On split Leibniz algebras

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Abstract

In order to begin an approach to the structure of arbitrary Leibniz algebras, (with no restrictions neither on the dimension nor on the base field), we introduce the class of split Leibniz algebras as the natural extension of the class of split Lie algebras. By developing techniques of connections of roots for this kind of algebras, we show that any of such algebras $\mathfrak{L}$ is of the form $\mathfrak{L} = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian subalgebra H and any I_j a well described ideal of $\mathfrak{L}$, satisfying $[I_j, I_k] = 0$ if $j \neq k$. In the case of $\mathfrak{L}$ being of maximal length we characterize the simplicity of the algebra in terms of connections of roots.

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1 Introduction and previous definitions

Throughout this paper, Leibniz algebras $\mathfrak{L}$ are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim \mathfrak{L}_\alpha$, the products $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$, or $\{k \in \mathbb{K} : k\alpha \in \Lambda, \text{ for a fixed } \alpha \in \Lambda\}$, where $\mathfrak{L}_\alpha$ denotes the root space associated to the root α, and Λ the set of nonzero roots of $\mathfrak{L}$.

Leibniz algebras were introduced as a nonantisymmetric analogue of Lie algebras by Loday [16]. The structure of this kind of algebras has been considered in the frameworks of low dimensional algebras, nilpotence and related problems [3, 6, 7, 8, 11, 15, 14], the simple case being introduced in [1, 2] where some results concerning special cases of simple Leibniz algebras were also obtained.

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All of these works necessarily concern finite dimensional Leibniz algebras. In the present paper we try to give an approach to the structure of infinite dimensional Leibniz algebras by introducing the class of split Leibniz algebras of arbitrary dimension as the natural extension of the class of split Lie algebras.

In §2 we develop techniques of connections of roots in the framework of split Leibniz algebras so as to show that $\mathfrak{L}$ is of the form $\mathfrak{L} = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian subalgebra H and any I_j a well described ideal of $\mathfrak{L}$ satisfying $[I_j, I_k] = 0$ if $j \neq k$. In §3 we concrete on split Leibniz algebras of maximal length $\mathfrak{L}$ by centering our attention on the simplicity of this kind of algebras. As a main result we give a characterization of the simplicity of $\mathfrak{L}$ in terms of connectivity on the set of its nonzero roots.

Definition 1.1. *A* Leibniz algebra $\mathfrak{L}$ *is a vector space over a field* $\mathbb{K}$ *endowed with a bilinear product* $[\cdot,\cdot]$ *satisfying the* Leibniz identity

$$[[y,z],x] = [[y,x],z] + [y,[z,x]],$$

for all $x, y, z \in \mathfrak{L}$.

Clearly Lie algebras are examples of Leibniz algebras.

For any $x \in \mathfrak{L}$, consider the *adjoint mapping* $\mathrm{ad}_x : \mathfrak{L} \longrightarrow \mathfrak{L}$ defined by $\mathrm{ad}_x(z) = [z, x]$. Observe that Leibniz identity is equivalent to assert that ad_x is a derivation for any $x \in \mathfrak{L}$. An *ideal* I of $\mathfrak{L}$ is a vector subspace such that $[I, \mathfrak{L}] + [\mathfrak{L}, I] \subset I$.

Let $\mathfrak{L}$ be a Leibniz algebra, the ideal $\mathfrak{I}$ generated by $\{[x,x] : x \in \mathfrak{L}\}$ plays an important role in the theory since it determines the (possible) non-Lie character of $\mathfrak{L}$. From the Leibniz identity, this ideal satisfies

$$[\mathfrak{L}, \mathfrak{I}] = 0. \tag{1}$$

Let us introduce the class of split algebras in the framework of Leibniz algebras. We recall that given an element x of a Lie algebra L, and by denoting also $\mathrm{ad}(x)$ for the adjoint mapping $\mathrm{ad}(x)(y) := [y, x]$. A splitting Cartan subalgebra H of a Lie algebra L is defined as a maximal abelian subalgebra, (MASA), of L satisfying that the adjoint mappings $\mathrm{ad}(h)$ for $h \in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a split Lie algebra, (see [9, 18]). This means that we have a root spaces decomposition $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ where $L_\alpha = \{v_\alpha \in L : [v_\alpha, h] = \alpha(h)v_\alpha$ for any $h \in H\}$ for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\} : L_\alpha \neq 0\}$. The subspaces L_α for $\alpha \in H^*$ are called root spaces of L and the elements $\alpha \in \Lambda \cup \{0\}$ are called roots of L. We introduce the concept of split Leibniz algebra in a analogous way. Given a subalgebra S of a Leibniz algebra $\mathfrak{L}$, we say that S is *abelian* if $[S, S] = 0$.

Definition 1.2. *Denote by* H *a maximal abelian subalgebra of a Leibniz algebra* $\mathfrak{L}$*. For a linear functional* $\alpha : H \longrightarrow \mathbb{K}$*, we define the* root space *of* $\mathfrak{L}$*, (respect* *to H), associated to α as the subspace*

$$\mathfrak{L}_\alpha = \{v_\alpha \in \mathfrak{L} : [v_\alpha, h] = \alpha(h)v_\alpha \ \ \textit{for any} \ \ h \in H\}.$$

The elements $\alpha \in H^*$ satisfying $\mathfrak{L}_\alpha \neq 0$ are called roots *of $\mathfrak{L}$ respect to H and we denote $\Lambda := \{\alpha \in H^* \backslash \{0\} : \mathfrak{L}_\alpha \neq 0\}$. We say that $\mathfrak{L}$ is a* split Leibniz algebra*, respect to H, if*

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha).$$

We also say that Λ is the root system *of $\mathfrak{L}$.*

Split Lie algebras are examples of split Leibniz algebras. Hence, the present paper extends the results in [9].

It is clear that the root space associated to the zero root satisfies $H \subset \mathfrak{L}_0$. Conversely, given any $v_0 \in \mathfrak{L}_0$ we can write $v_0 = h + \sum\limits_{i=1}^{n} v_{\alpha_i}$ with $h \in H$ and $v_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$ for $i = 1, ..., n$, being $\alpha_i \in \Lambda$ with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence $0 = [h + \sum\limits_{i=1}^{n} v_{\alpha_i}, h'] = \sum\limits_{i=1}^{n} \alpha_i(h')v_{\alpha_i}$ for any $h' \in H$. So, taking into account the direct character of the sum and that $\alpha_i \neq 0$, we have that any $v_{\alpha_i} = 0$ and then $v_0 \in H$. Consequently

$$H = \mathfrak{L}_0.$$

The below lemma is an immediate consequence of Leibniz identity.

