Coloring, location and domination of corona graphs

I. González Yero[1], D. Kuziak[2] and A. Rondón Aguilar[3]

[1]Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras
Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain.
ismael.gonzalez@uca.es
[2]Faculty of Applied Physics and Mathematics
Gdańsk University of Technology, ul. Narutowicza 11/12 80-233 Gdańsk, Poland.
dkuziak@mif.pg.gda.pl
[3]Filial Universitaria Municipal Media Luna, Universidad de Granma,
Carretera de Manzanillo Km. 1, 87700 Media Luna, Granma, Cuba.
arondona@udg.co.cu

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Abstract

A vertex coloring of a graph G is an assignment of colors to the vertices of G such that every two adjacent vertices of G have different colors. A coloring related property of a graphs is also an assignment of colors or labels to the vertices of a graph, in which the process of labeling is done according to an extra condition. A set S of vertices of a graph G is a dominating set in G if every vertex outside of S is adjacent to at least one vertex belonging to S. A domination parameter of G is related to those structures of a graph satisfying some domination property together with other conditions on the vertices of G. In this article we study several mathematical properties related to coloring, domination and location of corona graphs.

We investigate the distance-k colorings of corona graphs. Particularly, we obtain tight bounds for the distance-2 chromatic number and distance-3 chromatic number of corona graphs, throughout some relationships between the distance-k chromatic number of corona graphs and the distance-k chromatic number of its factors. Moreover, we give the exact value of the distance-k chromatic number of the corona of a path and an arbitrary graph. On the other hand, we obtain bounds for the Roman dominating number and the locating-domination number of corona graphs. We give closed formulaes for the k-domination number, the distance-k domination number, the independence domination number, the domatic number and the idomatic number of corona graphs.

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1 Introduction

Nowadays the studies about the behavior of several graph parameters in product graphs have become into an interesting topic of research in graph theory. For instance, is it well known the Hedetniemi's coloring conjecture [16, 21] for the categorical product (or direct product), which states that the chromatic number of categorial product graphs is equal to the minimum value between the chromatic numbers of its factors. Also, one of the oldest open problems in domination in graphs is related with product graphs. The problem was presented first by Vizing [26] in 1963. After that he pointed out as a conjecture in [27]. The conjecture states that the domination number of Cartesian product graphs is greater than or equal to the product of the domination numbers of its factors.

A graph labeling is an assignment of labels, traditionally represented by integers or colors, to the edges or vertices, or both, of a graph. Formally, given a graph G, a vertex labeling is a function mapping vertices of G to a set of labels. One of the most popular graphs labeling is the graph coloring, which is an assignment of colors to the vertices or edges, or both, of a graph. For instance, given a set of colors $C = \{c_1, c_2, ..., c_r\}$, a vertex coloring of a graph $G = (V, E)$ is a map $c : V \to C$ such that for every two adjacent vertices $u, v \in V$ it follows $c(u) \neq c(v)$. The minimum value $r = |C|$ for which G has a vertex coloring is called the *chromatic number* of G and it is denoted by $\chi(G)$. Nowadays, there are several kinds of investigations related to vertex colorings of graphs (for example [2, 5, 19]).

Coloring problems in graphs have been related to several number of scheduling problems [20]. For instance, the scheduling problem of assigning aircrafts to flights, the assignments of tasks to time slots or assigning frequency channels to different wireless applications [12]. Moreover, graph colorings can be applied to register allocation [4], pattern matching or some recreational games like the well known puzzles called Sudoku. On the other hand, the chromatic number has been related with several parameters of graphs, and as a consequence, there exists now different types of vertex colorings such as list coloring, total coloring, acyclic coloring, distance-k coloring, etc.

A set S of vertices of a graph G is an *independent set* of G if for every $v \in S$ it is satisfied that $\delta_S(v) = 0$. The minimum cardinality of any independent set in G is called the *independence number* and it is denoted by $\beta_0(G)$. Also, a set S is a *t-dependent set* in G, if for every vertex $v \in S$ it follows that $\delta_S(v) \le k$. Similarly, the minimum cardinality of any k-dependent set in G is the *k-dependence number* and it is denoted by $\beta_k(G)$.

The set of vertices $D \subset V$ is a *dominating set* if for every vertex $v \in \overline{D}$ it is satisfied that $\delta_D(v) \ge 1$ [14]. The minimum cardinality of any dominating of G is the *domination number* of G and it is denoted by $\gamma(G)$. Moreover, the set D is *k-dominating*, $k \ge 2$, if for every vertex $v \in \overline{D}$, it follows $\delta_S(v) \ge k$. The minimum cardinality of any k-dominating set in G is the *k-domination number* and it is denoted by $\gamma_k(G)$. The concept of domination has been related with several structures of the graph, which has led to different kind of domination parameters associated to some extra conditions. In this sense, some of the most popular cases are the independent dominating sets, connected dominating sets, convex dominating sets, distance-k dominating sets, domatic partitions, etc. For general notation and terminology in domination we follow the books [14, 15].

We begin by establishing the principal terminology and notation which we will use throughout the article. Hereafter $G = (V, E)$ represents a undirected finite graph without loops and multiple edges with set of vertices V and set of edges E. The order of G is $|V| = n(G)$ and the size $|E| = m(G)$ (If there is no ambiguity we will use only n and m).

We denote two adjacent vertices $u, v \in V$ by $u \sim v$ and in this case we say that uv is an edge of G or $uv \in E$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$ and the degree of v in X is denoted by $\delta_X(v) = |N_X(v)|$. In the case $X = V$ we will use only $N(v)$, which is also called the open neighborhood of a vertex $v \in V$, and $\delta(v)$ to denote the degree of v in G. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degrees of G are denoted by δ and Δ, respectively. The subgraph induced by $S \subset V$ is denoted by $\langle S \rangle$ and the complement of the set S in V is denoted by $\overline{S}$. The distance between two vertices $u, v \in V$ of G is denoted by $d_G(u, v)$ (or $d(u, v)$ if there is no ambiguity). The diameter of a graph is the maximum of the distances between any two vertices of G and it is denoted by $D(G)$. Given a vertex v of G, we denote by $M_t[v]$ the set of vertices of G whose distance to the vertex v is less than or equal to t, *i.e.* $M_t[v] = \{u \in V \ : \ d(u, v) \leq t\}$. Throughout the article, given the set of colors C, we will refer to the map $c : V \rightarrow C$ as a distance-k coloring of vertices of G.

The corona product graph (corona graph, for short) of two graphs was introduced first by Frucht and Harary in [10]. After that many works have been devoted to study its structure and to obtain some relationships between the corona graph and its factors [1, 6, 10, 17, 18, 28].

Let G and H be two graphs of order n_1 and n_2, respectively. The corona graph $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. Hereafter, we will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the i^{th} copy of H in $G \odot H$.

2 Distance-k coloring

A distance-k coloring of a graph G is an assignment of colors to the vertices of G such that every two different vertices a, b of G have different colors if the distance between a and b is less than or equal to k [9, 11, 12]. The minimum number of colors in a distance-k coloring of G is the *distance-k chromatic number* of G and it is denoted by $\chi_{\leq k}(G)$. Notice that the case $k = 1$ corresponds to the standard well known vertex coloring.

We begin this section by presenting the following almost straightforward result relative to the distance-1 chromatic number (*i.e.* the standard chromatic number) of any corona graph $G \odot H$ and further we will analyze the distance-k chromatic number of $G \odot H$, with $2 \leq k \leq D(G \odot H)$.

Remark 1. *For any graphs G and H,*

$$\chi(G \odot H) = \max\{\chi(G), \chi(H) + 1\}.$$

Proof. Since every vertex $u \in V_i$ is adjacent to $v_i \in V$ we obtain that $\chi(G \odot H) \geq \chi(H) + 1$. Also, it is clear that $\chi(G \odot H) \geq \chi(G)$.

