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On symmetries and conservation laws of a Gardner equation involving arbitrary functions

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ABSTRACT

In this work we study a generalized variable-coefficient Gardner equation from the point of view of Lie symmetries in partial differential equations. We find conservation laws by using the multipliers method of Anco and Bluman which does not require the use of a variational principle. We also construct conservation laws by using Ibragimov theorem which is based on the concept of adjoint equation for nonlinear differential equations.

1. Introduction

Nonlinear equations with variable coefficients have become increasingly important in recent years because these describe many nonlinear phenomena more realistically than equations with constant coefficients. The Gardner equation, for instance, is used in different areas of physics, such as fluid dynamics, plasma physics, quantum field theory, and it also describes a variety of wave phenomena in plasma and solid state.

In this paper, we consider the variable-coefficient Gardner equation with nonlinear terms given by

$$u_t + A(t)uu_x + C(t)u^2u_x + B(t)u_{xxx} + Q(t)u = 0, \tag{1}$$

where $A(t) \neq 0$, $B(t) \neq 0$, $C(t) \neq 0$ and $Q(t)$ are arbitrary smooth functions of t.

In [10], for $A(t) = 1$ and $C(t) = 0$, the optimal system of one-dimensional subalgebras was obtained. In [11], some conservation laws for Eq. (1) were constructed for some special forms of the functions $B(t)$ and $Q(t)$. Lie symmetries of Eq. (1) when $Q(t) = 0$, were derived in [15]. The classification of Lie symmetries obtained in [15] was enhanced in [19] by using the general extended equivalence group. In [23], adding to Eq. (1) the term $E(t)u_x$, where $E(t)$ is an arbitrary smooth function of t, the authors found new exact non-traveling solutions, which include soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions and combined Jacobi elliptic function solutions of Eq. (1). Soliton solutions of Eq. (1) were obtained in [20] transforming the equation to an homogeneous equation when $Q(t) = 0$ and a forcing term $R(t)$ has been added. Finally, in [9], exact solutions were obtained by using the general mapping deformation method adding a new term $E(t)u_x$ and a forcing term $R(t)$ to Eq. (1), where $E(t)$ and $R(t)$ are arbitrary smooth functions of t.

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Lie symmetries, in general, symmetry groups, have several applications in the context of nonlinear differential equations. It is noteworthy that they are used to obtain exact solutions and conservation laws of partial differential equations [5–7,16,22].

In [3] Anco and Bluman gave a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy–Kovaleskaya form

$$u_t = G(x, u, u_x, u_{xx}, \dots, u_{nx}).$$

Nontrivial conservation laws are characterized by a multiplier λ, which has no dependence on u_t and all derivatives of u_t, satisfying

$$\hat{E}[u](\lambda u_t - \lambda G(x, u, u_x, u_{xx}, \dots, u_{nx})) = 0.$$

Here

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots .$$

The conserved density T^t must satisfy

$$\lambda = \hat{E}[u]T^t,$$

and the flux T^x is given by

$$T^x = -D_x^{-1}(\lambda G) - \frac{\partial T^t}{\partial u_x}G + GD_x\left(\frac{\partial T^t}{\partial u_{xx}}\right) + \cdots .$$

In [13], Ibragimov introduced a general theorem on conservation laws which does not require the existence of a classical Lagrangian and it is used based on the concept of an adjoint equation for nonlinear equations. In [14], Ibragimov generalized the concept of linear self-adjointness by introducing the concept of nonlinear self-adjointness of differential equations. This concept has been recently used for constructing conservation laws [7,18].

The aim of this work is to obtain Lie symmetries of Eq. (1) and construct conservation laws by using both methods, the direct method proposed by Anco and Bluman [2,3] and Ibragimov theorem [13]. We have studied Lie symmetries of equation (1) for cases $Q \neq 0$ and $Q = 0$. In order to obtain conservation laws using Ibragimov theory we have determined the subclasses of Eq. (1) which are nonlinearly self-adjoint.

2. Classical symmetries

To apply the Lie classical method to Eq. (1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x, t, u) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x, t, u) + O(\epsilon^2),\\ u^* &= u + \epsilon\eta(x, t, u) + O(\epsilon^2), \end{aligned}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of Eq. (1). This yields an overdetermined, linear system of differential equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is formed by the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \tag{2}$$

Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (2) leads to a set of 18 determining equations. By simplifying this system we obtain that $\xi = \xi(x, t)$, $\tau = \tau(t)$, and $\eta = \eta(x, t, u)$ are related by the following conditions:

$$\begin{aligned} &\eta_{uu} = 0, \quad \eta_{ux} - \xi_{xx} = 0, \quad \eta_{uuu} = 0, \quad \eta_{uux} = 0, \quad -\tau B_t - \tau_t B + 3\xi_x B = 0, \quad \tau uBQ_t - \tau uB_tQ - \eta_u uBQ + 3\xi_x uBQ\\ &\quad + \eta BQ + \eta_x u^2BC + \eta_{xxx}B^2 + \eta_x uAB + \eta_t B = 0, \quad \tau u^3BC_t - \tau u^3B_tC + \xi_x u^3BC + 2\eta u^2BC - \tau u^2AB_t + 3\eta_{uxx}uB^2\\ &\quad - \xi_{xxx}uB^2 + \tau u^2A_tB + 2\xi_x u^2AB + \eta uAB - \xi_t uB = 0. \end{aligned} \tag{3}$$

In order to find Lie symmetries of the equation, we distinguish two cases: $Q \neq 0$ and $Q = 0$.