Lemma 1.1. *If $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \neq 0$ with $\alpha, \beta \in \Lambda \cup \{0\}$, then $\alpha + \beta \in \Lambda \cup \{0\}$ and $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha+\beta}$.*

Definition 1.3. *A root system Λ of a split Leibniz algebra $\mathfrak{L}$ is called* symmetric *if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.*

2 Decompositions

In the following, $\mathfrak{L}$ denotes a split Leibniz algebra with a symmetric root system Λ and $\mathfrak{L} = H \oplus (\bigoplus\limits_{\alpha \in \Lambda} \mathfrak{L}_\alpha)$ the corresponding root decomposition. We begin by developing connections of roots techniques in this setting.

Definition 2.1. *Let α and β be two nonzero roots. We say that α is* connected *to β if there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that*

1. $\alpha_1 = \alpha$.
2. $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, ..., \alpha_1 + \cdots + \alpha_{n-1}\} \subset \Lambda$.
3. $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \in \pm\beta$.

We also say that $\{\alpha_1, ..., \alpha_n\}$ is a connection *from α to β.*

Observe that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$ and so α is connected to $\pm\alpha$.

The next result shows the connection relation is of equivalence.

Proposition 2.1. *The relation $\sim$ in Λ defined by $\alpha \sim \beta$ if and only if α is connected to β is of equivalence.*

Proof. $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

Let us see the symmetric character of $\sim$: If $\alpha \sim \beta$, there exists a connection

$$\{\alpha_1, \alpha_2, \alpha_3..., \alpha_{n-1}, \alpha_n\} \subset \Lambda$$

from α to β. Then $\alpha_1 = \alpha$,

$$\{\alpha_1, \alpha_1 + \alpha_2, ..., \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\} \subset \Lambda, \tag{2}$$

and $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n \in \{\beta, -\beta\}$. Hence, we can distinguish two possibilities. In the first one

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n = \beta$$

and in the second one

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n = -\beta. \tag{3}$$

Suppose we have the first one. By the symmetry of Λ, we can consider the set of nonzero roots

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n, -\alpha_n, -\alpha_{n-1}, ..., -\alpha_3, -\alpha_2\} \subset \Lambda.$$

By equation (2), this family of elements in Λ clearly satisfy

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n = \beta,$$
$$(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) - \alpha_n = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \in \Lambda,$$
$$(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) - \alpha_n - \alpha_{n-1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} \in \Lambda,$$
$$\vdots$$
$$(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) - \alpha_n - \alpha_{n-1} \cdots - \alpha_3 = \alpha_1 + \alpha_2 \in \Lambda$$

and

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) - \alpha_n - \alpha_{n-1} \cdots - \alpha_3 - \alpha_2 = \alpha_1 = \alpha.$$

From here, β is connected to α, that is, $\beta \sim \alpha$.

Suppose now we are in the second possibility given by equation (3). In this case we have as above that $\{-\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1} - \alpha_n, \alpha_n, \alpha_{n-1}, ..., \alpha_2\}$ is a connection from β to α and $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, and write $\{\alpha_1, ..., \alpha_n\}$ for a connection from α to β and $\{\beta_1, ..., \beta_m\}$ for a connection from β to γ. If $m > 1$, then $\{\alpha_1, ..., \alpha_n, \beta_2, ..., \beta_m\}$ is a connection from α to γ in case $\alpha_1 + ... + \alpha_n = \beta$, and $\{\alpha_1, ..., \alpha_n, -\beta_2, ..., -\beta_m\}$ in case $\alpha_1 + ... + \alpha_n = -\beta$. If $m = 1$, then $\gamma \in \pm\beta$ and so $\{\alpha_1, ..., \alpha_n\}$ is a connection from α to γ. Therefore $\alpha \sim \gamma$ and $\sim$ is of equivalence. $\square$

Given $\alpha \in \Lambda$, we denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \beta \sim \alpha\}.$$

Clearly if $\beta \in \Lambda_\alpha$ then $-\beta \in \Lambda_\alpha$ and, by Proposition 2.1, if $\gamma \notin \Lambda_\alpha$ then $\Lambda_\alpha \cap \Lambda_\gamma = \emptyset$.

Our next goal is to associate an (adequate) ideal I_{Λ_α} to any Λ_α. For Λ_α, $\alpha \in \Lambda$, we define

$$H_{\Lambda_\alpha} := span_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] : \beta \in \Lambda_\alpha\} \subset H,$$

and

$$V_{\Lambda_\alpha} := \bigoplus_{\beta \in \Lambda_\alpha} \mathfrak{L}_\beta.$$

We denote by $\mathfrak{L}_{\Lambda_\alpha}$ the following subspace of $\mathfrak{L}$,

$$\mathfrak{L}_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.$$

Proposition 2.2. *Let $\alpha \in \Lambda$. Then the following assertions hold.*

1. $[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\alpha}] \subset \mathfrak{L}_{\Lambda_\alpha}$.
2. *If $\gamma \notin \Lambda_\alpha$ then $[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\gamma}] = 0$.*

Proof. 1. Taking into account $H = \mathfrak{L}_0$ and Lemma 1.1, we have

$$[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\alpha}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha} + \sum_{\beta,\delta \in \Lambda_\alpha} [\mathfrak{L}_\beta, \mathfrak{L}_\delta]. \tag{4}$$

If $\delta = -\beta$ then

$$[\mathfrak{L}_\beta, \mathfrak{L}_\delta] \subset H_{\Lambda_\alpha}. \tag{5}$$

If $\delta \neq -\beta$, by Lemma 1.1 we have that in case $[\mathfrak{L}_\beta, \mathfrak{L}_\delta] \neq 0$ then $\beta + \delta \in \Lambda$. From here, if $\{\alpha_1, ..., \alpha_n\}$ is a connection from α to β then $\{\alpha_1, ..., \alpha_n, \delta\}$ is a connection from α to $\beta + \delta$ in case $\alpha_1 + ... + \alpha_n = \beta$ and $\{\alpha_1, ..., \alpha_n, -\delta\}$ in case $\alpha_1 + ... + \alpha_n = -\beta$. Hence $\delta + \beta \in \Lambda_\alpha$ and so

$$[\mathfrak{L}_\beta, \mathfrak{L}_\delta] \subset V_{\Lambda_\alpha}. \tag{6}$$

From equations (4), (5) and (6) we conclude $[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\alpha}] \subset \mathfrak{L}_{\Lambda_\alpha}$.

