexagons, octagons and decagons) of opposition, as generalizations of Aristotle's square. More recently, [6] extended the previous work in more general structures of oppositions: the so-called graded cubes of opposition and 5-graded cubes of opposition.

6. Conclusions and future work

We have reformulated the Aristotelian square of opposition in terms of the f-indexes of inclusion and contradiction. We interpret the segment AO with the set of indexes of inclusion, and the segment EI as the set of indexes of contradiction, and the obtained results allow extracting information in terms of opposition. Firstly, we have allocated the extreme cases of the f-indexes of inclusion and contradiction in the vertexes of the square and have shown that it is a generalization of the Boolean case for crisp sets. Then, we have included the rest of f-indexes in the diagonals between the extreme cases and shown how the Contradiction, Contrariety, Subcontrariety, Subalternation and Superalternation relations in the Aristotelian square of opposition are extended in this context. The properties of these relations deal with the corresponding "degrees" determined by the f-indexes of inclusion and contradiction. In this line, the Contradiction relation is modeled by "exclusive indexes" whereas the rest are modeled in terms of upper and lower bounds of the respective f-indexes. Moreover, we have shown, with a set of examples, how to use that relations to infer information about one index from the other. Further work in this line will focus on trying to relate our work with other graded approaches of the opposition square such as [12,19], or to the cube of opposition as presented in [6,11] or other structures of opposition [5].

Our main interest within the lines of the f-index of inclusion and f-index of contradiction is aimed at relating them to logical implications, namely, as residuated implication. Then, we will be able to use them in a formal reasoning system and in the construction of knowledge databases. Another possible direction for future research is based on the possible consequences of the obtained results in terms of formal concept analysis, following the ideas in [5,9].

Declaration of competing interest

The authors declare that they have no conflicts of interest.

Data availability

No data was used for the research described in the article.

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References

[1] R. Abbruzzese, A. Gaeta, V. Loia, L. Lomasto, F. Orciuli, Detecting influential news in online communities: an approach based on hexagons of opposition generated by three-way decisions and probabilistic rough sets, Inf. Sci. 578 (2021) 364–377.
[2] J.Y. Béziau, D. Jacquette, Around and Beyond the Square of Opposition, Birkhäuser, Basel, 2012.
[3] J.Y. Béziau, G. Basti, The Square of Opposition: A Cornerstone of Thought, Birkhäuser, Cham, 2017.
[4] S. Boffa, D. Ciucci, P. Murinová, Comparing hexagons of opposition in probabilistic rough set theory, in: Information Processing and Management of Uncertainty in Knowledge-Based Systems. (IPMU 2022), Commun. Comput. Inf. Sci. 1601 (2022) 622–633.
[5] S. Boffa, P. Murinová, V. Novák, Graded polygons of opposition in fuzzy formal concept analysis, Int. J. Approx. Reason. 132 (2021) 128–153.
[6] S. Boffa, P. Murinová, V. Novák, P. Ferbas, Graded cubes of opposition in fuzzy formal concept analysis, Int. J. Approx. Reason. 145 (2022) 187–209.
[7] H. Bustince, N. Madrid, M. Ojeda-Aciego, A measure of contradiction based on the notion of n-weak-contradiction, in: IEEE Intl. Conf. on Fuzzy Systems (FUZZ-IEEE'13), 2013, pp. 1–6.
[8] H. Bustince, N. Madrid, M. Ojeda-Aciego, The notion of weak-contradiction: definition and measures, IEEE Trans. Fuzzy Syst. 23 (2015) 1057–1069.
[9] D. Dubois, H. Prade, From Blanche's hexagonal organization of concepts to formal concepts analysis and possibility theory, Log. Univers. 6 (2012) 149–169.
[10] D. Dubois, H. Prade, A. Rico, Graded cubes of opposition and possibility theory with fuzzy events, Int. J. Approx. Reason. 84 (2017) 168–185.

[11] D. Dubois, H. Prade, A. Rico, The cube of opposition: a structure underlying many knowledge representation formalisms, in: Proc. of International Joint Conference on Artificial Intelligence (IJCAI), 2015, pp. 2933–2939.

[12] D. Dubois, H. Prade, A. Rico, Structures of opposition and comparisons: Boolean and gradual cases, Log. Univers. 14 (2020) 115–149.

[13] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic of left-continuous t-norms, Fuzzy Sets Syst. 124 (2001) 271–288.

[14] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998.

[15] L.M. Kitainik, Fuzzy inclusions and fuzzy dichotomous decision procedures, in: J. Kacprzyk, S.A. Orlovski (Eds.), Optimization Models Using Fuzzy Sets and Possibility Theory, Springer, Netherlands, Dordrecht, 1987, pp. 154–170.

[16] N. Madrid, M. Ojeda-Aciego, A view of f-indexes of inclusion under different axiomatic definitions of fuzzy inclusion, Lect. Notes Artif. Intell. Sci. 10564 (2017) 307–318.

[17] N. Madrid, M. Ojeda-Aciego, Functional degrees of inclusion and similarity between L-fuzzy sets, Fuzzy Sets Syst. 390 (2020) 1–22.

[18] N. Madrid, M. Ojeda-Aciego, I. Perfilieva, f-inclusion indexes between fuzzy sets, in: Proc. of IFSA-EUSFLAT, 2015.

[19] P. Murinová, Graded structures of opposition in fuzzy natural logic, Log. Univers. 265 (2022) 495–522.

[20] P. Murinová, V. Novák, Analysis of generalized square of opposition with intermediate quantifiers, Fuzzy Sets Syst. 242 (2014) 89–113.

[21] T. Parsons, The traditional square of opposition, in: E.N. Zalta (Ed.), Stanford Encyclopedia of Philosophy, 2021.

[22] N. Pfeifer, G. Sanfilippo, Probabilistic squares and hexagons of opposition under coherence, Int. J. Approx. Reason. 88 (2017) 282–294.

[23] D. Sinha, E. Dougherty, Fuzzification of set inclusion: theory and applications, Fuzzy Sets Syst. 55 (1993) 15–42.

[24] E. Trillas, C. Alsina, J. Jacas, On contradiction in fuzzy logic, Soft Comput. 3 (4) (1999) 197–199.

[25] B. Xie, L.-W. Han, J.-S. Mi, Measuring inclusion, similarity and compatibility between Atanassov's intuitionistic fuzzy sets, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 20 (2) (2012) 261–280.

[26] V. Young, Fuzzy subsethood, Fuzzy Sets Syst. 77 (1996) 371–384.