



Review

Symmetry reductions of a generalized Kuramoto–Sivashinsky equation via equivalence transformations

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ARTICLE INFO

Article history:

Received 16 October 2016
 Revised 11 December 2017
 Accepted 27 February 2018
 Available online 7 March 2018

Keywords:

Partial differential equations
 Conservation laws
 Symmetries
 Equivalence transformations

ABSTRACT

In this paper we consider a generalized Kuramoto–Sivashinsky equation. The equivalence group of the class under consideration has been constructed. This group allows us to perform a comprehensive study and a clear and concise formulation of the results. We have constructed the optimal system of subalgebras of the projections of the equivalence algebra on the space formed by the dependent variable and the arbitrary functions. By using this optimal system, all nonequivalent equations admitting an extension by one of the principal Lie algebra of the class under consideration can be determined. Taking into account the additional symmetries obtained we reduce some partial differential equations belonging to the class into ordinary differential equations. We derive some exact solutions of these equations.

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1. Introduction

The Kuramoto–Sivashinsky (KS) equation

$$u_t + \beta_1 u_{xx} + \beta_2 u_{xxxx} + \delta u_{xxx} + uu_x = 0, \quad (1)$$

arises in the modelling of flow on an inclined plane, where β_1 , β_2 and δ are constants. This equation was introduced by Kuramoto [20] in one-spatial dimension for the study of phase turbulence in the Belousov–Zhabotinsky reaction. Sivashinsky derived it independently in the context of small thermal diffusive instabilities for laminar flame fronts. Some authors have previously studied this equation [18,19].

The study of dynamical behaviours such as chaos arises as an effective way to explain the physics of low-dimensional dynamical systems, i.e., dynamical systems with small number of state variables. Nevertheless most problems and phenomena in physics and other areas of science are described by partial differential equations. Among them we highlight the KS equation. Khellat and Vasegh [17] improved the understanding of the connection between low-dimensional systems and the KS equation which is subject to spatially periodic boundary condition and with arbitrary periodic initial condition given by

$$\begin{aligned} u_t + u_{xx} + k u_{xxxx} + uu_x &= 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x, 0) = f(x), & \quad u(x - \pi, t) = u(x + \pi, t), & \int_{-\pi}^{\pi} f(x) dx = 0, \end{aligned} \quad (2)$$

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where k represents the viscosity. The authors showed that the KS equation (2) leads respectively to stability, pitchfork bifurcation and a new type of behaviour. Sahoo and Saha Ray [27] considered the time-fractional KS equation given by

$$D_t^\alpha u + bu_{xx} + ku_{xxxx} + auu_x = 0, \tag{3}$$

which is a generalization of Eq. (2), where $0 < \alpha \leq 1$ and a, b and k are arbitrary constants. The authors obtained new types of exact analytical solutions of Eq. (3) by applying the tanh-sech method with the help of the fractional complex transform.

There have been several generalizations of the KS equation such as the generalized KS with dispersive effects [12]

$$u_t + (f(u))_x + \alpha u_{xx} + \phi(u)_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = g(u), \tag{4}$$

where α, β and γ are constants, $f(u), \phi(u)$ and $g(u)$ are functions of the dependent variable u . Here, $(f(u))_x = \frac{\partial f(u)}{\partial u} u_x = f_u u_x$. In [3] Bruzón et al. considered Eq. (4) and they made a full analysis of the symmetry reductions and proved that the nonclassical method applied to the equation leads to new reductions which cannot be obtained by Lie classical symmetries. Furthermore, making use of the new similarity solutions that the nonclassical method determines, they obtained some exact solutions which cannot be determined by applying a Lie classical reduction. If α and β are constants, $\gamma = 1, f(u) = \frac{\alpha_1}{2} u^2 + \beta u + \alpha_5, g(u) = 0$ and $\phi(u) = \frac{\alpha_2}{2} u^2 + (\alpha_3 - \alpha)u + \alpha_4$, with $\alpha_i, i = 1, \dots, 5$, arbitrary constants, Eq. (4) includes the Korteweg-de Vries equation supplements by additional terms of the KS equation which describes nonlinear convection and the input of energy produced by Marangoni forces on the long scales together with energy dissipation on short scales. For this equation Bruzón et al. obtained classical and nonclassical symmetries and the reduced equations [2]. We propose herein a generalized KS equation

$$u_t + (f(u))_x + \alpha(u)u_{xx} + (\phi(u))_{xx} + \beta(u)u_{xxx} + \gamma(u)u_{xxxx} = g(u), \tag{5}$$

with $f(u), g(u), \alpha(u), \beta(u), \gamma(u) \neq 0$ and $\phi(u)$ arbitrary functions. Here $(\phi(u))_{xx} = \frac{\partial^2 \phi(u)}{\partial u^2} u_x^2 + \frac{\partial \phi(u)}{\partial u} u_{xx} = \phi_{uu} u_x^2 + \phi_u u_{xx}$.

Since in different fields arise many equations that involve several arbitrary functions, such as industrial mathematics, mathematical biology and fluid mechanics, the study of variable-coefficient equations has increased in the last decades [4–6,9,10,15,16,22,33,35].