On the other hand, let $t = \max\{\chi(G), \chi(H) + 1\}$ and let us color the vertices v_i, $i \in \{1, ..., n_1\}$, of G by using t different colors $c_1, c_2, ..., c_t$. Now, if $c(v_i) = c_i$, then the copy H_i of H can be colored by using the set of colors $c_1, c_2, ..., c_{i-1}, c_{i+1}, ..., c_t$. Therefore, we obtain that $\chi(G \odot \mathrm{H}) \leq t = \max\{\chi(G), \chi(H) + 1\}$ and the result follows. □

Remark 2. *Let G be a graph of order n and let $k \geq 1$ be an integer. Then $\chi_{\leq k}(G) = n$ if and only if $D(G) \leq k$.*

Proof. If $D(G) \leq k$, then for every two different vertices u, v of G we have that $d(u, v) \leq D(G) \leq k$. So, u and v have different colors in any distance-k coloring of G and, as a consequence, $\chi_{\leq k}(G) = n$.

On the other side, let G be a graph such that $\chi_{\leq k}(G) = n$. Let us suppose $D(G) > k$ and let a, b be two different vertices of G such that $d(a, b) = D(G)$. Let $C = \{c_1, c_2, ..., c_{n-1}\}$ be a set of $n-1$ colors. Hence, we can color the set of vertices of $G - \{b\}$ with the $n-1$ colors in C. Now, as $k < D(G) = d(a, b)$ we can color vertex b by using the color of vertex a. Thus G can be colored with $n-1$ colors, which is a contradiction Therefore, we have that $D(G) \leq k$. □

As a consequence of the above remark, from now on we will focus on the cases $2 \leq k \leq D(G) - 1$. Also, as for every graph G and H we have that $D(G \odot H) = D(G) + 2$ we are interested here in obtaining the distance-k chromatic number of corona graphs $G \odot H$ for $2 \leq k \leq D(G) + 1$.

Theorem 3. *Let G be a graph of maximum degree Δ_1 and let H be a graph of order n_2. Then,*

$$\Delta_1 + n_2 + 1 \leq \chi_{\leq 2}(G \odot H) \leq \chi_{\leq 2}(G) + n_2.$$

Proof. The lower bound is a direct consequence of Theorem 22 (See Appendix) by taking into account that the maximum degree of $G \odot H$ is $n_2 + \Delta_1$.

On the other hand, let $t = \chi_{\leq 2}(G) + n_2$ and let $C = \{c_1, c_2, ..., c_t\}$ be a set of pairwise distinct colors. Let us color the vertices of G by using $\chi_{\leq 2}(G)$ different colors and let us suppose that for $v_i \in V$ we have $c(v_i) = c_i$. Hence, for every distinct vertices $v_j, v_l \in N_V[v_i]$ we have $c(v_j) \neq c(v_l)$. Also, as $\chi_{\leq 2}(G) \geq \delta_G(v_i) + 1$ for every $v_i \in V$ we obtain that

$$n_2 = t - \chi_{\leq 2}(G) \leq t - (\delta_G(v_i) + 1).$$

Now, let $C_i = \{c_{i_1}, c_{i_2}, ..., c_{i_r}\}$ be such that for every $c_{i_j} \in C_i$ there exists $v_j \in N_V[v_i]$ with $c(v_j) = c_{i_j}$. Since $|C - C_i| = t - (\delta_G(v_i) + 1) \geq n_2$ we obtain that the vertices of the copy H_i of H can be colored with the colors in $C - C_i$. Therefore, $\chi_{\leq 2}(G \odot H) \leq t = \chi_{\leq 2}(G) + n_2$. □

The following corollary shows that the above bounds are tight.

Corollary 4. *Let H be any graph of order n_2. Then,*

(i) *If $n_1 \geq 3$, then $\chi_{\leq 2}(P_{n_1} \odot H) = n_2 + 3$.*

(ii) *For any positive integer t, $\chi_{\leq 2}(C_{3t} \odot H) = n_2 + 3$.*

(iii) *For any tree T of maximum degree Δ_1, $\chi_{\leq 2}(T \odot H) = n_2 + \Delta_1 + 1$.*

Notice that, for instance, if G is a cycle of order $3t + 1$ or $3t + 2$, with $t \geq 1$ an integer, then $\chi_{\leq 2}(C_{3t+1}) = 4$ and $\chi_{\leq 2}(C_{3t+2}) = 5$. Thus, we have that $5 = \chi_{\leq 2}(C_{3t+1} \odot N_2) < 6$ and $5 = \chi_{\leq 2}(C_{3t+2} \odot N_2) < 6$. Also, for the complete bipartite graph $K_{s,t}$, with $2 < s \leq t$, we have that $\chi_{\leq 2}(K_{s,t}) = s + t$ and $t + 2 < \chi_{\leq 2}(K_{s,t} \odot K_1) = s + t < s + t + 1$.

Theorem 5. *Let G be a graph of minimum and maximum degree δ_1 and Δ_1, respectively and let H be a graph of order n_2. Then*

$$\chi_{\leq 3}(G \odot H) \leq \chi_{\leq 3}(G) + n_2(\Delta_1 + 1).$$

Moreover, if G is triangle free, then

$$\chi_{\leq 3}(G \odot H) \geq 2n_2 + \Delta_1 + \delta_1.$$

Proof. Let $t = \chi_{\leq 3}(G) + n_2(\Delta_1 + 1)$ and let $C = \{c_1, c_2, ..., c_t\}$ be a set of pairwise distinct colors. Let us color the vertices of G by using $\chi_{\leq 3}(G)$ different colors and let us suppose that for $v_i \in V$ we have that $c(v_i) = c_i$.

Now, let $C_i = \{c_{i_1}, c_{i_2}, ..., c_{i_r}\}$ be such that for every $c_{i_j} \in C_i$ there exists $v_j \in \bigcup_{v_l \in N_V[v_i]} N_V[v_l]$ with $c(v_j) = c_{i_j}$. Thus, $|C_i| \leq \chi_{\leq 3}(G)$ and we have that

$$|C - C_i| = t - |C_i| \geq t - \chi_{\leq 3}(G) = n_2(\Delta_1 + 1).$$

So, we obtain that the vertices of the $\Delta + 1$ copies of H corresponding to the vertices of G in $N_V[v_i]$ can be colored with the colors in $C - C_i$. Also, if $v \in V_l$ such that $v_l \notin N_V[v_i]$, then there exists a vertex $u \in V_r$, with $v_r \in N_V[v_i]$, such that $d_{G \odot H}(u, v) > 3$. Thus, v can be colored by using one color from the set of colors in $C - C_i$. Therefore, $\chi_{\leq 3}(G \odot H) \leq t = \chi_{\leq 3}(G) + n_2(\Delta_1 + 1)$.

On the other hand, let us suppose G is triangle free. Let $v_i \in V$ be a vertex of maximum degree in G and let $v_j \in N_V[v_i]$, $j \neq i$. Hence, for every two different vertices $u, v \in (V_i \cup V_j) \cup (N_V[v_i] \cup N_V[v_j])$ we have $d_{G \odot H}(u, v) \leq 3$. So, we have that $c(u) \neq c(v)$. Thus, we obtain that

$$\chi_{\leq 3}(G \odot H) \geq |(V_i \cup V_j) \cup (N_V[v_i] \cup N_V[v_j])| \geq 2n_2 + \Delta_1 + \delta_1.$$

□

Notice that the above bounds are tight. For instance, if H is a graph of order n_2, then the lower bound is achieved for the case of $C_4 \odot H$, where we have $\chi_{\leq 3}(C_4 \odot H) = 2n_2 + 4$. Moreover, the upper bound is tight for the corona graph $K_{n_1} \odot H$, in which case it is satisfied that $\chi_{\leq 3}(K_{n_1} \odot H) = n_1 n_2 + n_1$. Next we study the distance-k chromatic number of some particular cases of corona graphs.