Case 1. $Q \neq 0$.

For the sake of simplicity, in Case 1 we shall consider $C(t) = 1$, obtaining the following symmetries

$$\xi = k_1x + \beta, \quad \tau = \tau(t), \quad \eta = \frac{\beta_t}{A} + (k_1 + \alpha)u$$

where $A = A(t)$, $B = B(t)$, $Q = Q(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\tau = \tau(t)$ must satisfy the following conditions:

$$(3k_1 - \tau_t)B - \tau B_t = 0, \tag{4}$$

$$-\tau B_t + (4k_1 + 2\alpha)B = 0, \tag{5}$$

$$2\beta_t B + \tau A A_t B - \tau A^2 B_t + (3k_1 + \alpha)A^2 B = 0, \tag{6}$$

$$\tau B Q_t - \tau B_t Q + 3k_1 BQ + \alpha_t B = 0, \tag{7}$$

$$\beta_t (AQ - A_t) + \beta_{tt} A = 0. \tag{8}$$

Eq. (4) can be written as:

$$\frac{B_t}{B} = \frac{3k_1 - \tau_t}{\tau} = k, \tag{9}$$

where k is a constant. From (8) we get:

$$Q = \frac{A_t}{A} - \frac{\beta_{tt}}{\beta_t}. \tag{10}$$

Subcase 1.1. Setting $k \neq 0$, by solving (9) we obtain

$$B = b_0\, e^{kt}, \quad \tau(t) = k_4\, e^{-kt} + \frac{3\,k_1}{k}. \tag{11}$$

From (5) we obtain

$$\alpha(t) = \frac{k k_4 e^{-kt} - k_1}{2}. \tag{12}$$

Substituting (10), (11) and (12) into Eqs. (6) and (7), we get the following system

$$-2AA_t f_1 + \left(k k_1 e^{kt} + k^2 k_4\right)A^2 - 4ke^{kt}\beta_t = 0, \tag{13}$$

$$2\left(\frac{A_t}{A} - \frac{\beta_{tt}}{\beta_t}\right)_t f_1 - 2k^2 k_4\left(\frac{A_t}{A} - \frac{\beta_{tt}}{\beta_t}\right) - k^3 k_4 = 0. \tag{14}$$

where $f_1(t) = 3k_1 e^{kt} + kk_4$. This system admits two solutions

$$A(t) = \frac{\sqrt{e^{kt}}}{2d_0 + kk_1} f_1^{-\frac{d_0}{3kk_1} - \frac{1}{2}}\left(2a_1 k f_1^{\frac{d_0}{3kk_1} + \frac{1}{6}} + a_0(2d_0 + kk_1)\right), \tag{15}$$

$$\beta(t) = \frac{a_0}{8d_0 k(kk_1 - 2d_0)} f_1^{-\frac{2d_0}{3kk_1}}\left(a_0\left(4d_0^2 - k^2k_1^2\right) + 8a_1 d_0 k f_1^{\frac{d_0}{3kk_1} + \frac{1}{6}}\right), \tag{16}$$

and

$$A(t) = \frac{\sqrt{e^{kt}}}{2d_0 + kk_1} f_1^{-\frac{d_0}{3kk_1} - \frac{1}{2}}\left(a_0(2d_0 + kk_1) f_1^{\frac{d_0}{3kk_1} + \frac{1}{6}} - 2a_1 k\right), \tag{17}$$

$$\beta(t) = \frac{a_1}{2\left(4d_0^2 - k^2k_1^2\right)} f_1^{-\frac{2d_0}{3kk_1}}\left(2a_0(2d_0 + kk_1) f_1^{\frac{d_0}{3kk_1} + \frac{1}{6}} + \frac{a_1 k}{d_0}(kk_1 - 2d_0)\right). \tag{18}$$

Lastly, substituting solutions (15) and (16) or solutions (17) and (18) into (10), we obtain

$$Q(t) = \frac{2d_0 e^{kt} - k^2 k_4}{2f_1}. \tag{19}$$

In the above equations, the following appointments are introduced: $a_0 \neq 0$, $a_1 \neq 0$, $b_0 \neq 0$, $d_0 \neq 0$, $k_1 \neq 0$, k_4 are arbitrary constants, $2d_0 \pm kk_1 \neq 0$.