2. We have

$$[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\gamma}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\gamma} \oplus V_{\Lambda_\gamma}] \subset [H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]. \tag{7}$$

Consider the above third summand $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ and suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[\mathfrak{L}_\beta, \mathfrak{L}_\eta] \neq 0$. As necessarily $\beta \neq -\eta$, then $\beta + \eta \in \Lambda$. So $\{\beta, \eta, -\beta\}$ is a connection between β and η. By the transitivity of the connection relation we have $\gamma \in \Lambda_\alpha$, a contradiction. Hence $[\mathfrak{L}_\beta, \mathfrak{L}_\eta] = 0$ and so

$$[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0. \tag{8}$$

Consider now the first summand $[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ in equation (7) and suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}], \mathfrak{L}_\eta] \neq 0$. By Leibniz identity we get either $[\mathfrak{L}_\beta, \mathfrak{L}_\eta] \neq 0$ or $[\mathfrak{L}_{-\beta}, \mathfrak{L}_\eta] \neq 0$. From here $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] \neq 0$ in any case, what contradicts equation (8). Hence

$$[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0.$$

Finally, we note that the same above argument shows

$$[V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] = 0.$$

By equation (7) we conclude $[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_{\Lambda_\gamma}] = 0$. □

Proposition 2.2-1 let us assert that for any $\alpha \in \Lambda$, $\mathfrak{L}_{\Lambda_\alpha}$ is a subalgebra of $\mathfrak{L}$ that we call the subalgebra of $\mathfrak{L}$ *associated* to Λ_α.

The usual definition of simple algebra lack of interest in the case of Leibniz algebras because would imply the ideal $\mathfrak{I} = \mathfrak{L}$ or $\mathfrak{I} = 0$, being so $\mathfrak{L}$ an abelian or a Lie algebra respectively. Abdykassymova and Dzhumadil'daev introduced in [1, 2] the following adequate definition.

Definition 2.2. *A Leibniz algebra $\mathfrak{L}$ is said to be* simple *if its product is nonzero and its only ideals are $\{0\}$, $\mathfrak{I}$ and $\mathfrak{L}$.*

It should be noted that the above definition agrees with the definition of simple Lie algebra, since $\mathfrak{I} = \{0\}$ in this case.

Theorem 2.1. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the subalgebra*
$$\mathfrak{L}_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$$
of $\mathfrak{L}$ associated to Λ_α is an ideal of $\mathfrak{L}$.

2. *If $\mathfrak{L}$ is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$.*

Proof. 1. Since $[\mathfrak{L}_{\Lambda_\alpha}, H] + [H, \mathfrak{L}_{\Lambda_\alpha}] = [\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}_0] + [\mathfrak{L}_0, \mathfrak{L}_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}$, taking into account Proposition 2.2 we have

$$[\mathfrak{L}_{\Lambda_\alpha}, \mathfrak{L}] = [\mathfrak{L}_{\Lambda_\alpha}, H \oplus (\bigoplus_{\beta \in \Lambda_\alpha} \mathfrak{L}_\beta) \oplus (\bigoplus_{\gamma \notin \Lambda_\alpha} \mathfrak{L}_\gamma)] \subset \mathfrak{L}_{\Lambda_\alpha}$$

and

$$[\mathfrak{L}, \mathfrak{L}_{\Lambda_\alpha}] = [H \oplus (\bigoplus_{\beta \in \Lambda_\alpha} \mathfrak{L}_\beta) \oplus (\bigoplus_{\gamma \notin \Lambda_\alpha} \mathfrak{L}_\gamma), \mathfrak{L}_{\Lambda_\alpha}] \subset \mathfrak{L}_{\Lambda_\alpha}.$$

2. The simplicity of $\mathfrak{L}$ implies $\mathfrak{L}_{\Lambda_\alpha} \in \{\mathfrak{I}, \mathfrak{L}\}$ for any $\alpha \in \Lambda$. If some $\alpha \in \Lambda$ is such that $\mathfrak{L}_{\Lambda_\alpha} = \mathfrak{L}$, then $\Lambda_\alpha = \Lambda$. Hence, $\mathfrak{L}$ has all its nonzero roots connected and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$. Otherwise, if $\mathfrak{L}_{\Lambda_\alpha} = \mathfrak{I}$ for any $\alpha \in \Lambda$ then $\Lambda_\alpha = \Lambda_\beta$ for any $\alpha, \beta \in \Lambda$ and so $\Lambda_\alpha = \Lambda$. We also conclude that $\mathfrak{L}$ has all its nonzero roots connected and $H = \sum\limits_{\alpha \in \Lambda} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$. □

Theorem 2.2. *For a vector space complement* $\mathcal{U}$ *of* $span_{\mathbb{K}}\{[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] : \alpha \in \Lambda\}$ *in* H*, we have*

$$\mathfrak{L} = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ *is one of the ideals* $\mathfrak{L}_{\Lambda_\alpha}$ *of* $\mathfrak{L}$ *described in Theorem 2.1-1, satisfying* $[I_{[\alpha]}, I_{[\beta]}] = 0$ *if* $[\alpha] \neq [\beta]$.

Proof. By Proposition 2.1, we can consider the quotient set $\Lambda/\sim := \{[\alpha] : \alpha \in \Lambda\}$. Let us denote by $I_{[\alpha]} := \mathfrak{L}_{\Lambda_\alpha}$.We have $I_{[\alpha]}$ is well defined and by Theorem 2.1-1 an ideal of $\mathfrak{L}$. Therefore

$$L = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

By applying Proposition 2.2-2 we also obtain $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

Definition 2.3. *The* annihilator *of a Leibniz algebra* $\mathfrak{L}$ *is the set* $\mathrm{Z}(\mathfrak{L}) = \{x \in \mathfrak{L} : [x, \mathfrak{L}] + [\mathfrak{L}, x] = 0\}$.

Corollary 2.1. *If* $\mathrm{Z}(\mathfrak{L}) = 0$ *and* $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$*, then* $\mathfrak{L}$ *is the direct sum of the ideals given in Theorem 2.1-1,*

$$\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$ it is clear that $\mathfrak{L} = \sum\limits_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. The direct character of the sum now follows from the facts $[I_{[\alpha]}, I_{[\beta]}] = 0$, if $[\alpha] \neq [\beta]$, and $\mathrm{Z}(\mathfrak{L}) = 0$. □

3 Split Leibniz algebras of maximal length. The simple case.

In this section we focus on the simplicity of split Leibniz algebras by centering our attention in those of maximal length. This terminology is taking borrowed from the theory of gradations of Lie and Leibniz algebras, (see [4, 5, 12, 13]). See also [5, 9, 10, 18] for examples. From now on $\mathrm{char}(\mathbb{K}) = 0$.