On the other hand, equivalence transformations are used mainly for classifying classes of differential equations with arbitrary functions [7,8,11,21,24,25,28,29]. Just like symmetries of a differential equation transform solutions of the equation into other solutions, equivalence transformations map every equation of a class \mathfrak{C} of differential equations into an equation of the same class. Ovsiannikov [24] defined a methodology and notation for dealing with such transformations, for which he used the term equivalence transformations. He derived some results about them, including important properties:

- a) The transformations act on every equation in the class \mathfrak{C} .
- b) The transformations are fixed point transformations, in the sense that they do not depend on the arbitrary elements, and are realized on the point space (independent and dependent variables) associated with the differential equations.
- c) The transformations act on the arbitrary elements as point transformations of an augmented space of independent and dependent variables and additional variables representing values taken by the arbitrary elements.

The collection of all such transformations constitutes Ovsiannikov’s equivalence group. In [1] Akhatov et al., by using Ovsiannikov’s method, determined the infinitesimal form of transformations for a potential form of the nonlinear diffusion equation.

Ibragimov et al. used the equivalence group to give a preliminary symmetry group classification. They found the equivalence group for a large class of nonlinear hyperbolic equations and executed the preliminary classification for a finite-parameter subgroup [13]. By using the weak equivalence classification, Torrisi and Tracina [30] obtained a classification with respect to thermal conductivity and the internal energy at equilibrium of a system of partial differential equations (PDEs) which describe the unidimensional heat conduction in a homogeneous isotropic rigid body. In [26], Lie symmetries for a class of drift-diffusion systems were found following a different procedure based on the weak equivalence classification already introduced in [30–32]. They authors claimed that *even if this approach does not guarantee obtaining a complete symmetry classification, yet this method provides a systematic way for obtaining wide classes of symmetries when arbitrary functions appear*.

In [36], Vaneeva et al. discussed how point transformations can be used for the study of integrability, in particular, for deriving classes of integrable variable-coefficient differential equations. They described the procedure of finding the equivalence groupoid of a class of differential equations focusing on the case of evolution equations and they applied this study to a class of fifth-order variable-coefficient Korteweg-de Vries-like equations. Two alternative ways to solve completely the group classification problem for a variable coefficient Gardner equation were presented in [37]: the gauging of arbitrary elements of the class by the equivalence transformations and the method of mapping between classes. Recently, Tracina [34] proved a criterion, based on the property of nonlinear self-adjointness, for the existence of an invertible point transformation which maps a given PDE to a linear PDE.

On the other hand, Gandarias and Ibragimov [11] found and employed the Lie algebra of the generators of the equivalence transformations of a fourth-order non-linear evolutionary equations given by

$$u_t + f(u)u_{xxxx} + g(u)u_x u_{xxx} + h(u)u_{xx}^2 + d(u)u_x^2 u_{xx} - p(u)u_{xx} - q(u)u_x^2 = 0.$$

In this paper we obtain the equivalence transformations of the generalized KS Eq. (5). We determine the generators of the equivalence algebra. Equivalence algebra is used to perform a preliminary group classification. By using a theorem on projections we solve the problem of preliminary group classification of class (5) by means of the construction of the optimal system of subalgebras of the nonzero projections of the equivalence algebra on the space $(u, f, g, \alpha, \beta, \gamma, \phi)$. The optimal system allows us to obtain all nonequivalent equations (5) admitting an extension by one of the principal Lie algebra of class (5). Taking into account some nonequivalent equations which admit an additional symmetry, we have reduced Eq. (5) with the corresponding coefficients $f, g, \alpha, \beta, \gamma, \phi$ into ODEs. Finally, some exact solutions are obtained.

2. Equivalence transformations

In this section we determine the equivalence transformations of class (5). An equivalence transformation of class (5) is a nondegenerate point transformation from (t, x, u) to $(\tilde{t}, \tilde{x}, \tilde{u})$ with the property that it preserves the differential structure of the equation, that is, it transforms any equation of class (5) to an equation from the same class but with different arbitrary elements, $\tilde{f}(\tilde{u}), \tilde{g}(\tilde{u}), \tilde{\alpha}(\tilde{u}), \tilde{\beta}(\tilde{u}), \tilde{\gamma}(\tilde{u})$ and $\tilde{\phi}(\tilde{u})$ from the original ones. The set of equivalence transformations forms a group denoted by \mathcal{E} .

The equivalence transformations for our equation can be obtained by applying Lie’s infinitesimal criterion [24]. Unlike classical symmetries, we require not only the invariance of class (5) but also the invariance of the auxiliary system

$$f_t = f_x = g_t = g_x = \alpha_t = \alpha_x = \beta_t = \beta_x = \gamma_t = \gamma_x = \phi_t = \phi_x = 0. \tag{6}$$

We consider the one-parameter group of equivalence transformations in the augmented space $(t, x, u, f, g, \alpha, \beta, \gamma, \phi)$ given by

$$\begin{aligned} \tilde{t} &= t + \epsilon \tau(t, x, u) + O(\epsilon^2), \\ \tilde{x} &= x + \epsilon \xi(t, x, u) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon \eta(t, x, u) + O(\epsilon^2), \\ \tilde{f} &= f + \epsilon \omega^1(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \\ \tilde{g} &= g + \epsilon \omega^2(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \\ \tilde{\alpha} &= \alpha + \epsilon \omega^3(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \\ \tilde{\beta} &= \beta + \epsilon \omega^4(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \\ \tilde{\gamma} &= \gamma + \epsilon \omega^5(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \\ \tilde{\phi} &= \phi + \epsilon \omega^6(t, x, u, f, g, \alpha, \beta, \gamma, \phi) + O(\epsilon^2), \end{aligned} \tag{7}$$

where ϵ is the group parameter. In this case, the vector field takes the following form

$$Y = \tau \partial_t + \xi \partial_x + \eta \partial_u + \omega^1 \partial_f + \omega^2 \partial_g + \omega^3 \partial_\alpha + \omega^4 \partial_\beta + \omega^5 \partial_\gamma + \omega^6 \partial_\phi. \tag{8}$$