Subcase 1.2. Setting $k = 0$, by solving (9) we obtain

$$B = b_0, \quad \tau(t) = 3k_1 t + k_3. \tag{20}$$

From (5) we obtain

$$\alpha(t) = -2k_1. \tag{21}$$

Substituting (20), (21) and (10) into Eqs. (6) and (7), we get the following system

$$-2AA_t\tau - 2k_1 A^2 - 4\beta_t = 0, \tag{22}$$

$$2\tau\left(\frac{A_t}{A} - \frac{\beta_{tt}}{\beta_t}\right)_t + 6k_1\left(\frac{A_t}{A} - \frac{\beta_{tt}}{\beta_t}\right) = 0. \tag{23}$$

In this case, we have

$$A(t) = a_0\tau^{-\frac{d_0}{3k_1}} + a_1\tau^{-\frac{1}{3}}, \tag{24}$$

$$\beta(t) = \frac{1}{2}a_0(d_0 - k_1)\tau^{-\frac{2d_0}{3k_1}-\frac{1}{3}}\left(\frac{a_0\tau^{\frac{4}{3}}}{3k_1 - 2d_0} - \frac{a_1\tau^{\frac{d_0}{3k_1}+1}}{d_0 - 2k_1}\right). \tag{25}$$

Substituting solutions (24) and (25) into (10), we get that

$$Q(t) = \frac{d_0}{\tau}. \tag{26}$$

In the above equations, the following appointments are introduced: $a_0 \neq 0$, a_1, $b_0 \neq 0$, $d_0 \neq 0$, $k_1 \neq 0$, k_3 are arbitrary constants, $d_0 \neq \frac{3}{2}k_1$ and $d_0 \neq 2k_1$.

Case 2. $Q = 0$.

Now, the generators are given by:

$$\xi = k_1 x + \beta, \quad \tau = \tau(t), \quad \eta = (k_1 + k_3)u + k_2,$$

where $A = A(t)$, $B = B(t)$, $C = C(t)$, $\beta = \beta(t)$ and $\tau = \tau(t)$ must satisfy the following conditions:

$$(3k_1 - \tau_t)B - \tau B_t = 0, \tag{27}$$

$$\tau BC_t - \tau B_t C + (4k_1 + 2k_3)BC = 0, \tag{28}$$

$$\tau A_t B - \tau AB_t + 2k_2 BC + (3k_1 + k_3)AB = 0, \tag{29}$$

$$k_2 A - \beta_t = 0. \tag{30}$$

Subcase 2.1. We consider $k \neq 0$. In this case we have that $B(t)$ and $\tau(t)$ are given by (11). The remaining functions are defined as follows

$$C(t) = c_0 e^{kt} f_1^{-\frac{2k_3}{3k_1}-\frac{4}{3}}, \tag{31}$$

$$A(t) = \frac{1}{k_1 + k_3}e^{kt} f_1^{-\frac{2k_3}{3k_1}-\frac{4}{3}}\left(2c_0k_2 + a_0(k_1 + k_3)f_1^{\frac{k_3}{3k_1}+\frac{1}{3}}\right), \tag{32}$$

$$\beta(t) = \beta_0 - \frac{a_0k_2 f_1^{-\frac{k_3}{3k_1}}}{kk_3} - \frac{2c_0k_2^2 f_1^{-\frac{2k_3}{3k_1}-\frac{1}{3}}}{k(k_1 + k_3)(k_1 + 2k_3)}, \tag{33}$$

where f_1 has already been previously defined as $f_1(t) = 3k_1e^{kt} + kk_4$.

Subcase 2.2. We consider $k = 0$. Now, $B(t)$ and $\tau(t)$ are given by (20). In this case, the remaining functions are given by

$$C(t) = c_0\tau^{-\frac{2k_3}{3k_1}-\frac{4}{3}}, \tag{34}$$

$$A(t) = \frac{1}{k_1 + k_3}\tau^{-\frac{2k_3}{3k_1}-\frac{4}{3}}\left(2c_0k_2 + a_0(k_1 + k_3)\tau^{\frac{k_3}{3k_1}+\frac{1}{3}}\right), \tag{35}$$

$$\beta(t) = \beta_0 - \frac{a_0k_2\tau^{-\frac{k_3}{3k_1}}}{k_3} - \frac{2c_0k_2^2\tau^{-\frac{2k_3}{3k_1}-\frac{1}{3}}}{(k_1 + k_3)(k_1 + 2k_3)}. \tag{36}$$

In Case 2, the following designations are introduced: a_0, $b_0 \neq 0$, $c_0 \neq 0$, β_0, $k_1 \neq 0$, k_2, $k_3 \neq 0$, k_4 are arbitrary constants, $k_1 \neq -k_3$ and $k_1 \neq -2k_3$. For this case the symmetries were obtained in [15].