Definition 3.1. *We say that a split Leibniz algebra* $\mathfrak{L}$ *is of* maximal length *if* $\dim \mathfrak{L}_\alpha = 1$ *for any* $\alpha \in \Lambda$.

Our target is to characterize the simplicity of $\mathfrak{L}$ in terms of connectivity properties in Λ. We begin with a series of lemmas which hold for arbitrary split Leibniz algebras over a field of characteristic zero.

Lemma 3.1. *Let* $\mathfrak{L}$ *be a split Leibniz algebra with* $\mathrm{Z}(\mathfrak{L}) = 0$ *and* I *an ideal of* $\mathfrak{L}$*. If* $I \subset H$ *then* $I = \{0\}$.

Proof. Suppose there exists a nonzero ideal I of $\mathfrak{L}$ such that $I \subset H$. We have $[I, H] + [H, I] \subset [H, H] = 0$. We also have $[I, \bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha] + [\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha, I] \subset I \subset H$. Then, taking into account $H = \mathfrak{L}_0$, we get $[I, \bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha] + [\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha, I] \subset H \cap (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha) = 0$. From here $I \subset \mathrm{Z}(\mathfrak{L}) = 0$, a contradiction. □

Lemma 3.2. *Let $\mathfrak{L}$ be a split Leibniz algebra. For any $\alpha, \beta \in \Lambda$ with $\alpha \neq \epsilon\beta$, $\epsilon \in \mathbb{K}$, there exists $h_{\alpha,\beta} \in H$ such that $\alpha(h_{\alpha,\beta}) \neq 0$ and $\beta(h_{\alpha,\beta}) = 0$.*

Proof. As $\alpha \neq 0$, there exists $h \in H \setminus \{0\}$ such that $\alpha(h) \neq 0$. If $\beta(h) = 0$ we take $h_{\alpha,\beta} := h$. Suppose therefore $\beta(h) \neq 0$ and let us write $\epsilon = \alpha(h)\beta(h)^{-1}$. As $\alpha \neq \epsilon\beta$, there exists $h' \in H$ such that $\alpha(h') \neq \epsilon\beta(h')$. We can take $h_{\alpha,\beta} := \beta(h')h - \beta(h)h'$. □

Lemma 3.3. *Let $\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$ be a split Leibniz algebra. If I is an ideal of $\mathfrak{L}$ then $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda} (I \cap \mathfrak{L}_\alpha))$.*

Proof. Let $x \in I$ be. We can write $x = h_0 + \sum_{j=1}^{m} v_{\alpha_j}$ with $h_0 \in H$, $v_{\alpha_j} \in \mathfrak{L}_{\alpha_j}$ and $\alpha_j \neq \alpha_k$ if $j \neq k$. Let us show that any $v_{\alpha_j} \in I$. If $v_{\alpha_1} = 0$ then $v_{\alpha_1} \in I$. Suppose then $v_{\alpha_1} \neq 0$. For any $\alpha_{k_r} \neq \epsilon\alpha_1$, $\epsilon \in \mathbb{K}$ and $k_r \in \{2, ..., m\}$, Lemma 3.2 gives us $h_{\alpha_1,\alpha_{k_r}} \in H$ satisfying $\alpha_1(h_{\alpha_1,\alpha_{k_r}}) \neq 0$ and $\alpha_{k_r}(h_{\alpha_1,\alpha_{k_r}}) = 0$. From here,

$$[[...[[x, h_{\alpha_1,\alpha_{k_2}}], h_{\alpha_1,\alpha_{k_3}}], ...], h_{\alpha_1,\alpha_{k_s}}] =$$

$$\epsilon_1 v_{\alpha_1} + \sum_{t=1}^{p} \epsilon_{\mu_t} v_{\mu_t\alpha_1} \in I, \tag{9}$$

$\epsilon_1, \mu_t \in \mathbb{K} - \{0\}$, $\mu_t \neq 1$, $\epsilon_{\mu_t} \in \mathbb{K}$.

If any $\epsilon_{\mu_t} = 0$, $t = 1, ..., p$, then $\epsilon_1 v_{\alpha_1} \in I$ and so $v_{\alpha_1} \in I$. Let us suppose some $\epsilon_{\mu_t} \neq 0$ and write equation (9) as

$$\epsilon_1 v_{\alpha_1} + \sum_{t=1}^{r} \epsilon_{\mu_t} v_{\mu_t\alpha_1} \in I, \tag{10}$$

with $\epsilon_1, \mu_t, \epsilon_{\mu_t} \in \mathbb{K} - \{0\}$, $\mu_t \neq 1$, $r \leq p$.

Let $h \in H$ be such that $\alpha_1(h) \neq 0$, then

$$[\epsilon_1 v_{\alpha_1} + \sum_{t=1}^{r} \epsilon_{\mu_t} v_{\mu_t\alpha_1}, h] = \epsilon_1\alpha_1(h)v_{\alpha_1} + \sum_{t=1}^{r} \epsilon_{\mu_t}\mu_t\alpha_1(h)v_{\mu_t\alpha_1} \in I,$$

and so

$$\epsilon_1 v_{\alpha_1} + \sum_{t=1}^{r} \epsilon_{\mu_t}\mu_t v_{\mu_t\alpha_1} \in I, \ \mu_t \neq 1. \tag{11}$$

From equations (10) and (11) it follows easily that

$$\kappa_1 v_{\alpha_1} + \sum_{t=1}^{s} \kappa_{q_t} v_{q_t \alpha_1} \in I \tag{12}$$

with $\kappa_1, \kappa_{q_t} \in \mathbb{K} - \{0\}$, $q_t \in \{\mu_t : t = 1, ..., r\}$ and $s < r$. Following this process, (multiply equation (12) with h and compare the result with equation (12) taking into account $q_t \neq 1$, and so on), we obtain $v_{\alpha_1} \in I$. The same argument holds for any α_j, $j \neq 1$. From here, we deduce $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda} (I \cap M_\alpha))$. □

Let us return to a split Leibniz algebra of maximal length $\mathfrak{L}$. In fact, from now on $\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$ denotes a split Leibniz algebra of maximal length. We begin by observing that in this case the previous lemma let us assert that given any nonzero ideal I of $\mathfrak{L}$ then

$$I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda^I} \mathfrak{L}_\alpha) \text{ with } \Lambda^I \subset \Lambda\ . \tag{13}$$

In the particular, (an important), case $I = \mathfrak{I}$, we get

$$\mathfrak{I} = (\mathfrak{I} \cap H) \oplus (\bigoplus_{\alpha\in\Lambda^{\mathfrak{I}}} \mathfrak{L}_\alpha). \tag{14}$$