We notice that the fourth derivative with respect to x appears in the equation, therefore we need to take into account the fourth prolongation of the operator (8)

$$\tilde{Y} = Y + \zeta^t \partial_{u_t} + \zeta^x \partial_{u_x} + \zeta^{xx} \partial_{u_{xx}} + \zeta^{xxx} \partial_{u_{xxx}} + \zeta^{xxxx} \partial_{u_{xxxx}} + \tilde{\omega}_t^i \partial_{f_i} + \tilde{\omega}_x^i \partial_{f_x^i} + \tilde{\omega}_u^i \partial_{f_{u_i}}, \tag{9}$$

where $f^i, i = 1, \dots, 6$, represents each component $(f, g, \alpha, \beta, \gamma, \phi)$. The coefficients ζ^J is defined by

$$\zeta^J(t, x, u^{(4)}) = D_J(\eta - \tau u_t - \xi u_x) + \tau u_{Jt} + \xi u_{Jx},$$

with $J = (j_1, \dots, j_k), 1 \leq j_k \leq 2$ and $1 \leq k \leq 4, u^{(4)}$ denotes the sets of partial derivatives up to fourth order and D_t, D_x are the total derivatives with respect to t and x [23]. Lastly, the coefficients $\tilde{\omega}_r^i$ are given by

$$\tilde{\omega}_r^i = \tilde{D}_r(\omega^i) - f_t^i \tilde{D}_r(\tau) - f_x^i \tilde{D}_r(\xi) - f_u^i \tilde{D}_r(\eta),$$

where

$$\tilde{D}_r = \partial_r + f_r^i \partial_{f_i}.$$

For further information on how prolongations of higher order can be obtained, one can refer to references [28,29].

The invariance of system (5) and (6) under the one-parameter group of equivalence transformations (7) with infinitesimal generator (8) leads to a system of determining equations. After having solved the determining system, omitting tedious calculations, we obtain the associated equivalence algebra $\mathcal{L}_\mathcal{E}$ of class (5) which is finite-dimensional and is spanned by

$$\begin{aligned} Y_1 &= \partial_t, & Y_2 &= \partial_x, & Y_3 &= \partial_u, \\ Y_4 &= t \partial_t - f \partial_f - g \partial_g - \alpha \partial_\alpha - \beta \partial_\beta - \gamma \partial_\gamma - \phi \partial_\phi, \\ Y_5 &= x \partial_x + f \partial_f + 2\alpha \partial_\alpha + 3\beta \partial_\beta + 4\gamma \partial_\gamma + 2\phi \partial_\phi, \\ Y_6 &= u \partial_u + f \partial_f + g \partial_g + \phi \partial_\phi, \\ Y_7 &= t \partial_x + u \partial_f, & Y_8 &= \partial_\alpha - u \partial_\phi, \end{aligned}$$

$$Y_9 = \partial_f, \quad Y_{10} = \partial_\phi. \tag{10}$$

From generators (10) we get the finite form of the equivalence transformations:

Theorem 1. *The equivalence group of class (5) consists of the transformations*

$$\begin{aligned} \tilde{t} &= (t + \epsilon_1)e^{\epsilon_4}, \quad \tilde{x} = (x + \epsilon_7t + \epsilon_2)e^{\epsilon_5}, \quad \tilde{u} = (u + \epsilon_3)e^{\epsilon_6}, \\ \tilde{f} &= (f + \epsilon_7u + \epsilon_9)e^{-\epsilon_4 + \epsilon_5 + \epsilon_6}, \quad \tilde{g} = ge^{-\epsilon_4 + \epsilon_6}, \quad \tilde{\alpha} = (\alpha + \epsilon_8)e^{-\epsilon_4 + 2\epsilon_5}, \\ \tilde{\beta} &= \beta e^{-\epsilon_4 + 3\epsilon_5}, \quad \tilde{\gamma} = \gamma e^{-\epsilon_4 + 4\epsilon_5}, \quad \tilde{\phi} = (\phi - \epsilon_8u + \epsilon_{10})e^{-\epsilon_4 + 2\epsilon_5 + \epsilon_6}, \end{aligned} \tag{11}$$

where $\epsilon_i, i = 1, \dots, 10$, are arbitrary constants.

Proof. From (11) it follows that $u = e^{-\epsilon_6}\tilde{u} - \epsilon_3$. Thus,

$$u_t = e^{\epsilon_4 - \epsilon_6}\tilde{u}_{\tilde{t}} + \epsilon_7e^{\epsilon_5 - \epsilon_6}\tilde{u}_{\tilde{x}}, \quad u_x = e^{\epsilon_5 - \epsilon_6}\tilde{u}_{\tilde{x}}, \quad u_{xx} = e^{2\epsilon_5 - \epsilon_6}\tilde{u}_{\tilde{x}\tilde{x}}, \quad u_{xxx} = e^{3\epsilon_5 - \epsilon_6}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}, \quad u_{xxxx} = e^{4\epsilon_5 - \epsilon_6}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}. \tag{12}$$