3. Formal Lagrangian and adjoint equation

In [13] Ibragimov introduced a new theorem on conservation laws. This theorem is valid for any system of differential equations wherein the number of equations is equal to the number of dependent variables. The new theorem does not require the existence of a classical Lagrangian and it is based on the concept of adjoint equation for nonlinear equations. In order to obtain the adjoint equation we use the following definition:

Definition 1. Consider an qth-order partial differential equation

$$F(\mathrm{x}, u, u_{(1)}, \ldots, u_{(s)}) = 0 \tag{37}$$

with independent variables $\mathrm{x} = (x^1, \ldots, x^n)$ and a dependent variable u, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote the sets of partial derivatives of first, second, etc. order, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The formal Lagrangian is defined as

$$\mathcal{L} = v F(\mathrm{x}, u, u_{(1)}, \ldots, u_{(s)}), \tag{38}$$

where $v = v(\mathrm{x})$ is a new dependent variable. The adjoint equation to (37) is

$$F^*(\mathrm{x}, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{39}$$

with

$$F^*(\mathrm{x}, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta (v F)}{\delta u},$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}},$$

denotes the variational derivative (the Euler–Lagrange operator). Here

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots$$

represents the total differentiation.

Theorem 1. *The adjoint equation to Eq. (1) is*

$$F^* \equiv v\, Q - u^2\, v_x\, C - v_{xxx}\, B - u\, v_x\, A - v_t. \tag{40}$$

4. Nonlinearly self-adjoint equations

In this section we use the following definition given in [14].

Definition 2. Eq. (37) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint Eq. (39) by the substitution

$$v = \varphi(\mathrm{x}, u), \tag{41}$$

such that $\varphi(\mathrm{x}, u) \neq 0$, is identical with the original Eq. (37), i.e.

$$F^* |_{v=\varphi} = \lambda(\mathrm{x}, u, \ldots) F. \tag{42}$$

for some differential function $\lambda = \lambda(\mathrm{x}, u, \ldots)$. If $\varphi = u$ or $\varphi = \varphi(u)$ and $\varphi'(u) \neq 0$, Eq. (37) is said to be *self-adjoint* or *quasi self-adjoint*, respectively. If $\varphi_{x^i}(\mathrm{x}, u) \neq 0$ or $\varphi_u(\mathrm{x}, u) \neq 0$ Eq. (37) is said to be *weak self-adjoint* [8].

Taking into account expression (40) and using (41) and its derivatives, Eq. (42) can be written as

$$\begin{aligned} & u_x\, (\lambda(-u^2\, C - u\, A) - \varphi_u\, u^2\, C - 3\, \varphi_{uu}\, u_{xx}\, B - 3\, \varphi_{uxx}\, B - \varphi_u\, u\, A) \\ & + u_{xxx}\, (-\lambda\, B - \varphi_u\, B) - \lambda\, u\, Q + u_t\, (-\lambda - \varphi_u) + \varphi\, Q - \varphi_x\, u^2\, C - \varphi_t \\ & - 3\, \varphi_{ux}\, u_{xx}\, B - \varphi_{uuu}\, (u_x)^3\, B - 3\, \varphi_{uux}\, (u_x)^2\, B - \varphi_{xxx}\, B - \varphi_x\, u\, A = 0. \end{aligned} \tag{43}$$

Eq. (43) should be satisfied identically in all variables $u_t, u_x, u_{xx}, \ldots$ Requiring the vanishing of the coefficients of the derivatives of u we obtain:

Theorem 2. *Eq. (1) with $A(t) \neq 0$, $B(t) \neq 0$, $C(t) \neq 0$ and $Q(t)$ arbitrary functions, is nonlinearly self-adjoint and*

$$\varphi = c_1 e^{\int 2Q(t)dt} u + c_2 e^{\int Q(t)dt}, \tag{44}$$

with c_1 and c_2 arbitrary constants.

5. Conservation laws

Conservation laws appear in many of physical, chemical and mechanical processes, such laws enable us to solve problems in which certain physical properties do not change over time within an isolated physical system.

A conservation law of Eq. (1) is a space-time divergence such that

$$D_t T^t(x, t, u, u_x, u_t, \ldots) + D_x T^x(x, t, u, u_x, u_t, \ldots) = 0, \tag{45}$$

on all solutions $u(x, t)$ of Eq. (1). Here, T^t represents the conserved density and T^x the associated flux [4], and D_x, D_t denote the total derivative operators with respect to x and t respectively.

In this section we construct conservation laws of each case by using both methods.

5.1. Conservation laws by using the direct method of the multipliers of Anco and Bluman

We suppose, that T^t and T^x have no dependence on u_t and all derivatives of u_t.