From here, we can write

$$\Lambda = \Lambda^{\mathfrak{I}} \dot\cup \Lambda^{\neg\mathfrak{I}}, \tag{15}$$

where

$$\Lambda^{\mathfrak{I}} := \{\alpha \in \Lambda : \mathfrak{L}_\alpha \subset \mathfrak{I}\}$$

and

$$\Lambda^{\neg\mathfrak{I}} := \{\alpha \in \Lambda : \mathfrak{L}_\alpha \cap \mathfrak{I} = 0\}.$$

As consequence

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha) \oplus (\bigoplus_{\beta\in\Lambda^{\mathfrak{I}}} \mathfrak{L}_\beta). \tag{16}$$

We note that the fact $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, the split decomposition given by equation (16) and equation (1) show

$$H = \sum_{\alpha\in\Lambda^{\neg\mathfrak{I}}} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]. \tag{17}$$

Now, observe that the concept of connectivity of nonzero roots given in Definition 2.1 is not strong enough to detect if a given $\alpha \in \Lambda$ belongs to $\Lambda^{\mathfrak{I}}$ or to $\Lambda^{\neg\mathfrak{I}}$. Consequently we lose the information respect to whether a given root space $\mathfrak{L}_\alpha$ is contained in $\mathfrak{I}$ or not, which is fundamental to study the simplicity of $\mathfrak{L}$. So, we are going to refine the concept of connections of non-zero roots in the setup of maximal length split Leibniz algebras. The symmetry of $\Lambda^{\mathfrak{I}}$ and $\Lambda^{\neg\mathfrak{I}}$ will be understood as usual. That is, Λ^Υ, $\Upsilon \in \{\mathfrak{I}, \neg\mathfrak{I}\}$, is called *symmetric* if $\alpha \in \Lambda^\Upsilon$ implies $-\alpha \in \Lambda^\Upsilon$.

Definition 3.2. *Let $\alpha, \beta \in \Lambda^{\Upsilon}$ with $\Upsilon \in \{\mathfrak{I}, \neg\mathfrak{I}\}$. We say that α is* $\neg\mathfrak{I}$-connected *to β, denoted by $\alpha \sim_{\neg\mathfrak{I}} \beta$, if there exist*

$$\alpha_2, ..., \alpha_n \in \Lambda^{\neg\mathfrak{I}}$$

such that

$$\{\alpha_1, \alpha_1 + \alpha_2, ..., \alpha_1 + \cdots + \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\} \subset \Lambda^{\Upsilon},$$

$\alpha_1 = \alpha$ and $\alpha_1 + \cdots + \alpha_n \in \pm\beta$. The set $\{\alpha_1, ..., \alpha_n\}$ is called a $\neg\mathfrak{I}$-connection *from α to β.*

Proposition 3.1. *The following assertions hold.*

1. *If $\Lambda^{\neg\mathfrak{I}}$ is symmetric, then the relation $\sim_{\neg\mathfrak{I}}$ is of equivalence in $\Lambda^{\neg\mathfrak{I}}$.*
2. *If $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$ and $\Lambda^{\mathfrak{I}}$ is symmetric, then the relation $\sim_{\neg\mathfrak{I}}$ is of equivalence in $\Lambda^{\mathfrak{I}}$.*

Proof. 1. can be proved in a similar way to Proposition 2.1.

2. Let $\beta \in \Lambda^{\mathfrak{I}}$. Since $\beta \neq 0$, equation (17) gives us that there exists $\alpha \in \Lambda^{\neg\mathfrak{I}}$ such that $[\mathfrak{L}_\beta, [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]] \neq 0$. By Leibniz identity, either $[[\mathfrak{L}_\beta, \mathfrak{L}_\alpha], \mathfrak{L}_{-\alpha}] \neq 0$ or $[[\mathfrak{L}_\beta, \mathfrak{L}_{-\alpha}], \mathfrak{L}_\alpha] \neq 0$. In the first case, the $\neg\mathfrak{I}$-connection $\{\beta, \alpha, -\alpha\}$ gives us $\beta \sim_{\neg\mathfrak{I}} \beta$ while in the second one the $\neg\mathfrak{I}$-connection $\{\beta, -\alpha, \alpha\}$ gives us the same result. Consequently $\sim_{\neg\mathfrak{I}}$ is reflexive in $\Lambda^{\mathfrak{I}}$. The symmetric and transitive character of $\sim_{\neg\mathfrak{I}}$ in $\Lambda^{\mathfrak{I}}$ follows as in Proposition 2.1. $\square$

Let us introduce the notion of root-multiplicativity in the framework of split Leibniz algebras of maximal length, in a similar way to the ones for split Lie algebras and split Lie triple systems, (see [9, 10] for these notions and examples).

Definition 3.3. *We say that a split Leibniz algebra of maximal length $\mathfrak{L}$ is* root-multiplicative *if the below conditions hold.*

1. *Given $\alpha, \beta \in \Lambda^{\neg\mathfrak{I}}$ such that $\alpha + \beta \in \Lambda$ then $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \neq 0$.*
2. *Given $\alpha \in \Lambda^{\neg\mathfrak{I}}$ and $\gamma \in \Lambda^{\mathfrak{I}}$ such that $\alpha + \gamma \in \Lambda^{\mathfrak{I}}$ then $[\mathfrak{L}_\gamma, \mathfrak{L}_\alpha] \neq 0$.*

Another interesting notion related to split Leibniz algebras of maximal length $\mathfrak{L}$ is those of Lie-annihilator. Write $\mathfrak{L} = H \oplus (\bigoplus\limits_{\alpha \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha) \oplus (\bigoplus\limits_{\beta \in \Lambda^{\mathfrak{I}}} \mathfrak{L}_\beta)$, (see equation (16)).

Definition 3.4. *The* Lie-annihilator *of a split Leibniz algebra of maximal length $\mathfrak{L}$ is the set*

$$\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = \{x \in \mathfrak{L} : [x, H \oplus (\bigoplus_{\alpha \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha)] + [H \oplus (\bigoplus_{\alpha \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha), x] = 0\}.$$

Clearly the above definition agrees with the definition of annihilator of a Lie algebra, since in this case $\Lambda^{\mathfrak{I}} = \emptyset$. We also have $\mathrm{Z}(\mathfrak{L}) \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L})$.