Moreover,

$$\begin{aligned} f &= -\epsilon_7u - \epsilon_9 + e^{\epsilon_4 - \epsilon_5 - \epsilon_6}\tilde{f}, \quad g = e^{\epsilon_4 - \epsilon_6}\tilde{g}, \quad \alpha = -\epsilon_8 + e^{\epsilon_4 - 2\epsilon_5}\tilde{\alpha}, \\ \beta &= e^{\epsilon_4 - 3\epsilon_5}\tilde{\beta}, \quad \gamma = e^{\epsilon_4 - 4\epsilon_5}\tilde{\gamma}, \quad \phi = \epsilon_8u - \epsilon_{10} + e^{\epsilon_4 - 2\epsilon_5 - \epsilon_6}\tilde{\phi}. \end{aligned} \tag{13}$$

Hence,

$$f_u = -\epsilon_7 + e^{\epsilon_4 - \epsilon_5}\tilde{f}_{\tilde{u}}, \quad \phi_u = \epsilon_8 + e^{\epsilon_4 - 2\epsilon_5}\tilde{\phi}_{\tilde{u}}, \quad \phi_{uu} = e^{\epsilon_4 - 2\epsilon_5 + \epsilon_6}\tilde{\phi}_{\tilde{u}\tilde{u}}. \tag{14}$$

Substituting (12)–(14) and then dividing the result by $e^{\epsilon_4 - \epsilon_6}$ we obtain

$$\tilde{u}_{\tilde{t}} + \tilde{f}_{\tilde{u}}\tilde{u}_{\tilde{x}} + \tilde{\alpha}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{u}\tilde{u}}\tilde{u}_{\tilde{x}}^2 + \tilde{\phi}_{\tilde{u}}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{\beta}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{\gamma}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = \tilde{g}. \quad \square$$

3. Theorem on projections

At this point, it is useful to know the most general symmetry algebra admitted by Eq. (5) when the functions $f, g, \alpha, \beta, \gamma$ and ϕ are arbitrary. This symmetry algebra is known as the principal Lie algebra \mathcal{L}_p . Equivalence transformations are widely used in the study of PDEs, in particular, in nonlinear PDEs. For instance, they can be used for obtaining the principal Lie algebra as well as for finding extensions of the principal Lie algebra once the arbitrary elements are fixed. Let us introduce the following projections of the equivalence operator (8)

$$X = pr_{(x,u)}(Y) = \tau \partial_t + \xi \partial_x + \eta \partial_u, \tag{15}$$

$$Z = pr_{(u,\Psi)}(Y) = \eta \partial_u + \omega^1 \partial_f + \omega^2 \partial_g + \omega^3 \partial_\alpha + \omega^4 \partial_\beta + \omega^5 \partial_\gamma + \omega^6 \partial_\phi, \tag{16}$$

where $x = (t, x)$ and $\Psi = (f, g, \alpha, \beta, \gamma, \phi)$ represent the set of independent variables and the set of arbitrary functions respectively. The equivalence generator Y of class (5) can be written as a linear combination of the elements of \mathcal{L}_E

$$\begin{aligned} Y &= \sum_{i=1}^{10} c_i Y_i = (c_1 + c_4t)\partial_t + (c_2 + c_5x + c_7t)\partial_x + (c_3 + c_6u)\partial_u + (c_9 + c_7u + (-c_4 + c_5 + c_6)f)\partial_f \\ &\quad + (-c_4 + c_6)g\partial_g + (c_8 + (-c_4 + 2c_5)\alpha)\partial_\alpha + (-c_4 + 3c_5)\beta\partial_\beta \\ &\quad + (-c_4 + 4c_5)\gamma\partial_\gamma + (c_{10} - c_8u + (-c_4 + 2c_5 + c_6)\phi)\partial_\phi. \end{aligned} \tag{17}$$

In order to obtain the principal Lie algebra we apply the following theorem:

Theorem 2. *Let Y be an equivalent operator for Eq. (5), the projection $X = pr_{(x,u)}(Y) \in \mathcal{L}_p$ if and only if the projection $Z = pr_{(u,\Psi)}(Y)$ is identically zero.*

Taking into account Theorem 2, we require the vanishing of the projection Z

$$\begin{aligned} Z &= (c_3 + c_6u)\partial_u + (c_9 + c_7u + (-c_4 + c_5 + c_6)f)\partial_f + (-c_4 + c_6)g\partial_g + (c_8 + (-c_4 + 2c_5)\alpha)\partial_\alpha \\ &\quad + (-c_4 + 3c_5)\beta\partial_\beta + (-c_4 + 4c_5)\gamma\partial_\gamma + (c_{10} - c_8u + (-c_4 + 2c_5 + c_6)\phi)\partial_\phi, \end{aligned} \tag{18}$$

this leads us to

$$c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0.$$

Thus, we get that the principal Lie algebra \mathcal{L}_p is given by

$$X = pr_{(x,u)}(Y) = c_1 Y_1 + c_2 Y_2.$$

Table 1
Table of commutators.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
Z_1	0	0	0	Z_1	Z_7	$-Z_8$	0	0
Z_2	0	0	0	0	Z_5	Z_6	Z_7	Z_8
Z_3	0	0	0	0	$-Z_5$	$-2Z_6$	$-Z_7$	$-2Z_8$
Z_4	$-Z_1$	0	0	0	0	0	$-Z_7$	$-Z_8$
Z_5	$-Z_7$	$-Z_5$	Z_5	0	0	0	0	0
Z_6	Z_8	$-Z_6$	$2Z_6$	0	0	0	0	0
Z_7	0	$-Z_7$	Z_7	Z_7	0	0	0	0
Z_8	0	$-Z_8$	$2Z_8$	Z_8	0	0	0	0

Consequently, for arbitrary functions $f, g, \alpha, \beta, \gamma$ and ϕ , Eq. (5) admits the two-dimensional Lie algebra generated by

$$X_1 = \partial_t, \quad X_2 = \partial_x. \tag{19}$$

We are interested in obtaining extensions of the principal Lie algebra, for this purpose, we will make use of the following theorem:

Theorem 3. Let Y be an equivalent operator for Eq. (5). The operator $X = pr_{(x,u)}(Y)$ is admitted by Eq. (5) with specific functions

$$f = F(u), \quad g = G(u), \quad \alpha = A(u), \quad \beta = B(u), \quad \gamma = \Gamma(u), \quad \phi = \Phi(u), \tag{20}$$

if and only if these functions are invariant with respect to the projection $Z = pr_{(u,\Psi)}(Y)$.