Each conservation law (45) has an equivalent characteristic form in which has been eliminated u_t and its differential consequences from T^t and T^x by using Eq. (1)

$$\widehat{T^t} = T^t \mid_{u_t=\Delta} = T^t - \Phi,$$
$$\widehat{T^x} = T^x \mid_{u_t=\Delta} = T^x - \Psi,$$

where $\Delta = -Auu_x - Cu^2u_x - Bu_{xxx} - Qu$, so that

$$\left(D_t\widehat{T^t}(x,t,u,u_x,u_{xx},\ldots) + D_x\widehat{T^x}(x,t,u,u_x,u_{xx},\ldots)\right) \mid_{u_t=\Delta} = 0,$$

is verified on all solutions of Eq. (1), and where

$$D_t \mid_{u_t=\Delta} = \partial_t + \Delta\partial_u + D_x(\Delta)\partial_{u_x} + \cdots$$
$$D_x \mid_{u_t=\Delta} = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \cdots = D_x.$$

In particular, moving off of solutions, we have the identity

$$D_t = D_t \mid_{u_t=\Delta} + (u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu)\partial_u + D_x(u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu)\partial_{u_x} + \cdots$$

These expressions yield the characteristic form of conservation law (45)

$$\begin{aligned} &D_t\widehat{T^t}(x,t,u,u_x,u_{xx},\ldots) + D_x(\widehat{T^x}(x,t,u,u_x,u_{xx},\ldots) + \widehat{\Psi}(x,t,u,u_x,u_t,\ldots)) \\ &\quad = (u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu)\Lambda(x,t,u,u_x,u_{xx},\ldots), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \widehat{\Psi}(x,t,u,u_x,u_t,\ldots) = &E_{u_x}(\widehat{T^t})(u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu) \\ &+ E_{u_{xx}}(\widehat{T^t})D_x(u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu) + \cdots \end{aligned}$$

is a trivial flux [4], and the function

$$\Lambda = E_u(\widehat{T^t}),$$

is a multiplier, where $E_u = \partial_u - D_x\partial_{u_x} + D_x^2\partial_{u_{xx}} - \cdots$, denotes the (spatial) Euler operator with respect to u.

A function $\Lambda(x,t,u,u_x,u_{xx},\ldots)$ is called multiplier if it verifies that $(u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu)\Lambda$ is a divergence expression for all functions $u(x, t)$, not only solutions of Eq. (1).

In order to obtain conservation laws, we use (46) from which is deduced that all nontrivial conserved densities in the form (45) arise from multipliers Λ of Eq. (1), where Λ depends only on x, t, u and x derivatives of u. Divergence condition can be characterized as follows

$$\frac{\delta}{\delta u}((u_t + Auu_x + Cu^2u_x + Bu_{xxx} + Qu)\Lambda) = 0, \tag{47}$$

where $\frac{\delta}{\delta u} = \partial_u - D_x\partial_{u_x} - D_t\partial_{u_t} + D_xD_t\partial_{u_{xt}} + D_x^2\partial_{u_{xx}} + \cdots$, denotes the variational derivative.

Eq. (47) is linear in u_t, u_{tx}, u_{txx}, ...taking the coefficient of u_t and x derivatives of u_t we obtain a system of determining equations for Λ, which yields the following equivalent equations [1–3]

$$-D_t\Lambda - D_x((Au + Cu^2)\Lambda) - D_x^3(B\Lambda) + (Au_x + 2Cuu_x + Q)\Lambda = 0,$$

and

$$\Lambda_u = E_u(\Lambda), \quad \Lambda_{u_x} = -E_u^{(1)}(\Lambda), \quad \Lambda_{u_{xx}} = -E_u^{(2)}(\Lambda), \ldots$$

which are verified for all solutions $u(x, t)$ of Eq. (1).

Given a multiplier Λ, we can obtain the conserved density using a standard method [21]

$$T^t = \int_0^1 d\lambda u\Lambda(x,t,\lambda u,\lambda u_x,\lambda u_{xx},\ldots).$$

We have considered multipliers up to second order, i.e., $\Lambda(x, t, u, u_x, u_{xx})$. In this section we proceed to obtain conservation laws for the values obtained for $A(t)$, $B(t)$, $C(t)$ and $Q(t)$ in Case 1, Section 2.