Proposition 6. *Let H be a graph of order n_2 and let $T = (V, E)$ be a tree. Let $v_i, v_j \in V$ such that $\Delta_{ij}(T) = \delta(v_i) + \delta(v_j) = \max\{\delta(v_l) + \delta(v_r) \ : \ v_l, v_r \in V, \ v_l \sim v_r\}$. Then*

$$\chi_{\leq 3}(T \odot H) = 2n_2 + \Delta_{ij}(T).$$

Proof. Let $B = V_i \cup V_j \cup N_V[v_i] \cup N_V[v_j]$ be the set of vertices of $T \odot H$. Since T is a tree we have that $(N_V[v_i] - \{v_j\}) \cap (N_V[v_j] - \{v_i\}) = \emptyset$. Thus, $|B| = |V_i| + |V_j| + |N_V[v_i]| + |N_V[v_j]| - 2$. Also, for every two different vertices $a, b \in B$ we have that $d_{T \odot H}(a, b) \leq 3$ and, as a consequence, we obtain that $c(a) \neq c(b)$. Therefore,

$$\begin{aligned} \chi_{\leq 3}(T \odot H) &\geq |B| \\ &= |V_i| + |V_j| + |N_V[v_i]| + |N_V[v_j]| - 2 \\ &= 2n_2 + \delta(v_i) + \delta(v_j) \\ &= 2n_2 + \Delta_{ij}(T). \end{aligned}$$

On the other hand, let $t = 2n_2 + \Delta_{ij}(T)$ and let $C = \{c_1, c_2, ..., c_t\}$ be a set of pairwise distinct colors. Let us color the set of vertices of T by using the minimum number of colors from the set C and let us suppose that $c(v_i) = c_i$ and $c(v_j) = c_j$. Now, let $C_{ij} = \{c_{ij_1}, c_{ij_2}, ...c_{ij_r}\} \subset C$ be such that for every $c_{ij_l} \in C_{ij}$ there exists $a \in N_V[v_i] \cup N_V[v_j]$ with $c(a) = c_{ij_l}$. Notice that $|C_{ij}| = \Delta_{ij}(T)$ and also, any vertex belonging to $V_i \cup V_j$ can be colored by using the colors in $C - C_{ij}$.

Now, if $v_q \notin N_V[v_i] \cup N_V[v_j]$ there exists a vertex $a \in N_V[v_i] \cup N_V[v_j]$ such that $d_{T\odot H}(a, v_q) > 3$. Thus, v_q can be colored by using one of the colors in C_{ij}. Also, if $b \in V_f$, with $f \neq i, j$, then there exist a vertex $b' \in V_i \cup V_j$ such that $d_{T\odot H}(b', b) > 3$. Thus, b' can be colored by using one color from the set of colors in $C - C'$. Therefore, we have that $T \odot H$ can be colored with t colors. As a consequence, $\chi_{\leq 3}(T \odot H) \leq t = 2n_2 + \Delta_{ij}(T)$ and the result follows. □

Proposition 7. *Let H be a graph of order n_2 and let $n_1 \geq 2$. Then for every $2 \leq k \leq n_1$,*

$$\chi_{\leq k}(P_{n_1} \odot H) = \begin{cases} n_2(k-1) + k + 1, & \text{if } k \leq n_1 - 1, \\ n_2(k-1) + k, & \text{if } k = n_1. \end{cases}$$

Proof. Let us suppose $k \leq n_1 - 1$ and let $P_{n_1} = v_1 v_2 ... v_{n_1}$ in $P_{n_1} \odot H$. Hence, there exists a vertex v_i of degree two in P_{n_1} such that v_{i+k-2} has degree two and for every two different vertices $a, b \in A = \{v_{i-1}, v_i, v_{i+1}, ..., v_{i+k-2}, v_{i+k-1}\}$ we have that $d_{P_{n_1}\odot H}(a, b) \leq k$. Thus, $c(a) \neq c(b)$. Now, let $B = (\bigcup_{j=i}^{i+k-2} V_j) \cup A$. Hence, for every $a, b \in B$ we have that $d_{P_{n_1}\odot H}(a, b) \leq k$. Thus, $c(a) \neq c(b)$. Therefore,

$$\chi_{\leq k}(P_{n_1} \odot H) \geq |B| = |A| + \left| \bigcup_{j=i}^{i+k-2} V_j \right| = k + 1 + n_2(k-1).$$

On the other hand, let $t = k + 1 + n_2(k-1)$ and let $C = \{c_1, c_2, ..., c_t\}$ be a set of pairwise distinct colors. Now, let $Q = \{v_i, v_{i+1}, ..., v_{i+k-1}, v_{i+k}\}$ be any $k+1$ consecutive vertices in P_{n_1}. Hence, the vertices in Q can be colored by using $k+1$ different colors of C. Now, let $C' \subset C$ be such that for every $c_l \in C'$ there exists $v \in Q$ with $c(v) = c_l$. Hence, since $|C - C'| = n_1(k-1)$ we obtain that the vertices in $\bigcup_{j=1}^{k-1} V_{i+j}$ can be colored by using the colors in $C - C'$. Now, if $v_r \notin Q$, then there exists a vertex $v_q \in Q$ such that $d_{P_{n_1}\odot H}(v_r, v_q) > k$. So, v_r can be colored by using a color from the set C'. Also, if $u \in V_f$ such that $v_f \notin Q - \{v_i, v_{i+k}\}$, then there exists a vertex $v \in V_y$ such that $v_y \in Q - \{v_i, v_{i+k}\}$ for which $d_{P_{n_1}\odot H}(u, v) > k$. So, u can be colored by using a color from the set $C - C'$. Therefore, $\chi_{\leq t}(P_{n_1} \odot H) \leq t = k + 1 + n_2(k-1)$ and the result follows.

Now, let us suppose that $k = n_1$. So, for every different vertices v_i, v_j of P_{n_1} in $P_{n_1} \odot H$ we have that $c(v_i) \neq c(v_j)$. Now, let $B = (\bigcup_{i=1}^{n_1-1} V_i) \cup V$. Hence, we have that, for every two different vertices $a, b \in B$, $d_{P_{n_1}\odot H}(a, b) \leq k$. Thus, $c(a) \neq c(b)$ and, as a consequence,

$$\chi_{\leq k}(P_{n_1} \odot H) \geq |B| = n_1 + \left| \bigcup_{i=1}^{n_1-1} V_i \right| = n_1 + n_2(n_1 - 1) = n_2(k-1) + k.$$

On the other hand, let $t = n_2(k-1) + k$ and let $C = \{c_1, c_2, ..., c_t\}$. Since $k = n_1$ we can color the set of vertices of P_{n_1} by using k colors. Let C' be the set of colors used to color the set V. Since the distance between the vertices in V_1 and V_{n_1} is greater than k, these sets can be assigned the same colors and the rest of the vertices in $\bigcup_{i=1}^{n_1-1} V_i$ can be colored by using the colors in $C - C'$. Therefore, we obtain that $\chi_{\leq k}(P_{n_1} \odot H) \leq t = n_2(k-1) + k$ and the result follows. □

3 Roman domination

The concepts about Roman domination were introduced first by Steward in [25] and studied further by some authors, for instance we mention [8]. A map $f : V \to \{0, 1, 2\}$ is a *Roman* *dominating function* for a graph G if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of a Roman dominating function is given by $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and it is denoted by $\gamma_R(G)$. In this section we study the Roman domination number of corona graphs.

Let f be a Roman dominating function on G and let $\Pi(G) = \{B_0, B_1, B_2\}$ be the ordered partition of the vertices of G induced by f, where $B_i = \{v \in V \ : \ f(v) = i\}$ and let $b_i(G) = |B_i|$, with $i \in \{0, 1, 2\}$. Frequently, a Roman dominating function f is represented by its induced partition $\{B_0, B_1, B_2\}$. It is clear that for any Roman dominating function f on the graph $G = (V, E)$ of order n we have that $f(V) = \sum_{u \in V} f(u) = 2b_2(G) + b_1(G)$ and $b_0(G) + b_1(G) + b_2(G) = n$. The following lemma obtained in [8] will be useful into proving some of the results in this section.