The proof of Theorems 2 and 3 can be found in [14].

4. Optimal system of one-dimensional subalgebras

Generally, it is not usually feasible to obtain all the possible group-invariant solutions, since there can be an infinite number of Lie subgroups of the Lie group of symmetries G of a given equation. Therefore, we would like to classify all the possible invariant solutions into different classes such as two solutions belonging to the same class are equivalent (one solution can be transformed into the other under the action of an element of the Lie symmetry group) and solutions belonging to different classes are not equivalent (these solutions are not related by any element of the Lie symmetry group). This classification problem can be solved by constructing the optimal system of subalgebras [23,24].

An optimal system of one-dimensional Lie subalgebras includes essential information about different types of invariant solutions [23,24]. Let \mathcal{G} be the Lie algebra of G . We say two one-dimensional subalgebras h and \tilde{h} are equivalent if $\tilde{h} = \text{Ad } g(h)$ where $\text{Ad } g$ is the adjoint action of g on \mathcal{G} . Moreover, for each $V \in \mathcal{G}$ is defined a linear operator $\text{ad } V : \mathcal{G} \rightarrow \mathcal{G}$, $\text{ad } V(W) = [V, W]$, where $[\cdot, \cdot]$ represents the Lie bracket. A set of one-parameter subalgebras forms an optimal system if every one-parameter subalgebra of \mathcal{G} is equivalent to an exclusive element of the set under some component of the adjoint representation $\tilde{h} = \text{Ad } g(h), g \in G$.

Through the employment of the exponential map Exp from \mathcal{G} to G it is possible to construct the adjoint action $\text{Ad } G$ of the underlying Lie group by means of the formula [23]

$$\text{Ad}(\exp(\epsilon V))W = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} (\text{ad } V)^n(W) = W - \epsilon[V, W] + \frac{\epsilon^2}{2}[V, [V, W]] - \dots$$

The classification of all nonequivalent equations (with respect to a given equivalence group \mathcal{E}) admitting extensions of the principal Lie algebra is known as a preliminary group classification. The method of preliminary group classification is simple and effective when the classification is based on finite-dimensional equivalence algebra $\mathcal{L}_{\mathcal{E}}$. In our case, we take into account the ten-dimensional subalgebra (10) and use it for obtaining a preliminary group classification. According to Theorem 3 the problem of preliminary group classification of Eq. (5) with respect to the finite-dimensional algebra (10) is equivalent to constructing the optimal system of subalgebras given by

$$\begin{aligned} Z_1 &= \partial_u, & Z_2 &= -f\partial_f - g\partial_g - \alpha\partial_\alpha - \beta\partial_\beta - \gamma\partial_\gamma - \phi\partial_\phi, & Z_3 &= f\partial_f + 2\alpha\partial_\alpha + 3\beta\partial_\beta + 4\gamma\partial_\gamma + 2\phi\partial_\phi, \\ Z_4 &= u\partial_u + f\partial_f + g\partial_g + \phi\partial_\phi, & Z_5 &= u\partial_f, & Z_6 &= \partial_\alpha - u\partial_\phi, & Z_7 &= \partial_f, & Z_8 &= \partial_\phi, \end{aligned} \tag{21}$$

where $Z_i, i = 1, \dots, 8$, are the nonzero projections of (10) on the space (u, Ψ) .

Therefore, we consider \mathcal{G} the symmetry algebra with basis $\{Z_1, \dots, Z_8\}$. First, we construct the commutator table for \mathcal{G} which is a 8×8 table whose (i, j) th entry expresses the Lie bracket $[Z_i, Z_j]$. Commutator table is shown in Table 1. From the commutator table for this algebra, we construct the adjoint action $\text{Ad } G$ on the basis $\{Z_1, \dots, Z_8\}$ of \mathcal{G} . We show the adjoint action in Table 2, where the (i, j) th entry gives $\text{Ad}(\exp(\epsilon Z_i))Z_j$.

In substance, the optimal system can be constructed by taking a general element $Z = a_1Z_1 + a_2Z_2 + \dots + a_8Z_8 \in \mathcal{G}$, where $a_i, i = 1, \dots, 8$, are arbitrary constants, and simplifying it as much as possible through well-considered applications of adjoint

Table 2
The adjoint action on the basis $\{Z_1, \dots, Z_8\}$.