Proposition 3.2. *Suppose $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$ and $\mathfrak{L}$ is root-multiplicative. If $\Lambda^{\neg\mathfrak{I}}$ has all of its roots $\neg\mathfrak{I}$-connected, then any ideal I of $\mathfrak{L}$ such that $I \nsubseteq H \oplus \mathfrak{I}$ satisfies $I = \mathfrak{L}$.*

Proof. By equations (13) and (15) we can write

$$I = (I \cap H) \oplus (\bigoplus_{\alpha_i \in \Lambda^{\neg\mathfrak{I},I}} \mathfrak{L}_{\alpha_i}) \oplus (\bigoplus_{\beta_j \in \Lambda^{\mathfrak{I},I}} \mathfrak{L}_{\beta_j}) \tag{18}$$

with $\Lambda^{\neg\mathfrak{I},I} \subset \Lambda^{\neg\mathfrak{I}}$ and $\Lambda^{\mathfrak{I},I} \subset \Lambda^{\mathfrak{I}}$. As $I \nsubseteq H \oplus \mathfrak{I}$ we have $\Lambda^{\neg\mathfrak{I},I} \neq \emptyset$ and so we can fix some $\alpha_0 \in \Lambda^{\neg\mathfrak{I},I}$ such that

$$\mathfrak{L}_{\alpha_0} \subset I. \tag{19}$$

For any $\beta \in \Lambda^{\neg\mathfrak{I}}$, $\beta \neq \pm\alpha_0$, the fact that α_0 and β are $\neg\mathfrak{I}$-connected gives us a $\neg\mathfrak{I}$-connection $\{\gamma_1, ..., \gamma_r\} \subset \Lambda^{\neg\mathfrak{I}}$ from α_0 to β such that

$$\gamma_1 = \alpha_0,$$

$$\gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3, ..., \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_{r-1} \in \Lambda^{\neg\mathfrak{I}}$$

and

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_r \in \pm\beta.$$

Consider $\alpha_0 = \gamma_1$, γ_2 and $\gamma_1 + \gamma_2$. Since $\gamma_1, \gamma_2 \in \Lambda^{\neg\mathfrak{I}}$, the root-multiplicativity and maximal length of $\mathfrak{L}$ show $[\mathfrak{L}_{\alpha_0}, \mathfrak{L}_{\gamma_2}] = \mathfrak{L}_{\alpha_0+\gamma_2}$, and by equation (19)

$$\mathfrak{L}_{\alpha_0+\gamma_2} \subset I.$$

We can argue in a similar way from $\alpha_0 + \gamma_2$, γ_3 and $\alpha_0 + \gamma_2 + \gamma_3$ to get

$$\mathfrak{L}_{\alpha_0+\gamma_2+\gamma_3} \subset I.$$

Following this process with the $\neg\mathfrak{I}$-connection $\{\gamma_1, ..., \gamma_r\}$ we obtain that

$$\mathfrak{L}_{\alpha_0+\gamma_2+\gamma_3+\cdots+\gamma_r} \subset I$$

and so either $\mathfrak{L}_\beta \subset I$ or $\mathfrak{L}_{-\beta} \subset I$. That is,

$$\mathfrak{L}_{\epsilon\beta} \subset I \text{ for some } \epsilon \in \pm 1 \text{ and any } \beta \in \Lambda^{\neg\mathfrak{I}}.$$

Since $H = \sum_{\beta \in \Lambda^{\neg\mathfrak{I}}} [\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}]$, (see equation (17)), we get

$$H \subset I. \tag{20}$$

Now, given any $\delta \in \Lambda$, the facts $\delta \neq 0$, $H \subset I$ and the maximal length of $\mathfrak{L}$ show

$$[\mathfrak{L}_\delta, H] = \mathfrak{L}_\delta \subset I. \tag{21}$$

From equations (20) and (21) we conclude $I = \mathfrak{L}$. □

Proposition 3.3. *Suppose $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, $\mathrm{Z}(\mathfrak{L}) = 0$ and $\mathfrak{L}$ is root-multiplicative. If $\Lambda^{\neg\mathfrak{I}}$, $\Lambda^{\mathfrak{I}}$ are symmetric and $\Lambda^{\mathfrak{I}}$ has all of its roots $\neg\mathfrak{I}$-connected, then any nonzero ideal I of $\mathfrak{L}$ such that $I \subset \mathfrak{I}$ satisfies either $I = \mathfrak{I}$ or $\mathfrak{I} = I \oplus K$ with K an ideal of $\mathfrak{L}$.*

Proof. By equations (13), and (15) we can write

$$I = (I \cap H) \oplus \Big(\bigoplus_{\alpha_i \in \Lambda^{\mathfrak{I},I}} \mathfrak{L}_{\alpha_i} \Big)$$

with $\Lambda^{\mathfrak{I},I} \subset \Lambda^{\mathfrak{I}}$. Observe that the fact $\mathrm{Z}(\mathfrak{L}) = 0$ implies

$$\mathfrak{I} \cap H = \{0\}. \tag{22}$$

Indeed, $[\mathfrak{L}, \mathfrak{I} \cap H] + [\mathfrak{I} \cap H, \mathfrak{L}_\alpha] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ for any $\alpha \in \Lambda^{\mathfrak{I}}$, $[\mathfrak{I} \cap H, H] \subset [H, H] = 0$ and $[\mathfrak{I} \cap H, \mathfrak{L}_\beta] = 0$ for any $\beta \in \Lambda^{\neg\mathfrak{I}}$ because in the opposite case $\mathfrak{L}_\beta \subset \mathfrak{I}$, being then $\beta \in \Lambda^{\mathfrak{I}}$. From here $\mathfrak{I} \cap H \subset \mathrm{Z}(\mathfrak{L}) = 0$. Hence, we can write

$$I = \bigoplus_{\alpha_i \in \Lambda^{\mathfrak{I},I}} \mathfrak{L}_{\alpha_i},$$

with $\Lambda^{\mathfrak{I},I} \neq \emptyset$, and so we can take some $\alpha_0 \in \Lambda^{\mathfrak{I},I}$ such that $\mathfrak{L}_{\alpha_0} \subset I$. We can argue with the root-multiplicativity and the maximal length of $\mathfrak{L}$ as in Proposition 3.2 to conclude that given any $\beta \in \Lambda^{\mathfrak{I}}$, there exists a $\neg\mathfrak{I}$-connection $\{\gamma_1, ..., \gamma_r\}$ from α_0 to β such that

$$[[\cdots[\mathfrak{L}_{\alpha_0}, \mathfrak{L}_{\gamma_2}], \cdots], \mathfrak{L}_{\alpha_n}] \in \mathfrak{L}_{\pm\beta}$$

and so $\mathfrak{L}_{\epsilon\beta} \subset I$ for some $\epsilon \in \pm 1$. That is

$$\epsilon_\beta \beta \in \Lambda^{\mathfrak{I},I} \text{ for any } \beta \in \Lambda^{\mathfrak{I}} \text{ and some } \epsilon_\beta \in \pm 1.$$

Suppose $-\alpha_0 \in \Lambda^{\mathfrak{I},I}$. Then we also have that $\{-\gamma_1, ..., -\gamma_r\}$ is a $\neg\mathfrak{I}$-connection from $-\alpha_0$ to β satisfying

$$[[\cdots[\mathfrak{L}_{-\alpha_0}, \mathfrak{L}_{-\gamma_2}], \cdots], \mathfrak{L}_{-\alpha_n}] = \mathfrak{L}_{-\epsilon_\beta\beta} \subset I$$

and so $\mathfrak{L}_\beta + \mathfrak{L}_{-\beta} \subset I$. Equations (14) and (22) let us now conclude $I = \mathfrak{I}$.