Lemma 8. [8] *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Theorem 9. *Let G and H be two graphs of order n_1 and $n_2 \geq 2$, respectively. Then,*

$$\gamma_R(G \odot H) = 2n_1.$$

Proof. Let f be a Roman dominating function on $G \odot H$ and let v_i be a vertex of G. We have the following cases.

Case 1: $f(v_i) = 0$ or $f(v_i) = 1$. If there is a vertex $v \in V_i$, such that $f(v) = 0$, then there exists other vertex $x \in V_i$ with $x \sim v$ and $f(x) = 2$. On the contrary, if for every $u \in V_i$ we have that $f(u) \neq 0$, then $f(u) = 1$ or $f(u) = 2$ for any vertex $u \in V_i$. As a consequence, since $n_2 \geq 2$, in both cases we have that $f(V_i \cup \{v_i\}) = \sum_{u \in V_i \cup \{v_i\}} f(u) \geq 2$.

Case 2: $f(v_i) = 2$. It is clear that $f(V_i \cup \{v_i\}) = \sum_{u \in V_i \cup \{v_i\}} f(u) \geq 2$.

Thus, we obtain that

$$\gamma_R(G \odot H) = \sum_{v \in V \cup \left(\cup_{i=1}^{n_1} V_i\right)} f(v) = \sum_{i=1}^{n_1} \sum_{v \in V_i \cup \{v_i\}} f(v) \geq 2n_1.$$

On the other side, since $\gamma(G \odot H) = n_1$, by Lemma 8 we have that

$$\gamma_R(G \odot H) \leq 2\gamma(G \odot H) = 2n_1.$$

Therefore, the result follows. □

Next we analyze the corona graph $G \odot K_1$.

Theorem 10. *Let G be a graph of order n. Then there exists a Roman dominating function $\{B_0, B_1, B_2\}$ of minimum weight such that*

$$\gamma_R(G \odot K_1) = \gamma_R(G) + n - b_2(G).$$

Proof. Let f' be a Roman dominating function on G of minimum weight. Let v_i be a vertex of G and we will denote by u_i the pendant vertex of v_i corresponding to the i^{th} copy of K_1 in $G \odot K_1$. Hence, we define a function f on $G \odot K_1$ in the following way:

- For every $v_i \in V$, we have that $f(v_i) = f'(v_i)$.
- If $f'(v_i) = 0$ or $f'(v_i) = 1$ for some $i \in \{1, ..., n\}$, then $f(u_i) = 1$.

- If $f'(v_j) = 2$ for some $j \in \{1, ..., n\}$, then $f(u_j) = 0$.

Thus, it is clear that f is a Roman dominating function for $G \odot K_1$ and the weight of f is given by

$$\begin{aligned} f(V \cup (\cup_{i=1}^n \{u_i\})) &= \sum_{u \in V \cup (\cup_{i=1}^n \{u_i\})} f(u) \\ &= f'(V) + f(\cup_{i=1}^n \{u_i\}) \\ &= 2b_2(G) + b_1(G) + f\left(\cup_{i=1}^n \{u_i\}\right) \\ &= 2b_2(G) + 2b_1(G) + b_0(G). \end{aligned}$$

Now, since $\gamma_R(G) = 2b_2(G) + b_1(G)$ and $b_0(G) + b_1(G) + b_2(G) = n$ we obtain that

$$\gamma_R(G \odot K_1) \leq f(V \cup (\cup_{i=1}^n \{u_i\})) = \gamma_R(G) + n - b_2(G).$$

On the other side, let h be a Roman dominating function on $G \odot K_1$ of minimum weight and let h' be a function on G such that for every $v_i \in V$ we have that $h'(v_i) = h(v_i)$. Now, if $h'(v_l) = 0$, for some $l \in \{1, ..., n\}$, then as for every vertex u_i, $i \in \{1, ..., n\}$, it is satisfied that $h(u_i) \neq 2$, there exists $v_j \in N_G(v_l)$, $j \neq l$, such that $h(v_j) = 2$. So, $h'(v_j) = 2$ and h' is a Roman dominating function on G.

Moreover, we have the following facts:

- If $h(u_j) = 0$ for some $j \in \{1, ..., n\}$, then $h(v_i) = h'(v_i) = 2$.
- If $h(u_l) = 1$ for some $l \in \{1, ..., n\}$, then $h(v_l) = h'(v_l) = 0$ or $h(v_l) = h'(v_l) = 1$.

Thus, we obtain that

$$\begin{aligned} \gamma_R(G \odot K_1) &= \sum_{u \in V \cup (\cup_{i=1}^n \{u_i\})} h(u) \\ &= \sum_{u \in V} h(u) + \sum_{u \in \cup_{i=1}^n \{u_i\}} h(u) \\ &= \sum_{u \in V} h'(u) + \sum_{u \in \cup_{i=1}^n \{u_i\}} h(u) \\ &\geq \gamma_R(G) + \sum_{u \in \cup_{i=1}^n \{u_i\}} h(u) \\ &= \gamma_R(G) + b_0(G) + b_1(G) \\ &= \gamma_R(G) + n - b_2(G). \end{aligned}$$

Therefore, the result follows. □

Notice that the above result gives a formula for the Roman domination number of $G \odot K_1$, but such a formula depends on $b_2(G)$, which is unknown in general. Thus, the formula leads to the conclusion that obtaining the Roman domination number of $G \odot K_1$ could be very difficult even if it is knew the Roman domination number of G.

Corollary 11. *Let G be a graph of order $n \geq 2$, different from $\overline{K_n}$. Then,*

$$\gamma_R(G) + \frac{n}{2} \leq \gamma_R(G \odot K_1) \leq \gamma_R(G) + n - 1.$$

Proof. The result follows from the above theorem and the fact that $1 \leq b_2(G) \leq \frac{n}{2}$. □

Notice that if G is the star graph $S_{1,n}$, then the upper bound of the above theorem is attained.

4 Location-domination

The concepts about resolvability and location in graphs were introduced independently by Harary and Melter [13] and Slater [22], respectively, to define the same structure in a graph. In the present work we will use the terminology of [22]. The domination parameter related to location in graphs can be seen in two different ways as it was presented in [23, 24] and [3], respectively. Given a set $S = \{v_1, v_2, ..., v_t\}$ of vertices of a graph G, we say that S is a *locating set* (or resolving set) if for every two different vertices u, v of G it is satisfied that $(d(u, v_1), d(u, v_2), ..., d(u, v_t)) \neq (d(v, v_1), d(v, v_2), ..., d(v, v_t))$ where $d(x, v_i)$ represents the distance between the vertices x and v_i, for every $x \in \{u, v\}$ and $i \in \{1, ..., t\}$. The minimum cardinality of any locating set of G is called the *location number* (or *metric dimension*) of G and it is denoted by $dim(G)$.

Also, a set D of vertices of G is *locating dominating* (or resolving dominating) if it is locating (or resolving) and dominating. The minimum cardinality of any locating dominating set of G is called the *location domination number* of G and it is denoted by γ_{ld}. On the other hand the set D is *locating-dominating* if for every pair of different vertices $u, v \in \overline{D}$ it is satisfied that $N_D(u) \neq N_D(v)$. The minimum cardinality of any locating-dominating set of G is called the *location-domination number* of G and it is denoted by $\gamma_{l-d}(G)$. In this sense the following inequalities chain is satisfied for any connected graph G.

$$dim(G) \leq \gamma_{ld}(G) \leq \gamma_{l-d}(G).$$

For the case of corona graphs, as it was studied in [31], we have that every locating set is also a locating dominating set of G. Thus, we obtain that $dim(G) = \gamma_{ld}(G)$. So, in the present section we will be centered into studying the locating-dominating number of corona graphs.