Subcase 1.1. In this case, Eq. (1) is given by

$$u_t + A(t)uu_x + u^2u_x + b_0e^{kt}u_{xxx} + \frac{2d_0e^{kt} - k^2k_4}{2f_1}u = 0, \tag{48}$$

where $A(t)$ is given by (15) or (17). For Eq. (48) multiplier is given by

$$\Lambda = e^{-\frac{kt}{2}}f_1^{\frac{2d_0}{3kk_1}+1}\left(c_1 + c_2\,e^{-\frac{kt}{2}}\,u\right),$$

where $f_1(t) = 3k_1 e^{kt} + kk_4$. The conserved density obtained from this multiplier is

$$T^t = e^{-\frac{kt}{2}} f_1^{\frac{2d_0}{3kk_1}+1} u\left(c_1 + \frac{c_2}{2} e^{-\frac{kt}{2}} u\right).$$

The flux obtained from this multiplier depends on function $A(t)$. If $A(t)$ is given by (15), we obtain

$$T^x = \frac{e^{-\frac{kt}{2}}}{12} f_1^{\frac{d_0}{3kk_1}} \left(3\,c_2\, f_1^{\frac{d_0}{3kk_1}+1}\left(u^4 e^{-\frac{kt}{2}} + 2\,b_0\, e^{\frac{kt}{2}} \left(2\,u\,u_{xx} - u_x^2\right)\right)\right.$$
$$\left. + 4(a_0 c_2 + c_1)u^3 f_1 + 6\,a_0\, c_1\, u^2 e^{\frac{kt}{2}} f_1^{-\frac{d_0}{3kk_1}} + 12\, b_0\, c_1\, u_{xx}\, e^{kt} f_1^{\frac{1}{2}} + \frac{4\,a_1\,k}{2\,d_0 + k\,k_1} u^2 \left(2\,c_2\,u\, f_1^{\frac{d_0}{3kk_1}+\frac{2}{3}} + 3\,c_1\, e^{\frac{kt}{2}} f_1^{\frac{1}{6}}\right)\right).$$

If $A(t)$ is given by (17), we get

$$T^x = \frac{e^{-\frac{kt}{2}}}{12} f_1^{\frac{d_0}{3kk_1}} \left(3\,c_2\, f_1^{\frac{d_0}{3kk_1}+1}\left(u^4 e^{-\frac{kt}{2}} + 2\,b_0\, e^{\frac{kt}{2}} \left(2\,u\,u_{xx} - u_x^2\right)\right)\right.$$
$$\left. + 4\,c_1\, f_1^{\frac{1}{2}} \left(u^3 + 3\,b_0\, u_{xx}\, e^{kt}\right) + 6\,a_0\,c_1\,u^2 e^{\frac{kt}{2}} f_1^{\frac{1}{6}} + 4\,a_0\,c_2\,u^3 f_1^{\frac{d_0}{3kk_1}+\frac{2}{3}} - \frac{4\,a_1\,k}{2\,d_0 + k\,k_1} u^2 \left(2\,c_2\,u\, f_1^{\frac{1}{2}} + 3\,c_1\, e^{\frac{kt}{2}}\right)\right).$$

Subcase 1.2. In this case, Eq. (1) is given by

$$u_t + \left(a_0 \tau^{-\frac{d_0}{3k_1}} + a_1 \tau^{-\frac{1}{3}}\right) u u_x + u^2 u_x + b_0 u_{xxx} + \frac{d_0}{\tau} u = 0, \tag{49}$$

where τ is given by (20). For Eq. (49) has been obtained the following multiplier

$$\Lambda = \tau^{\frac{d_0}{3k_1}} \left(c_1 + c_2\, \tau^{\frac{d_0}{3k_1}}\right).$$

The conserved density and the flux obtained from this multiplier are:

$$T^t = \tau^{\frac{d_0}{3k_1}} u\left(c_1 + \frac{c_2}{2} \tau^{\frac{d_0}{3k_1}} u\right),$$

$$T^x = \frac{1}{12}\, \tau^{\frac{d_0}{3k_1}} \left(\tau^{\frac{d_0}{3k_1}} \left(6\,b_0\,c_2\left(2\,u\,u_{xx} - u_x^2\right) + 3\,c_2\,u^4\right) + 6\,a_0\,c_1\,u^2\, \tau^{-\frac{d_0}{3k_1}}\right.$$
$$\left. + 6\,a_1\,c_1\,u^2\,\tau^{-\frac{1}{3}} + 4\,a_1\,c_2\,u^3\,\tau^{\frac{d_0}{3k_1}-\frac{1}{3}} + 4\,c_1\left(u^3 + 3\,b_0\,u_{xx}\right) + 4\,a_0\,c_2\,u^3\right).$$

5.2. Conservation laws by using a general theorem on conservation laws proved by Ibragimov

In this section we construct conservation laws for the values obtained for the functions $A(t)$, $B(t)$, $C(t)$ and $Q(t)$ in Case 2, Section 2 using the following theorem on conservation laws proved in [13].