Now suppose there is not any $\alpha_0 \in \Lambda^{\mathfrak{I},I}$ such that $-\alpha_0 \in \Lambda^{\mathfrak{I},I}$. Equation (15) let us write $\Lambda^{\mathfrak{I}} = \Lambda^{\mathfrak{I},I} \dot\cup (-\Lambda^{\mathfrak{I},I})$ and, (joint with equations (14) and (22)), assert that by denoting $K = \bigoplus_{\alpha_i \in \Lambda^{\mathfrak{I},I}} \mathfrak{L}_{-\alpha_i}$ we have

$$\mathfrak{I} = I \oplus K.$$

Let us finally show that K is an ideal of $\mathfrak{L}$. We have $[\mathfrak{L}, K] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ and

$$[K, \mathfrak{L}] \subset [K, H] + [K, \bigoplus_{\beta \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\beta] + [K, \bigoplus_{\gamma \in \Lambda^{\mathfrak{I}}} \mathfrak{L}_\gamma] \subset K + [K, \bigoplus_{\beta \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\beta].$$

Let us consider the last summand $[K, \bigoplus_{\beta\in\Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha]$ and suppose there exist $\alpha_i \in \Lambda^{\mathfrak{I},I}$ and $\beta \in \Lambda^{\neg\mathfrak{I}}$ such that $[\mathfrak{L}_{-\alpha_i}, \mathfrak{L}_\beta] \neq 0$. Since $\mathfrak{L}_{-\alpha_i} \subset K \subset \mathfrak{I}$, we get $-\alpha_i + \beta \in \Lambda^{\mathfrak{I}}$. By the root-multiplicativity of $\mathfrak{L}$, the symmetries of $\Lambda^{\neg\mathfrak{I}}$ and $\Lambda^{\mathfrak{I}}$, and the fact $\mathfrak{L}_{\alpha_i} \subset I$ we obtain $0 \neq [\mathfrak{L}_{\alpha_i}, \mathfrak{L}_{-\beta}] = \mathfrak{L}_{\alpha_i-\beta} \subset I$, that is $\alpha_i - \beta \in \Lambda^{\mathfrak{I},I}$. Hence, $-\alpha_i + \beta \in -\Lambda^{\mathfrak{I},I}$ and so $[\mathfrak{L}_{-\alpha_i}, \mathfrak{L}_\beta] \subset K$. Consequently $[K, \bigoplus_{\beta\in\Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\alpha] \subset K$ and K is an ideal of $\mathfrak{L}$. □

We introduce the definition of primeness in the framework of Leibniz algebras following the same motivation that in the case of simplicity (see Definition 2.2 and the above paragraph).

Definition 3.5. *A Leibniz algebra $\mathfrak{L}$ is said to be* prime *if given two ideals I, K of $\mathfrak{L}$ satisfying $[I, K] + [K, I] = 0$, then either $I \in \{0, \mathfrak{I}, \mathfrak{L}\}$ or $K \in \{0, \mathfrak{I}, \mathfrak{L}\}$.*

We also note that the above definition agrees with the definition of prime Lie algebra, since $\mathfrak{I} = \{0\}$ in this case.

Under the hypotheses of Proposition 3.3 we have:

Corollary 3.1. *If furthermore $\mathfrak{L}$ is prime, then any nonzero ideal I of $\mathfrak{L}$ such that $I \subset \mathfrak{I}$ satisfies $I = \mathfrak{I}$.*

Proof. Observe that, by Proposition 3.3, we could have $\mathfrak{I} = I \oplus K$ with I, K ideals of $\mathfrak{L}$, being $[I, K] + [K, I] = 0$ as consequence of $I, K \subset \mathfrak{I}$. The primeness of $\mathfrak{L}$ completes the proof. □

Proposition 3.4. *Suppose $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ and $\mathfrak{L}$ is root-multiplicative. If $\Lambda^{\neg\mathfrak{I}}$ has all of its roots $\neg\mathfrak{I}$-connected, then any ideal I of $\mathfrak{L}$ such that $I \nsubseteq \mathfrak{I}$ satisfies $I = \mathfrak{L}$.*

Proof. Taking into account Proposition 3.2 we just have to study the case in which

$$I = (I \cap H) \oplus (\bigoplus_{\beta_j\in\Lambda^{\mathfrak{I},I}} \mathfrak{L}_{\beta_j}),$$

with $I \cap H \neq 0$ (see equation (18)). But this possibility never happens. Indeed, observe that $[I \cap H, \mathfrak{L}_\alpha] + [\mathfrak{L}_\alpha, I \cap H] = 0$ for any $\alpha \in \Lambda^{\neg\mathfrak{I}}$. Indeed, in the opposite case $\mathfrak{L}_\alpha \subset I$ and so $\alpha \in \Lambda^{\neg\mathfrak{I}} \cap \Lambda^{\mathfrak{I},I} \subset \Lambda^{\neg\mathfrak{I}} \cap \Lambda^{\mathfrak{I}} = \emptyset$. Since we also have $[I\cap H, H] + [I\cap H, H] \subset [H, H] = 0$ we get $I\cap H \subset \mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$, a contradiction. Proposition 3.2 completes the proof. □

Given any $\alpha \in \Lambda^{\Upsilon}$, $\Upsilon \in \{\Lambda^{\mathfrak{I}}, \Lambda^{\neg\mathfrak{I}}\}$ we denote by

$$\Lambda^{\Upsilon}_\alpha := \{\beta \in \Lambda^{\Upsilon} : \beta \sim_{\neg\mathfrak{I}} \alpha\}.$$

If $\alpha \in \Lambda^{\Upsilon}$, let us write $H_{\Lambda^{\Upsilon}_\alpha} := span_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] : \beta \in \Lambda^{\Upsilon}_\alpha\} \subset H$, and $V_{\Lambda^{\Upsilon}_\alpha} := \bigoplus_{\beta\in\Lambda^{\Upsilon}_\alpha} \mathfrak{L}_\beta$. We denote by $\mathfrak{L}_{\Lambda^{\Upsilon}_\alpha}$ the following subspace of $\mathfrak{L}$, $\mathfrak{L}_{\Lambda^{\Upsilon}_\alpha} := H_{\Lambda^{\Upsilon}_\alpha} \oplus V_{\Lambda^{\Upsilon}_\alpha}$.