Lemma 12. *Let G and H be connected graphs. Let S be a locating-dominating set of minimum cardinality in $G \odot H$ and let v_i be a vertex of G. Then,*

(i) *If $v_i \notin S$, then $S \cap V_i$ is a locating-dominating set in the copy H_i of H in $G \odot H$.*

(ii) *If $v_i \in S$, then $S \cap (V_i \cup \{v_i\})$ is a locating-dominating set in the subgraph $K_1 \odot H_i$.*

Proof. Let $S_i = S \cap V_i$. If $v_i \notin S$, then as every vertex $x \in V_i$ is adjacent to only one vertex not in V_i (the vertex v_i) it is satisfied that $N_S(x) = N_{S_i}(x)$. So, for every two different vertices $u, v \in \overline{S_i}$ in H_i we have that $N_{S_i}(u) \neq N_{S_i}(v)$ and, as a consequence, (i) follows.

Now, let us suppose that $v_i \in S$ and let $S'_i = S \cap (V_i \cup \{v_i\})$. Let v_i be the vertex of K_1. Hence, for every vertex $x \in \overline{S'_i}$ in $K_1 \odot H_i$ it is satisfied that $N_{S'_i}(x) = \{v_i\} \cup (S \cap V_i) = N_S(x)$. Thus, for every two different vertices $u, v \in \overline{S'_i}$ we have

$$N_{S'}(u) = \{v_i\} \cup (S \cap V_i) = N_S(u) \neq N_S(v) = \{v_i\} \cup (S \cap V_i) = N_{S'}(v).$$

Thus, (ii) follows. □

Lemma 13. *For any graph H,*

(i) *If there exist a locating-dominating set A of minimum cardinality in H such that for every vertex $v \in \overline{A}$ it is satisfied that $N_A(v) \subsetneq A$, then $\gamma_{l-d}(K_1 \odot H) = \gamma_{l-d}(H)$.*

(ii) *If for any locating-dominating set B of minimum cardinality in H there exists a vertex $u \in \overline{B}$ such that $N_B(u) = B$, then $\gamma_{l-d}(K_1 \odot H) = \gamma_{l-d}(H) + 1$.*

Proof. Let S be a locating-dominating set of minimum cardinality in $K_1 \odot H$ and let v be the vertex of K_1. Now, if there exist a locating-dominating set A of minimum cardinality in H such that for every vertex $v \in \overline{A}$, $N_A(v) \subsetneq A$, then since $v \sim u$ for every vertex u of H, it is satisfied that $v \notin S$. So, for any two different vertices $x, y \in \overline{S} - \{v\}$ we have that $N_S(x) \neq N_S(y)$. Thus, S is a locating-dominating set in H and $\gamma_{l-d}(K_1 \odot H) \geq \gamma_{l-d}(H)$. Now, if there exist a locating-dominating set A of minimum cardinality in H such that for every vertex $v \in \overline{A}$ it is satisfied that $N_A(v) \subsetneq A$, then it is clear that A is also a locating-dominating set in $K_1 \odot H$. So, $\gamma_{l-d}(K_1 \odot H) \leq \gamma_{l-d}(H)$ and (i) follows.

On the other side, let us suppose that for every locating-dominating set B of minimum cardinality in H there exists a vertex $u \in \overline{B}$ such that $N_B(u) = B$. Since for the vertex v of K_1 it is satisfied that $N_B(v) = B$ we obtain that any locating-dominating set of minimum cardinality in $K_1 \odot H$ must contain the set B and either the vertex v or the other vertex u of H such that $N_B(u) = B$. So, $\gamma_{l-d}(K_1 \odot H) \geq \gamma_{l-d}(H) + 1$. On the other hand, if B' is a locating-dominating set of minimum cardinality in H such that there exists a vertex $u' \in \overline{B'}$ with $N_{B'}(u') = B'$, then it is easy to check that $B' \cup \{u'\}$ is a locating-dominating set in $K_1 \odot H$. Therefore, $\gamma_{l-d}(K_1 \odot H) \leq \gamma_{l-d}(H) + 1$ and (ii) follows. □

Theorem 14. *For any connected graph G of order n and any connected graph H,*

(i) *If there exist a locating-dominating set A of minimum cardinality in H such that for every vertex $v \in \overline{A}$ it is satisfied that $N_A(v) \subsetneq A$, then $\gamma_{l-d}(G \odot H) = n\gamma_{l-d}(H)$.*

(ii) *If for any locating-dominating set B of minimum cardinality in H there exists a vertex $u \in \overline{B}$ such that $N_B(u) = B$, then $\gamma_{l-d}(G \odot H) = n\gamma_{l-d}(H) + \gamma(G)$.*

Proof. Let S_i, $i \in \{1, ..., n\}$, be a locating-dominating set of minimum cardinality in the copy $H_i = (V_i, E_i)$ of H in $G \odot H$ such that for every vertex $v \in \overline{S_i}$ it is satisfied that $N_{S_i}(v) \subsetneq S_i$. Now, let $S = \bigcup_{i=1}^{n} S_i$ and let $x, y \in \overline{S}$ be two different vertices of $G \odot H$. If x is a vertex of G and y is a vertex of a copy H_i of H in $G \odot H$ with $x \sim y$, then we have that

$$N_S(y) = N_{S_i}(y) \subsetneq S_i = N_S(x).$$

On the contrary, in any other cases for x, y we have that $N_S(y) \neq N_S(x)$. Thus, S is a locating-dominating set in $G \odot H$ and $\gamma_{l-d}(G \odot H) \leq n\gamma_{l-d}(H)$.

On the other hand, let S be a locating-dominating set in $G \odot H$. So, by Lemma 12, for every $i \in \{1, ..., n\}$ we have either $|S \cap V_i| \geq \gamma_{l-d}(H)$ or $|S \cap (V_i \cap \{v_i\})| \geq \gamma_{l-d}(K_1 \odot H)$. Also, by Lemma 13 we have that $\gamma_{l-d}(K_1 \odot H) \geq \gamma_{l-d}(H)$. Therefore,

$$\gamma_{l-d}(G \odot H) = |S| \geq \sum_{i=1}^{n} \gamma_{l-d}(H) = n\gamma_{l-d}(H).$$

Therefore (i) follows. In order to prove (ii), let S'_i, $i \in \{1, ..., n\}$, be a locating-dominating set in the copy H_i of H in $G \odot H$. Hence, there exists a vertex $u \in \overline{S'_i}$ such that $N_{S'_i}(u) = S'_i$. Let D be a dominating set of minimum cardinality in G and let $S' = D \cup (\bigcup_{i=1}^{n} S'_i)$. Now, let $x, y \in \overline{S'}$ such that x is a vertex of G, y is a vertex of a copy H_i of H and $x \sim y$. So, if $y = u$, then $N_{S'}(y) = N_{S'_i}(y)$. Since D is a dominating set in G, there exists at least a vertex $v \in D \subset S$ such that $x \sim v$. Thus, $S_i \cup \{v\} \subseteq N_{S'}(x)$ and as a consequence, $N_{S'}(x) \neq N_{S'}(y)$. On the contrary, in any other case for $x, y \in \overline{S'}$ we have that $N_{S'}(x) \neq N' S(y)$. Therefore, S' is a locating-dominating set in $G \odot H$ and $\gamma_{l-d}(G \odot H) \leq n\gamma_{l-d}(H) + \gamma(G)$.

On the other side, let S' be a locating-dominating set of minimum cardinality in $G \odot H$. For any locating-dominating set L in H there exists a vertex $u \in \overline{L}$ such that $N_L(u) = L$ and also for every $v_i \in V$ in $G \odot H$, $N_L(v_i) = L$. So, for every $v_i \in V$ there is $u_i \in V_i$ such that $N_L(u_i) = N_L(v_i) = L$. Since S' is a locating-dominating set in $G \odot H$ we have either,

- $u_i \in S'$ and $v_i \notin S'$. In such a case by taking the set $S'' = (S' - \{u_i\}) + \{v_i\}$ we have that S'' is also a locating-dominating set of minimum cardinality in $G \odot H$,
- or $v_i \in S'$ and $u_i \notin S'$.