Ad	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
Z_1	Z_1	Z_2	Z_3	$Z_4 - \epsilon Z_1$	$Z_5 - \epsilon Z_7$	$Z_6 + \epsilon Z_8$	Z_7	Z_8
Z_2	Z_1	Z_2	Z_3	Z_4	$e^{-\epsilon} Z_5$	$e^{-\epsilon} Z_6$	$e^{-\epsilon} Z_7$	$e^{-\epsilon} Z_8$
Z_3	Z_1	Z_2	Z_3	Z_4	$e^{\epsilon} Z_5$	$e^{2\epsilon} Z_6$	$e^{\epsilon} Z_7$	$e^{2\epsilon} Z_8$
Z_4	$e^{\epsilon} Z_1$	Z_2	Z_3	Z_4	Z_5	Z_6	$e^{\epsilon} Z_7$	$e^{\epsilon} Z_8$
Z_5	$Z_1 + \epsilon Z_7$	$Z_2 + \epsilon Z_5$	$Z_3 - \epsilon Z_5$	Z_4	Z_5	Z_6	Z_7	Z_8
Z_6	$Z_1 - \epsilon Z_8$	$Z_2 + \epsilon Z_6$	$Z_3 - 2\epsilon Z_6$	Z_4	Z_5	Z_6	Z_7	Z_8
Z_7	Z_1	$Z_2 + \epsilon Z_7$	$Z_3 - \epsilon Z_7$	$Z_4 - \epsilon Z_7$	Z_5	Z_6	Z_7	Z_8
Z_8	Z_1	$Z_2 + \epsilon Z_8$	$Z_3 - 2\epsilon Z_8$	$Z_4 - \epsilon Z_8$	Z_5	Z_6	Z_7	Z_8

maps to Z , $\text{Ad}(\exp(\epsilon Z_i))Z$, and discrete symmetries. For further information on how optimal system can be constructed, one can refer, as example, to reference [23].

To conclude, we obtain the following optimal system of one-dimensional subalgebras of (21)

$$\begin{aligned}
 Z^{(1)} &= Z_1, & Z^{(2)} &= Z_2, & Z^{(3)} &= Z_1 + Z_2, & Z^{(4)} &= Z_1 + Z_8, & Z^{(5)} &= aZ_2 + Z_3, & Z^{(6)} &= Z_7 + aZ_8, \\
 Z^{(7)} &= Z_1 + aZ_2 + Z_3, & Z^{(8)} &= aZ_2 + bZ_3 + Z_4, & Z^{(9)} &= cZ_2 + dZ_3 + Z_5, & Z^{(10)} &= aZ_2 + bZ_3 + Z_6, \\
 Z^{(11)} &= Z_2 + Z_3 + Z_7, & Z^{(12)} &= 2Z_2 + Z_3 + Z_8, & Z^{(13)} &= Z_4 + Z_5 + aZ_6, & Z^{(14)} &= Z_4 - Z_5 + aZ_6, \\
 Z^{(15)} &= Z_1 + aZ_2 + bZ_3 + Z_5, & Z^{(16)} &= -Z_1 + aZ_2 + bZ_3 + Z_5, & Z^{(17)} &= Z_1 + aZ_2 + bZ_3 + Z_6, \\
 Z^{(18)} &= -Z_1 + aZ_2 + bZ_3 + Z_6, & Z^{(19)} &= Z_1 + Z_2 + Z_3 + Z_7, & Z^{(20)} &= Z_1 + 2Z_2 + Z_3 + Z_8, \\
 Z^{(21)} &= mZ_2 + mZ_3 + Z_4 + Z_5, & Z^{(22)} &= 2aZ_2 + aZ_3 + Z_4 + Z_6, & Z^{(23)} &= (1 + a)Z_2 + aZ_3 + Z_4 + Z_7, \\
 Z^{(24)} &= (1 + 2a)Z_2 + aZ_3 + Z_4 + Z_8, & Z^{(25)} &= aZ_2 + aZ_3 + Z_6 + Z_7, & Z^{(26)} &= Z_2 + Z_4 + Z_7 + mZ_8, \\
 Z^{(27)} &= Z_1 + 2aZ_2 + aZ_3 + Z_5 + mZ_6, & Z^{(28)} &= -Z_1 + 2aZ_2 + aZ_3 + Z_5 + mZ_6, \\
 Z^{(29)} &= Z_1 + aZ_2 + aZ_3 + Z_6 + mZ_7, & Z^{(30)} &= -Z_1 + aZ_2 + aZ_3 + Z_6 + mZ_7, \\
 Z^{(31)} &= -Z_2 - Z_3 + Z_4 + Z_5 + Z_8, & Z^{(32)} &= 2Z_2 + Z_3 + Z_4 + mZ_6 + Z_7, \\
 Z^{(33)} &= 2aZ_2 + aZ_3 + Z_5 + bZ_6 + Z_8, & Z^{(34)} &= 2aZ_2 + aZ_3 + Z_5 + bZ_6 - Z_8,
 \end{aligned} \tag{22}$$

where $a, b, c \neq 2d, m \neq 0$ are arbitrary constants.

5. Reductions and exact solutions

The symmetry group of a partial differential equation (PDE) is the largest transformation group which acts on dependent and independent variables of the equation so it transforms solutions of the equation into other solutions. Lie symmetry groups are considered to be one of the most powerful methods to construct exact solutions of PDEs. Local symmetries admitted by a PDE are useful for obtaining invariant solutions. This technique is based on the following: if a differential equation is invariant under a Lie group of transformations, then a reduction transformation exists. In the case of PDEs with two independent variables, this technique yields a similarity variable and a similarity solution which allow us to transform the PDE into an ODE, which is generally easier to solve.

Lie point symmetries can be obtained by using the invariance criterion [23,24] which leads to an overdetermined linear system of equations called determining system. The problem lies in the fact that, when arbitrary functions appear, the determining system usually becomes very difficult to solve. In these cases, the method of preliminary group classification is an effective way to give a group classification. This method does not guarantee a priori a complete group classification, however, it allows us to obtain symmetries when the determining system is intricate.