Lemma 3.4. *If $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, then $\mathfrak{L}_{\Lambda^{\mathfrak{I}}_\alpha}$ is an ideal of $\mathfrak{L}$ for any $\alpha \in \Lambda^{\mathfrak{I}}$.*

Proof. By equation (17) we get $H_{\Lambda_\alpha^{\mathfrak{I}}} = 0$ and so

$$\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}} = \bigoplus_{\beta \in \Lambda_\alpha^{\mathfrak{I}}} \mathfrak{L}_\beta.$$

We have $[\mathfrak{L}, \mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}] + [\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}, \mathfrak{I}] \subset [\mathfrak{L}, \mathfrak{I}] = 0$ and $[\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}, H] \subset \mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}$. Finally $[\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}, \mathfrak{L}_\gamma] \subset \mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}$ for any $\gamma \in \Lambda_\alpha^{\neg\mathfrak{I}}$. Indeed, given any $\beta \in \Lambda_\alpha^{\mathfrak{I}}$ such that $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \neq 0$ we have $\beta + \gamma \in \Lambda^{\mathfrak{I}}$ and so $\{\beta, \gamma\}$ is a $\neg\mathfrak{I}$-connection from β to $\beta + \gamma$. By the symmetry and transitivity of $\sim_{\neg\mathfrak{I}}$ in $\Lambda^{\mathfrak{I}}$ we get $\beta + \gamma \in \Lambda_\alpha^{\mathfrak{I}}$. Hence $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \subset \mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}$, taking into account equation (16) we conclude $\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}$ is an ideal of $\mathfrak{L}$. □

Theorem 3.1. *Suppose $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, $\mathrm{Z}_{\mathrm{Lie}}(\mathfrak{L}) = 0$ and $\mathfrak{L}$ is root-multiplicative. If $\Lambda^{\neg\mathfrak{I}}$, $\Lambda^{\mathfrak{I}}$ are symmetric then $\mathfrak{L}$ is simple if and only if it is prime and $\Lambda^{\neg\mathfrak{I}}$, $\Lambda^{\mathfrak{I}}$ have all of their roots $\neg\mathfrak{I}$-connected.*

Proof. Suppose $\mathfrak{L}$ simple. If $\Lambda^{\mathfrak{I}} \neq \emptyset$ and we take $\alpha \in \Lambda^{\mathfrak{I}}$, Lemma 3.4 gives us $\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}}$ is a nonzero ideal of $\mathfrak{L}$ and so, (by simplicity), $\mathfrak{L}_{\Lambda_\alpha^{\mathfrak{I}}} = \mathfrak{I} = \bigoplus_{\beta \in \Lambda^{\mathfrak{I}}} \mathfrak{L}_\beta$ (see equations (14) and (22)). Hence, $\Lambda_\alpha^{\mathfrak{I}} = \Lambda^{\mathfrak{I}}$ and consequently

$$\Lambda^{\mathfrak{I}} \text{ has all of its roots } \neg\mathfrak{I}\text{-connected.}$$

Consider now any $\gamma \in \Lambda^{\neg\mathfrak{I}}$ and the subspace $\mathfrak{L}_{\Lambda_\gamma^{\neg\mathfrak{I}}}$. Let us denote by $I(\mathfrak{L}_{\Lambda_\gamma^{\neg\mathfrak{I}}})$ the ideal of $\mathfrak{L}$ generated by $\mathfrak{L}_{\Lambda_\gamma^{\mathfrak{I}}}$. Observe that the fact $\mathfrak{I}$ is an ideal of $\mathfrak{L}$ let us assert that $I(\mathfrak{L}_{\Lambda_\gamma^{\neg\mathfrak{I}}}) \cap (\bigoplus_{\delta \in \Lambda^{\neg\mathfrak{I}}} \mathfrak{L}_\delta)$ is contained in the linear span of the set

$$\{[[\cdots[v_{\gamma'}, v_{\alpha_1}], \cdots], v_{\alpha_n}];\ [v_{\alpha_n}, [\cdots[v_{\alpha_1}, v_{\gamma'}], \cdots]];$$
$$[[\cdots[v_{\alpha_1}, v_{\gamma'}], \cdots], v_{\alpha_n}];\ [v_{\alpha_n}, [\cdots[v_{\gamma'}, v_{\alpha_1}], \cdots]] \text{ with } 0 \neq v_{\gamma'} \in \mathfrak{L}_{\Lambda_\gamma^{\neg\mathfrak{I}}},$$
$$0 \neq v_{\alpha_i} \in \mathfrak{L}_{\alpha_i},\ \alpha_i \in \Lambda^{\neg\mathfrak{I}} \text{ and } n \in \mathbb{N}\ \}.$$

By simplicity $I(\mathfrak{L}_{\Lambda_\gamma^{\neg\mathfrak{I}}}) = \mathfrak{L}$. From here, given any $\delta \in \Lambda^{\neg\mathfrak{I}}$, the above observation and Leibniz identity give us we can write $\delta = \gamma' + \alpha_1 + \cdots + \alpha_m$ with $\gamma' \in \Lambda_\gamma^{\neg\mathfrak{I}}$, $\alpha_i \in \Lambda^{\neg\mathfrak{I}}$ and being the partial sums nonzero. Hence $\{\gamma', \alpha_1, ..., \alpha_m\}$ is a $\neg\mathfrak{I}$-connection from γ' to δ. By the symmetry and transitivity of $\sim_{\neg\mathfrak{I}}$ in $\Lambda^{\neg\mathfrak{I}}$ we deduce γ is $\neg\mathfrak{I}$-connected to any $\delta \in \Lambda^{\neg\mathfrak{I}}$. Consequently, Proposition 3.1 let us assert

$$\Lambda^{\neg\mathfrak{I}} \text{ has all of its roots } \neg\mathfrak{I}\text{-connected.}$$

Finally, since $\mathfrak{L}$ is simple then is prime.

The converse is consequence of Corollary 3.1 and Proposition 3.4. □

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