Thus, let A be the set of vertices of G such that for every vertex $v_j \in A$ we have either $v_j \in S$ or there exists $u_j \in V_j$ such that $u_j \in S$ and $N_L(u_j) = L$ for any locating-dominating set L of minimum cardinality in H_j. Thus, A is a dominating set in G and we obtain that

$$\begin{aligned}
|S| &= \sum_{i=1}^{n} (S \cap (V_i \cup \{v_i\})) \\
&= \sum_{i=1}^{|A|} (S \cap (V_i \cup \{v_i\})) + \sum_{j=1}^{n-|A|} (S \cap V_i) \\
&\geq |A|(\gamma_{l-d}(H) + 1) + (n - |A|)\gamma_{l-d}(H) \ \text{(By Lemma 13)} \\
&= |A| + n\gamma_{l-d}(H) \\
&= \gamma(G) + n\gamma_{l-d}(H).
\end{aligned}$$

Therefore, the proof of (ii) is complete. □

5 Other kinds of domination related parameters

A set of vertices D of a graph G is a (*connected*[1], *convex*[2] or *independent*) *dominating set* in G if D is a dominating set and a (connected, convex or independent) set in G. The minimum cardinality of any (connected, convex or independent) dominating set in G is called the (*connected, convex* or *independent*) *domination number* of G and it is denoted by ($\gamma_c(G), \gamma_{con}(G)$ or $i(G)$). A set D is a *distance-k dominating set* in G if for every vertex $v \in \overline{D}$ it follows that $d(u,v) \leq k$ for some $v \in D$, where $d(u,v)$ represents the distance between the vertices u and v. The minimum cardinality of any distance-k dominating set of a graph is called the *distance-k dominating* number of G and it is denoted by $\gamma_{\leq k}(G)$.

There are some domination parameters whose value is very easy to observe for the case of corona graph. For instance, it is clear that $\gamma(G \odot H) = \gamma_c(G \odot H) = \gamma_{con}(G \odot H) = n_1$ and $\beta_0(G \odot H) = n_1\beta_0(H)$. At next we obtain the exact value of some domination related parameters of corona graphs.

Theorem 15. *For any connected graph G of order n and for any graph H, if $k \geq 2$, then*

$$\gamma_k(G \odot H) = n \min\{\gamma_k(H), \gamma_{k-1}(H) + 1\}.$$

[1] D is connected in G if for any two different vertices $u, v \in D$ there exists a path P of length $d(u,v)$ between u and v such that every vertex of P belongs to D.

[2] D is convex in G if for any two different vertices $u, v \in D$ all the vertices of all paths of length $d(u,v)$ between u and v belong to D.

Proof. Let S be a k-dominating set of minimum cardinality in $G \odot H$. If $v_i \in V \cap S$, then for every $v \in \overline{V_i \cap S}$ in H_i we have that $k \leq \delta_S(v) = \delta_{V_i \cap S}(v) + 1$. Thus, $V_i \cap S$ is a $(k-1)$-dominating set in H_i. Also, if $v_j \notin V \cap S$, then for every $v \in \overline{V_j \cap S}$ in H_j we have that $k \leq \delta_S(v) = \delta_{V_j \cap S}(v)$ and we obtain that $V_j \cap S$ is a k-dominating set in H_j. Let $A = S \cap V$. Hence, we have that

$$|S| \geq |A| + |A|\gamma_{k-1}(H) + (n - |A|)\gamma_k(H). \tag{1}$$

Now, if $\gamma_{k-1}(H) = \gamma_k(H)$ or $\gamma_{k-1}(H) = \gamma_k(H) - 1$, then by (1) we obtain that $|S| \geq n\gamma_k(H)$. On the contrary, if $\gamma_{k-1}(H) \leq \gamma_k(H) - 2$ then by (1) we have

$$\begin{aligned}|S| &\geq |A| + |A|\gamma_{k-1}(H) + (n - |A|)\gamma_k(H)\\ &\geq |A| + |A|\gamma_{k-1}(H) + (n - |A|)(\gamma_{k-1}(H) + 2)\\ &= 2n - |A| + n\gamma_{k-1}(H)\\ &\geq 2n - n + n\gamma_{k-1}(H)\\ &= n(\gamma_{k-1}(H) + 1).\end{aligned}$$

Therefore, we obtain that $\gamma_k(G \odot H) \geq n \min\{\gamma_k(H), \gamma_{k-1}(H) + 1\}$.

On the other hand, let A_i be a k-dominating set in H_i, $i \in \{1, ..., n\}$ and let $A = \bigcup_{i=1}^n A_i$. So, for every vertex $v_i \in V$ we have that $\delta_A(v_i) = \delta_{A_i}(v_i) = |A_i| \geq k$. Also, for every vertex $u \in \overline{A_i}$ in H_i we have that $\delta_A(u) = \delta_{A_i}(v) \geq k$. Thus, A is a k-dominating set in $G \odot H$ and, as a consequence, $\gamma_k(G \odot H) \leq n\gamma_k(H)$.

Also, let B_i be a $(k-1)$-dominating set in H_i, $i \in \{1, ..., n\}$ and let $B = (\bigcup_{i=1}^n B_i) \cup V$. So, for every vertex $u \in \overline{B_i}$ in H_i we have that $\delta_B(u) = \delta_{B_i}(u) + 1 \geq k$. Thus, B is a k-dominating set in $G \odot H$ and, as a consequence, $\gamma_k(G \odot H) \leq n + n\gamma_{k-1}(H) = n(\gamma_{k-1}(H) + 1)$.

Therefore, $\gamma_k(G \odot H) \leq n \min\{\gamma_k(H), (\gamma_{k-1}(H) + 1)\}$ and the result follows. □

Theorem 16. *For any connected graph G and any graph H, if $k \geq 2$, then*

$$\gamma_{\leq k}(G \odot H) = \gamma_{\leq k-1}(G).$$

Proof. Let S be a distance-$(k-1)$ dominating set in G with order n. Hence, for every vertex $v_i \in V$, with $i \in \{1, ..., n\}$, we have that $d_{G \odot H}(v_i, S) = d_G(v_i, S) \leq k - 1$. Also, for every vertex $u \in V_i$, $i \in \{1, ..., n\}$, in $G \odot H$ we have that $d_{G \odot H}(u, S) = d_G(v_i, S) + 1 \leq k$. So, S is a distance-k dominating set in $G \odot H$ and, as a consequence, $\gamma_{\leq k}(G \odot H) \leq \gamma_{\leq k-1}(G)$.

On the other hand, let B be a distance-k dominating set in $G \odot H$ of minimum cardinality. Now, let $A = \{v_{i_1}, v_{i_2}, ... v_{i_r}\}$ be the set of vertices of G such that $(V_{i_j} \cup \{v_{i_j}\}) \cap B \neq \emptyset$, for every $j \in \{1, ..., r\}$. Since, $\gamma_{\leq k}(G \odot H) \leq \gamma_{\leq k-1}(G) < n$, we have that $r \leq n - 1$ and for every vertex $v_l \in \overline{A}$ in G we have,

$$d_G(v_l, A) \leq d_{G \odot H}(v_l, B) \leq k.$$

Now, if $d_G(v_l, A) = k$, then $d_G(v_l, A) = d_{G \odot H}(v_l, B) = k$ and for any vertex $u \in V_l$ we have that

$$d_{G \odot H}(u, B) = d_{G \odot H}(u, v_l) + d_{G \odot H}(v_l, B) = d_G(v_l, A) + 1 = k + 1,$$

which is a contradiction because B is a distance-k dominating set in $G \odot H$. Thus, for every vertex $v_l \in \overline{A}$ in G, we have that $d_G(v_l, A) \leq k - 1$ and, as a consequence, A is a distance-$(k-1)$ dominating set in G.