Thus, by applying Theorem 3 to the optimal system (22) it is possible to obtain all nonequivalent equations (5) which admit extensions of the principal Lie algebra by one, namely, those equations belonging to (5) such that they admit, besides the primary operators X_1 and X_2 (19), an additional operator X_3 . We note that due to the extension of the optimal system, we do not show all nonequivalent equations.

From the associated equivalence algebra (10) we can separate equivalence generators for class (5) into three different groups: translations (Y_1, Y_2, Y_3, Y_9 and Y_{10}), dilatations (Y_4, Y_5 and Y_6) and generalized Galilean transformations (Y_7 and Y_8). In order to illustrate the algorithm of passing from operators of the equivalence algebra (10) or, more specifically, from operators (22) to the form of the functions $f, g, \alpha, \beta, \gamma$ and ϕ , and the additional operator X_3 , we consider, by way of example, some generators involving representatives of the three different groups. Once we have obtained the new generator X_3 , we determine the similarity variable and the similarity solution which allow us to transform the PDE which admits X_3 into an ODE. Finally, we obtain some exact solutions of the corresponding ODE.

Example 1. To begin with, we take the generator

$$Y = Y_3 + Y_4 = t\partial_t + \partial_u - f\partial_f - g\partial_g - \alpha\partial_\alpha - \beta\partial_\beta - \gamma\partial_\gamma - \phi\partial_\phi. \tag{23}$$

Following [Theorem 3](#), we require the invariance of (20) with respect to the operator

$$Z^{(3)} = pr_{(u,\Psi)}(Y) = \partial_u - f\partial_f - g\partial_g - \alpha\partial_\alpha - \beta\partial_\beta - \gamma\partial_\gamma - \phi\partial_\phi. \tag{24}$$

That means,

$$\begin{aligned} Z^{(3)}(f - F(u))|_{f=F(u)} &= 0, & Z^{(3)}(g - G(u))|_{g=G(u)} &= 0, & Z^{(3)}(\alpha - A(u))|_{\alpha=A(u)} &= 0, \\ Z^{(3)}(\beta - B(u))|_{\beta=B(u)} &= 0, & Z^{(3)}(\gamma - \Gamma(u))|_{\gamma=\Gamma(u)} &= 0, & Z^{(3)}(\phi - \Phi(u))|_{\phi=\Phi(u)} &= 0, \end{aligned}$$

which yields

$$f = c_1 e^{-u}, \quad g = c_2 e^{-u}, \quad \alpha = c_3 e^{-u}, \quad \beta = c_4 e^{-u}, \quad \gamma = c_5 e^{-u}, \quad \phi = c_6 e^{-u},$$

where c_i are arbitrary constants. Therefore, [Eq. \(5\)](#) takes the following form

$$u_t + e^{-u}(-c_1 u_x + (c_3 - c_6)u_{xx} + c_6 u_x^2 + c_4 u_{xxx} + c_5 u_{xxxx} - c_2) = 0, \tag{25}$$

which admits the additional symmetry

$$X_3 = t\partial_t + \partial_u. \tag{26}$$

The corresponding invariant solution is written in the form

$$z = x, \quad u = \ln|t| + h(z), \tag{27}$$

where $h(z)$ must satisfy

$$c_5 h'''' + c_4 h''' + (c_3 - c_6) h'' + c_6 (h')^2 - c_1 h' + e^h - c_2 = 0. \tag{28}$$

Since that $c_5 \neq 0$, [Eq. \(29\)](#) can be written as an autonomous equation of fourth order $h'''' = F(h, h', h'', h''')$. The substitution $w(h) = (h')^2$ leads to the following third-order equation

$$2c_5 w w''' + (c_5 w' \pm 2c_4 \sqrt{w})w'' + 2(c_3 - c_6)w' + 4c_6 w \mp 4c_1 \sqrt{w} - 4c_2 + 4e^h = 0. \tag{29}$$

Example 2. Let us consider generator

$$Y = Y_4 + Y_6 = t\partial_t + u\partial_u - \alpha\partial_\alpha - \beta\partial_\beta - \gamma\partial_\gamma. \tag{30}$$

By applying [Theorem 3](#), we require the invariance of (20) with respect to the projection

$$Z = Z^{(8)}|_{a=1, b=0} = u\partial_u - \alpha\partial_\alpha - \beta\partial_\beta - \gamma\partial_\gamma, \tag{31}$$

which yields

$$f = c_1, \quad g = c_2, \quad \alpha = \frac{c_3}{u}, \quad \beta = \frac{c_4}{u}, \quad \gamma = \frac{c_5}{u}, \quad \phi = c_6,$$

where c_i are arbitrary constants. Thus, [Eq. \(5\)](#) is given by

$$u u_t + c_3 u_{xx} + c_4 u_{xxx} + c_5 u_{xxxx} - c_2 u = 0, \tag{32}$$

which admits the additional symmetry

$$X_3 = t\partial_t + u\partial_u. \tag{33}$$

The corresponding similarity variable and similarity solution are written in the form

$$z = x, \quad u = t h(z), \tag{34}$$

where $h(z)$ must satisfy

$$c_5 h'''' + c_4 h''' + c_3 h'' + h^2 - c_2 h = 0. \tag{35}$$

In this case, [Eq. \(36\)](#) does not admit any nontrivial Lie symmetry but the substitution $w(h) = (h')^2$ leads to the following third-order equation

$$2c_5 w w''' + (c_5 w' \pm 2c_4 \sqrt{w})w'' + 2c_3 w' + 4h^2 - 4c_2 h = 0. \tag{36}$$