Theorem 3. *Any Lie point, Lie–Bäcklund or non-local symmetry*

$$\mathbf{v} = \xi^i(\mathrm{x}, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta(\mathrm{x}, u, u_{(1)}, \ldots) \frac{\partial}{\partial u}, \tag{50}$$

of Eq. (37) provides a conservation law $D_i(T^i) = 0$ for the simultaneous system (37) and (39). The conserved vector is given by

$$T^i = \xi^i \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) - \cdots\right]$$
$$+ D_j(W)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) + \cdots\right] + D_j D_k(W)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - \cdots\right] + \cdots, \tag{51}$$

where W and $\mathcal{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = v F(\mathrm{x}, u, u_{(1)}, \ldots, u_{(s)}). \tag{52}$$

In order to apply Theorem 3 to our equation we perform the following change of notation:

$$(x^1, x^2) = (t, x), \quad (\xi^1, \xi^2) = (\tau, \xi), \quad (T^1, T^2) = (T^t, T^x).$$

Subcase 2.1. In this case, we consider Eq. (1) with $Q(t) = 0$, $A(t)$, $B(t)$ and $C(t)$ given by (32), (11) and (31), respectively. Consequently, the equation admits the following generator

$$v = (k_1 x + \beta(t))\partial_x + \left(k_4 e^{-kt} + \frac{3k_1}{k}\right)\partial_t + ((k_1 + k_3)u + k_2)\partial_u, \tag{53}$$

where $\beta(t)$ is given by (33). From Theorem 2, we have that φ is given by

$$\varphi = c_1 u + c_2. \tag{54}$$

Thus, we obtain conservation law (45) with the conserved vector

$$\begin{aligned} T^t &= (k_3\, u + k_2)(c_1\, u + c_2) + k_1\, u\left(\frac{3}{2}\, c_1\, u + 2\, c_2\right), \\ T^x &= \frac{e^{kt}}{12\,(k_3 + k_1)}(6\, b_0\, c_1\, (2\, u\, u_{xx} - u_x^2)(2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2) + 12\, b_0\, u_{xx}(c_2\, (k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2) + c_1\, (k_3 + k_1)) \\ &\quad + f_1^{-\frac{k_3}{3k_1}-1}(4\, a_0\, c_1\, u^3\, (2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2) + 6\, a_0\, c_2\, u^2\, (k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2) \\ &\quad + 6\, a_0\, c_1\, k_2\, u^2\, (k_3 + k_1)) + f_1^{-\frac{2k_3}{3k_1}-\frac{4}{3}}(3\, c_0\, c_1\, u^4\, (2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2) + 4\, c_0\, c_1\, k_1\, u^3\, (5\, k_3 + 7\, k_2) \\ &\quad + 4\, c_0\, c_2\, u^3\, (k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2) + 12\, c_0\, k_2\, u^2\, (c_2\, k_3 + c_1\, k_2 + 2\, c_2\, k_1))). \end{aligned} \tag{55}$$

Subcase 2.2. Now, we consider Eq. (1) with $Q(t) = 0$, $A(t)$, $B(t)$ and $C(t)$ given by (35), (20) and (34), respectively. Consequently, the equation admits the following generator

$$v = (k_1 x + \beta(t))\partial_x + \tau\, \partial_t + ((k_1 + k_3)u + k_2)\partial_u, \tag{56}$$

where $\tau(t)$ and $\beta(t)$ are given by (20) and (36), respectively. As in the previous case, φ is given by (54). Thus, we obtain conservation law (45) with the conserved vector

$$\begin{aligned} T^t &= (k_3\, u + k_2)(c_1\, u + c_2) + k_1\, u\left(\frac{3}{2}\, c_1\, u + 2\, c_2\right), \\ T^x &= \frac{1}{12\,(k_3 + k_1)}\Big(6\, b_0\, c_1\, \big(2\, u\, u_{xx} - u_x^2\big)\big(2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2\big) + 12\, b_0\, u_{xx}\big(c_2\, \big(k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2\big) + c_1\, (k_3 + k_1)\big) \\ &\quad + \tau^{-\frac{k_3}{3k_1}-1}\big(4\, a_0\, c_1\, u^3\, \big(2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2\big) + 6\, a_0\, c_2\, u^2\, \big(k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2\big) + 6\, a_0\, c_1\, k_2\, u^2\, (k_3 + k_1)\big) \\ &\quad + \tau^{-\frac{2k_3}{3k_1}-\frac{4}{3}}\big(3\, c_0\, c_1\, u^4\, \big(2\, k_3^2 + 5\, k_1\, k_3 + 3\, k_1^2\big) + 4\, c_0\, c_1\, k_1\, u^3\, (5\, k_3 + 7\, k_2) \\ &\quad + 4\, c_0\, c_2\, u^3\, \big(k_3^2 + 3\, k_1\, k_3 + 2\, k_1^2\big) + 12\, c_0\, k_2\, u^2\, (c_2\, k_3 + c_1\, k_2 + 2\, c_2\, k_1)\big)\Big). \end{aligned}$$

We remark that some of these conservation laws yield conserved integrals with physical meaning. Setting in (55) $k_1 = 1$, $c_2 = \frac{1}{2}$ and $k_2 = k_3 = c_1 = 0$ we have

$$C_1 = \int_{-\infty}^{\infty} u\, dx,$$

which is the conserved mass for Eq. (1). The conserved integral arising from (55) setting $k_1 = 1$, $c_1 = \frac{2}{3}$ and $k_2 = k_3 = c_2 = 0$ gives the energy

$$C_2 = \int_{-\infty}^{\infty} u^2 dx,$$

for Eq. (1).