Therefore, $\gamma_{\leq k}(G \odot H) = |B| \geq |A| \geq \gamma_{\leq k-1}(G)$ and the result follows. □

Theorem 17. *For any connected graph G of order n and for any graph H,*

$$i(G \odot H) = ni(H) - \beta_0(G)(i(H) - 1).$$

Proof. Let S be an independent dominating set of minimum cardinality in $G \odot H$. If $v_i \in V \cap S$, then for every $v \in V_i$ we have that $v \notin S$. Also, if $v_i \notin V \cap S$, then there exists $S_i \subset V_i$, such that $S_i \subset S$ and $|S_i| \geq i(H)$. Thus, we obtain that there exist the sets $A \subset V$ and $S_i \subset V_i$, $i \in \{1, ..., t\}$, such that $n = t + |A|$, $|A| \leq \beta_0(G)$ and $S = (\bigcup_{i=1}^{t} S_i) \cup A$. Thus, $t \geq n - \beta_0(G)$ and we have

$$\begin{aligned} |S| &= |A| + \sum_{i=1}^{t} |S_i| \\ &= n - t + \sum_{i=1}^{t} |S_i| \\ &\geq n - t + ti(H) \\ &= n + t(i(H) - 1) \\ &\geq n + (n - \beta_0(G))(i(H) - 1) \\ &= ni(H) - \beta_0(G)(i(H) - 1). \end{aligned}$$

Therefore, $i(G \odot H) = |S| \geq ni(H) - \beta_0(G)(i(H) - 1)$.

On the other hand, let A be an independent set of maximum cardinality in G. Now, for every $v_i \in \overline{A}$, let $S_i \subset V_i$ be an independent dominating set in H_i. Let $S = A \cup (\bigcup_{v_i \in \overline{A}} S_i)$. It is easy to see that S is independent and dominating. So, $i(G \odot H) \leq ni(H) - \beta_0(G)(i(H) - 1)$ and the result follows. □

A *domatic partition* of a graph G is a vertex partition of G in which every set is a dominating set [7, 29, 30]. The maximum number of sets in any domatic partition of G is called the *domatic number* of G and it is denoted by $d(G)$. If G has a domatic partition, then G is a *domatic graph* (or G is domatic). Similarly, if there exists a vertex partition of G into independent dominating sets, then such a partition is called *idomatic* [30]. The maximum number of sets in any partition of a graph G into independent dominating sets is called the *idomatic number* of G and it is denoted by $d_i(G)$. If G has an idomatic partition, then G is an *idomatic graph* (or G is idomatic). Next we study the domatic and idomatic numbers of corona graphs.

Remark 18. *For any connected graph G and for any graph H,*

$$d(G \odot H) = d(H) + 1.$$

Proof. Let $\Pi_i = \{S_{i1}, S_{i2}, ..., S_{id(H)}\}$ be a domatic partition for H_i. Now, let $S_i = \cup_{j=1}^{n_1} S_{ji}$, where n_1 is the order of G and $i \in \{1, ..., d(G)\}$. Since every S_{ij}, $j \in \{1, ..., d(H)\}$ is a dominating set in H_i, $i \in \{1, ..., n_1\}$ we obtain that $S_i = \cup_{j=1}^{n_1} S_{ji}$, $i \in \{1, ..., n_1\}$ is a dominating set in $G \odot H$. Also, as V is a dominating set in $G \odot H$ we have that $d(G \odot H) \geq d(H) + 1$.

On the other hand, let $\Pi = \{A_1, A_2, ..., A_t\}$ be a domatic partition of maximum cardinality for $G \odot H$. If $v_l \in V$ belongs to $A_j \in \Pi$, then $A_j \cap V_l = \emptyset$ and for every $A_i \in \Pi$, $i \neq j$, $A_i \cap V_l \neq \emptyset$ and also $A_i \cap V_l$ is an independent dominating set in H_l for every $i \in \{1, ..., r\} - \{j\}$. Thus, $\Pi' = \{A_1 \cap V_l, A_2 \cap V_l, ..., A_{j-1} \cap V_l, A_{j+1} \cap V_l, ..., A_r \cap V_l\}$ is a domatic partition for H_l. Therefore $d(G \odot H) = t \leq d(H) + 1$ and the result follows. □

Theorem 19. *Let G be a connected graph and let H be an idomatic graph. Then $d_i(G \odot H) = d_i(H) + 1$ if and only if G has a partition into $d_i(H) + 1$ independent sets.*

Proof. Since $v_i \sim u$ for every $u \in V_i$, $i \in \{1, ..., n\}$, where n is the order of G, we have that for every independent dominating set S it is satisfied that $v_i \in S$ if and only if $V_i \cap S = \emptyset$.

($\Leftarrow$) Let us suppose that G has a partition into $t = d_i(H) + 1$ independent sets and let $\{A_1, A_2, ..., A_t\}$ be the partition of G into t independent sets. Now, for every $v_{i_j} \in A_i$, $i \in \{1, ..., t\}$, let $\{B_{i_j 1}, B_{i_j 2}, ..., B_{i_j, i_j - 1}, B_{i_j, i_j + 1}, ..., B_{i_j t}\}$ be an idomatic partition of H_{i_j}.

Let us form a partition $\Pi = \{S_1, S_2, ..., S_t\}$ of $G \odot H$ such that

$$S_i = A_i \cup \left(\bigcup_{v_l \notin A_i} B_{li} \right).$$

Thus, every $S_i \in \Pi$ is an independent dominating set in $G \odot H$ and, a consequence, Π is an idomatic partition in $G \odot H$ and $d_i(G \odot H) \geq t = d_i(H) + 1$.

Now, let $\Pi = \{S_1, S_2, ..., S_r\}$ be an idomatic partition of maximum cardinality in $G \odot H$. Let $v_i \in V$ be a vertex of G. Hence, there exists $S_l \in \Pi$ such that $v_i \in S_l$ and $S_l \cap V_i = \emptyset$. Moreover, for every $S_j \in \Pi$, with $j \neq l$, we have that $S_j \cap V_i \neq \emptyset$. Let $\Pi_i = \{S'_1, S'_2, ..., S'_{l-1}, S'_{l+1}, ..., S_r\}$ the partition of H_i obtained from Π in such a way that $S'_j = S_j \cap V_i$, for every $j \in \{1, ..., r\} - \{l\}$. Since every vertex of H_i is not adjacent to any vertex outside of $V_i \cup \{v_i\}$ we have that Π is an idomatic partition in H_i. Thus, we have

$$d_i(H) \geq r - 1 = d_i(G \odot H) - 1.$$

Therefore, we obtain that $d_i(G \odot H) = d_i(H) + 1$.

($\Rightarrow$) Let $\Pi = \{S_1, S_2, ..., S_r\}$ be an idomatic partition in $G \odot H$, with $r = d_i(G \odot H) = d_i(H) + 1$. If there exists $S_j \in \Pi$ such that $S_j \cap V = \emptyset$, then we have that for every $S_i \in \Pi$, $S_i \cap V_j \neq \emptyset$. Since, $\Pi_j = \{S_1 \cap V_j, S_2 \cap V_j, ..., S_r \cap V_j\}$ is an idomatic partition in H_j we obtain a contradiction. So, for every $S_i \in \Pi$, $S_i \cap V \neq \emptyset$. As every $S_i \in \Pi$ is an independent dominating set we have that $\{S_1 \cap V, S_2 \cap V, ..., S_r \cap V\}$ is a partition of V into $r = d_i(H) + 1$ independent sets. □

Appendix

In this extra section we present some results which are useful into proving some of the above theorems or propositions. The girth $g(G)$ of the graph G is the length of a shortest cycle contained in G.