Example 3. Now, we consider the generator

$$Y = Y_5 + Y_6 = x\partial_x + u\partial_u + 2f\partial_f + g\partial_g + 2\alpha\partial_\alpha + 3\beta\partial_\beta + 4\gamma\partial_\gamma + 3\phi\partial_\phi. \tag{37}$$

The projection (16) is given by

$$Z = Z^{(8)}|_{a=0, b=1} = u\partial_u + 2f\partial_f + g\partial_g + 2\alpha\partial_\alpha + 3\beta\partial_\beta + 4\gamma\partial_\gamma + 3\phi\partial_\phi. \tag{38}$$

Requiring the invariance of (20) with respect to the operator (38) we obtain that the functions are given by

$$f = c_1 u^2, \quad g = c_2 u, \quad \alpha = c_3 u^2, \quad \beta = c_4 u^3, \quad \gamma = c_5 u^4, \quad \phi = c_6 u^3,$$

where c_i are arbitrary constants. Thus, we conclude that Eq. (5) with the above functions

$$u_t + 2uu_x(c_1 + 3c_6u_x) + u^2u_{xx}(c_3 + 3c_6) + c_4u^3u_{xxx} + c_5u^4u_{xxxx} - c_2u = 0, \tag{39}$$

admits the additional symmetry

$$X_3 = x\partial_x + u\partial_u. \tag{40}$$

The corresponding invariant solution is written in the form

$$z = t, \quad u = xh(z), \tag{41}$$

where $h(z)$ must satisfy

$$h' = -6c_6h^3 - 2c_1h^2 + c_2h. \tag{42}$$

If $c_6 = 0$, Eq. (42) is the Bernoulli equation, thus we can obtain the corresponding solution h of the ODE

$$h(z) = \frac{c_2e^{c_2z}}{2c_1e^{c_2z} - e^{c_0c_2}},$$

where c_0 is a constant of integration. In the case that $c_6 \neq 0$, Eq. (42) is an Abel equation of the first kind, more specifically is a separable equation whose implicit solution is given by

$$\log(2h(3c_6y + c_1) - c_2) - 2\log(h) + \frac{2c_1 \arctan\left(\frac{6c_6h + c_1}{\sqrt{-c_1^2 - 6c_2c_6}}\right)}{\sqrt{-c_1^2 - 6c_2c_6}} = -2c_2z + c_0,$$

where c_0 is an arbitrary constant.

Example 4. Finally, we take into account generator

$$Y = Y_3 + Y_7 + Y_8 = t\partial_x + \partial_u + u\partial_f + \partial_\alpha - u\partial_\phi, \tag{43}$$

whose projection is

$$Z = Z^{(27)}|_{a=0, m=1} = \partial_u + u\partial_f + \partial_\alpha - u\partial_\phi. \tag{44}$$

The invariance of (20) with respect to (44) leads us to

$$f = \frac{u^2}{2} + c_1, \quad g = c_2, \quad \alpha = u + c_3, \quad \beta = c_4, \quad \gamma = c_5, \quad \phi = -\frac{u^2}{2} + c_6,$$

where c_i are arbitrary constants. Eq. (5) with the above specific functions

$$u_t + uu_x + c_3u_{xx} - u_x^2 + c_4u_{xxx} + c_5u_{xxxx} - c_2 = 0, \tag{45}$$

admits the following additional operator

$$X_3 = t\partial_x + \partial_u. \tag{46}$$

The corresponding invariant solution is written in the form

$$z = t, \quad u = \frac{x}{t} + h(z), \tag{47}$$

where $h(z)$ must satisfy

$$z^2h' - c_2z^2 + hz - 1 = 0. \tag{48}$$

For this reduction, the ODE is a first order equation whose solution is

$$h(z) = \frac{c_2z}{2} + \frac{\ln(z) + c_0}{z},$$

where c_0 is a constant of integration. Undoing transformation (47), we obtain a solution of Eq. (45) which is given by

$$u(t, x) = \frac{c_2t}{2} + \frac{x + \ln(t) + c_0}{t}.$$

6. Conclusions

In this paper we have considered a fourth-order nonlinear partial differential equation containing six arbitrary functions depending on the dependent variable. We have constructed the equivalence group of the class which is exploited to simplify the classifying equations. We have determined that class (5) has a ten-dimensional algebra (10). We have used this algebra to perform a preliminary group classification. According to a theorem on projections, the problem of preliminary group classification of class (5) is reduced to the construction of the optimal system of subalgebras of the nonzero projections of (10) on the space $(u, f, g, \alpha, \beta, \gamma, \phi)$. From the optimal system is possible to obtain all nonequivalent equations (5) which admit an additional symmetry. In order to clarify the algorithm, we have taken some nonequivalent equations admitting extensions of the principal Lie algebra. Moreover, we have shown the corresponding coefficients $f, g, \alpha, \beta, \gamma, \phi$, and the extra symmetry operator. Finally, by using these symmetries, we have reduced the PDE into ODEs and some exact solutions have been derived.

Acknowledgements

The authors gratefully acknowledge helpful discussions on equivalence transformations and the collaboration received by Dr. Rita Tracina (Università degli studi di Catania). The authors also acknowledge the financial support from Junta de Andalucía group FQM-201, [Universidad de Cádiz](#) and they express their sincere gratitude to the Plan Propio de Investigación de la Universidad de Cádiz.

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