Another powerful application of conservation laws taking into account the relationship between Lie symmetries and conservation laws it is the so called double reduction method given by Sjöberg [17]. This method allow us to reduce the Gardner equation to a second order ordinary differential equation. Sjöberg introduced this method in order to get solutions of a qth partial differential Eq. (37) from the solutions of an ordinary differential equation of order $q-1$. This method can be applied when a symmetry $\mathbf{v}$ is associated to a conserved vector T. In accordance with the definition given by Kara and Mahomed [12] we will establish that a symmetry $\mathbf{v}$ is associated to T if the following equation holds

$$\mathbf{v}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0. \tag{57}$$

In the terms of the canonical variables r, s and w, symmetry (2) becomes a translation on s, $\mathbf{v} = \frac{\partial}{\partial s}$. Thus, the conservation law can be rewritten as

$$D_s T^s + D_r T^r = 0, \tag{58}$$

with

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \tag{59}$$

and

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}. \tag{60}$$

Due to the fact that solutions of the Eq. (37) written in canonical variables must be invariant with respect to **v** and T is associated with **v**, equation (58) becomes

$D_r T^r = 0,$

so that

$$T^r(r, w, w_r, w_{rr}, \dots, w_{r^{q-1}}) = k, \quad k = const. \tag{61}$$

We stress that Eq. (61) is an ordinary differential equation of order $q-1$, whose solutions are solutions of Eq.(37), by writing this solution in terms of x, t and u.

In order to show the above explained procedure, let us consider the equation

$$F \equiv u_t + uu_x + u^2 u_x + u_{xxx} = 0, \tag{62}$$

It can be easily chequed that Eq. (62) admits the following generator

$$\mathbf{v} = \left(k_1 x - \frac{k_1}{2} t\right)\partial_x + (3 k_1 t + k_2)\partial_t + \left(-k_1 u - \frac{k_1}{2}\right)\partial_u, \tag{63}$$

where k_1 and k_2 are arbitrary constants. From theorem (50), one can get the following conserved vector for generator (63)

$$\begin{aligned} T^t &= -\frac{c_1 k_1}{2} u^2 - \frac{c_1 k_1}{2} u - \frac{c_2 k_1}{2}, \\ T^x &= -c_1 k_1 u u_{xx} - \frac{c_1 k_1}{2} u_{xx} + \frac{c_1 k_1}{2} u_x^2 - \frac{c_1 k_1}{4} u^4 - \frac{c_1 k_1}{2} u^3 - \frac{c_1 k_1}{4} u^2, \end{aligned} \tag{64}$$

where c_1 and c_2 are arbitrary constants. The conserved vector (64) is not associated to symmetry (63). It can be easily seen that generator

$$\mathbf{v} = c\,\partial_x + \partial_t.$$

is associated to (64). The canonical coordinates are

$$r = x - ct, \quad s = t, \quad w = u.$$

We suppose without loss of generality that $k_1 = c_1 = 1$ and $c_2 = 0$. From (60) we get

$$T^r = \frac{1}{4}((4w + 2) w_{rr} - 2 w_r^2 + w^4 + 2 w^3 + (1 - 2c) w^2 - 2cw). \tag{65}$$

Setting $T^r = \frac{k}{4}$, $k =$ const., we obtain

$$(4w + 2) w_{rr} - 2 w_r^2 - 2cw + (1 - 2c) w^2 + 2 w^3 + w^4 = k. \tag{66}$$

Due to the fact that (66) is an autonomous equation the substitution $w_r = p(w)$ yields the following first order ordinary differential equation

$$p' = \frac{k + 2p^2 + 2cw + (2c - 1) w^2 - 2w^3 - w^4}{4w + 2},$$

whose solutions are solutions of (62) once written in terms of x, t and u.

6. Conclusions

In this paper we have considered a generalized variable-coefficient Gardner equation. Classical symmetries of Eq. (1) have been obtained involving different arbitrary functions which can be used to determine similarity and exact solutions. Symmetries obtained in this work generalize those already obtained by other authors in equations which belong to the family of Eq. (1), such as KdV equation and other Gardner equations with time-dependent coefficients. In particular, we have considered two different cases, which in turn would lead to different subcases.

We have determined the subclasses of Eq. (1) which are nonlinearly self-adjoint, as well as the multipliers, of Anco and Bluman method. We have derived conservation laws by using both methods. We have shown that some of these conservation laws yields conserved integrals with physical meaning, such as mass and energy. Finally, as an example of another application of the conserved vectors, we have applied the double reduction method to get exact solutions of the Gardner equation from solutions of a second order reduced ordinary differential equation.

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