Lemma 20. *Let G be a graph of minimum degree δ and let $t \geq 1$ be an integer. If $g(G) \geq 2t + 1$ then, for any vertex $v \in V$*

$$|M_t[v]| \geq 1 + \delta(v) \sum_{i=0}^{t-1} (\delta - 1)^i.$$

Proof. Let $v \in V$ be a vertex of G. Hence, we have that $|M_1[v]| = 1 + \delta(v)$ and

$$\begin{aligned}
|M_2[v]| &= |M_1[v]| + \sum_{u \in M_1[v]-\{v\}} (\delta(u) - 1) \\
&\geq 1 + \delta(v) + \sum_{u \in M_1[v]-\{v\}} (\delta - 1) \\
&= 1 + \delta(v) + \delta(v)(\delta - 1) \\
&= 1 + \delta(v) \sum_{i=0}^{1} (\delta - 1)^i
\end{aligned}$$

Now, let us proceed by induction on t. Let us suppose that $|M_{t-1}[v]| \geq 1 + \delta(v) \sum_{i=0}^{t-2} (\delta - 1)^i$, hence

$$\begin{aligned}
|M_t[v]| = |M_{t-1}[v]| &+ \sum_{u \in M_{t-1}[v]-M_{t-2}[v]} (\delta(u) - 1) \\
&\geq 1 + \delta(v) \sum_{i=0}^{t-2} (\delta - 1)^i + \sum_{u \in M_{t-1}[v]-M_{t-2}[v]} (\delta(u) - 1) \\
&\geq 1 + \delta(v) \sum_{i=0}^{t-2} (\delta - 1)^i + \sum_{u \in M_{t-1}[v]-M_{t-2}[v]} (\delta - 1) \\
&\geq 1 + \delta(v) \sum_{i=0}^{t-2} (\delta - 1)^i + \delta(v)(\delta - 1)^{t-1} \\
&= 1 + \delta(v) \sum_{i=0}^{t-1} (\delta - 1)^i
\end{aligned}$$

□

Lemma 21. *Let G be a graph of minimum degree δ and let $t \geq 1$ be an integer. If $g(G) \geq 2t + 2$ then, for any edge $uv \in E$*

$$|M_t[u] \cup M_t[v]| \geq \begin{cases} 2 + 2\delta \displaystyle\sum_{i=1}^{t/2} (\delta - 1)^{2i-1}, & \text{if } t \text{ is even,} \\ 2\delta \displaystyle\sum_{i=0}^{(t-1)/2} (\delta - 1)^{2i}, & \text{if } t \text{ is odd.} \end{cases}$$

Proof. Let $uv \in E$ be an edge of G. By Lemma 20 we have that

$$|M_t[u]| \geq 1 + \delta(u) \sum_{i=0}^{t-1} (\delta - 1)^i \geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i$$

and

$$|M_t[v]| \geq 1 + \delta(v) \sum_{i=0}^{t-1} (\delta - 1)^i \geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i.$$

Since $g(G) \geq 2t+2$ and $u \sim v$ we also have that

$$\begin{aligned}
|M_t[u] \cup M_t[v]| &= |M_t[u]| + |M_t[v]| - |M_t[u] \cap M_t[v]| \\
&= |M_t[u]| + |M_t[v]| - (|M_{t-1}[u]| + |M_{t-1}[v]| - |M_{t-1}[u] \cap M_{t-1}[v]|) \\
&= |M_t[u]| + |M_t[v]| - |M_{t-1}[u]| - |M_{t-1}[v]| + (|M_{t-2}[u]| + |M_{t-2}[v]| - |M_{t-3}[u] \cap M_{t-3}[v]|) \\
&= \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= |M_t[u]| + |M_t[v]| - |M_{t-1}[u]| - |M_{t-1}[v]| + |M_{t-2}[u]| + |M_{t-2}[v]| - \ \ldots \\
&\quad + (-1)^{t-2}|M_2[u]| + (-1)^{t-2}|M_2[v]| + (-1)^{t-1}|M_1[u]| + (-1)^{t-1}|M_1[v]| + (-1)^t 2 \\
&\geq 2 + 2\delta\sum_{i=0}^{t-1}(\delta-1)^i - 2 - 2\delta\sum_{i=0}^{t-2}(\delta-1)^i + 2 + 2\delta\sum_{i=0}^{t-3}(\delta-1)^i - 2 - 2\delta\sum_{i=0}^{t-4}(\delta-1)^i + \ \ldots \\
&\quad + (-1)^{t-2}\left(2 + 2\delta\sum_{i=0}^{1}(\delta-1)^i\right) + (-1)^{t-1}\left(2 + 2\delta\sum_{i=0}^{0}(\delta-1)^i\right) + (-1)^t 2
\end{aligned}$$

Now, if t is even, then we obtain

$$\begin{aligned}
|M_t[u] \cup M_t[v]| &\geq 2\delta(\delta-1)^{t-1} + 2\delta(\delta-1)^{t-3} + \ \ldots\ + 2\delta(\delta-1)^3 + 2\delta(\delta-1) + 2 \\
&= 2 + 2\delta\sum_{i=1}^{t/2}(\delta-1)^{2i-1}.
\end{aligned}$$

On the contrary, if t is odd, then we have

$$\begin{aligned}
|M_t[u] \cup M_t[v]| &\geq 2\delta(\delta-1)^{t-1} + 2\delta(\delta-1)^{t-3} + \ \ldots\ + 2\delta(\delta-1)^2 + 2\delta \\
&= 2\delta\sum_{i=0}^{(t-1)/2}(\delta-1)^{2i}.
\end{aligned}$$

□

Theorem 22. *Let G be a graph of minimum degree δ and maximum degree Δ. Let $k \geq 2$ be an integer. If $g(G) \geq k+1$ and k is even, then*

$$\chi_{\leq k}(G) \geq 1 + \Delta\sum_{i=0}^{\frac{k}{2}-1}(\delta-1)^i.$$

Proof. Let us suppose k is even and let v be a vertex of maximum degree in G. Let $A \subset V$ be the set of vertices of G such that for every $u \in A$ we have $d(u,v) \leq \frac{k}{2}$. Now, since $g(G) \geq k+1$, by Lemma 20 we have that $|A| \geq 1 + \Delta\sum_{i=0}^{k/2-1}(\delta-1)^i$. Also, for every two vertices $x, y \in A$ we have that $d(x,y) \leq k$. So, we obtain that $c(x) \neq c(y)$ and, as a consequence, $\chi_{\leq k}(G) \geq |A|$. Thus, the result follows. □

Theorem 23. *Let G be a graph of minimum degree δ and maximum degree Δ. Let $k \geq 2$ be an integer. If $g(G) \geq k+1$ and k is odd, then*

$$\chi_{\leq k}(G) \geq \begin{cases} 2 + 2\delta\displaystyle\sum_{i=1}^{(k-1)/4}(\delta-1)^{2i-1}, & \text{if } \frac{k-1}{2} \text{ is even,} \\ 2\delta\displaystyle\sum_{i=0}^{(k-3)/4}(\delta-1)^{2i}, & \text{if } \frac{k-1}{2} \text{ is odd.} \end{cases}$$

Proof. Let us suppose k is odd and let $uv \in E$ be an edge of G. Let $B \subset V$ be the set of vertices of G such that for every $x \in B$ we have either $d(x,v) \leq \frac{k-1}{2}$ or $d(x,u) \leq \frac{k-1}{2}$. Let $r = \frac{k-1}{2}$. Since, $g(G) \geq k+1$, by Lemma 21 we obtain that

$$|A| \geq \begin{cases} 2 + 2\delta \sum_{i=1}^{r/2} (\delta-1)^{2i-1}, & \text{if } r \text{ is even}, \\ 2\delta \sum_{i=0}^{(r-1)/2} (\delta-1)^{2i}, & \text{if } r \text{ is odd}. \end{cases}$$

Now, for every two vertices $a, b \in B$ we have $d(a,b) \leq k$. Thus, $c(a) \neq c(b)$ and, as a consequence, $\chi_{\leq k}(G) \geq |B|$. Therefore, the result follows. □

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