Strong resolving graphs: the realization and the characterization problems

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Abstract

The strong resolving graph G_{SR} of a connected graph G was introduced in [Discrete Applied Mathematics 155 (1) (2007) 356–364] as a tool to study the strong metric dimension of G. Basically, it was shown that the problem of finding the strong metric dimension of G can be transformed to the problem of finding the vertex cover number of G_{SR}. Since then, several articles on the strong metric dimension of graphs which are using this tool have been published. However, the tool itself has remained unnoticed as a properly structure. In this paper, we survey the state of knowledge on the strong resolving graphs, and also derive some new results regarding its properties.

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1. Introduction

Graphs are basic combinatorial structures, and transformations of structures are fundamental to the development of mathematics. Particularly, in

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graph theory, some elementary transformations generate a new graph from an original one by some simple local changes, such as addition or deletion of a vertex or of an edge, merging and splitting of vertices, edge contraction, etc. Other advanced transformations create a new graph from the original one by complex changes, such as complement graph, line graph, total graph, graph power, dual graph, strong resolving graph, etc.

Some of these transformations of graphs emerged as a natural tool to solve practical problems. In other cases, the problem of finding a specific parameter of a graph has become the problem of finding another parameter of another graph obtained from the original one. This is the case of the strong resolving graph G_{SR} of a connected graph G which was introduced in Oellermann and Peters-Fransen (2007) as a tool to study the strong metric dimension of G. Basically, it was shown that the problem of finding the strong metric dimension of G can be transformed to the problem of finding the vertex cover number of G_{SR}. Since then, several articles dealing with the strong resolving graph have been published. However, in almost all these works the results related to the strong resolving graph are not explicit, as they implicitly appear as a part of the proofs of main results concerning the strong metric dimension. In this sense, this interesting construction has passed in front of researchers's eyes without the attention that should require. In this paper, we would like to motivate the graph theory community to have a deeper look into this graph transformation. Accordingly, herein we survey the state of knowledge on the strong resolving graph and also derive some new results.

For a graph transformation, there are two general problems Grünbaum (1969), which we shall formulate in terms of strong resolving graphs:

- **Realization Problem.**[1] Determine which graphs have a given graph as their strong resolving graphs.

- **Characterization Problem.** Characterize those graphs that are strong resolving graphs of some graphs.

The majority of results presented in this paper concerns the above mentioned problems. Basically, we focus on the following graph equation

$$G_{SR} \cong H, \tag{1}$$

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[1]This problem was called Determination Problem in Grünbaum (1969).

i.e., the goal is to find all pairs of graphs G and H satisfying (1).

The remainder of the paper is structured as follows. Subsection 1.1 covers general notation and terminology. Subsection 1.2 is devoted to introduce the strong metric dimension, whereas Subsection 1.3 introduces the strong resolving graph. In Section 2 we study the realization problem for some specific families of graphs, while in Section 3 we collect the known results related to the characterization problem of product graphs. We close our exposition with a collection of open problems to be dealt with.

1.1. Notation and Terminology

We continue by establishing the basic terminology and notations which is used throughout this work. For the sake of completeness we refer the reader to the books Diestel (2005); Hammack et al. (2011); West (1996). Graphs considered herein are undirected, finite and contain neither loops nor multiple edges. Let G be a graph of order $n = |V(G)|$. A graph is nontrivial if $n \geq 2$. We use the notation $u \sim v$ for two adjacent vertices u and v of G. For a vertex v of G, $N_G(v)$ denotes the set of neighbors that v has in G, *i.e.*, $N_G(v) = \{u \in V(G) : u \sim v\}$. The set $N_G(v)$ is called the *open neighborhood of a vertex v* in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood of a vertex v* in G. The *degree* of a vertex v of G is denoted by $\delta_G(v)$, *i.e.*, $\delta_G(v) = |N_G(v)|$. The *open neighborhood of a set S* of vertices of G is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the *closed neighborhood of S* is $N_G[S] = N_G(S) \cup S$.

We use the notation K_n, C_n, P_n, and N_n for the *complete graph*, *cycle*, *path*, and *empty graph*, respectively. Moreover, we write $K_{s,t}$ for the *complete bipartite graph* of order $s+t$ and in particular $K_{1,n}$ for the *star* of order $n+1$. A vertex of degree one in a tree T is called a *leaf* and the number of leaves in T is denoted by $l(T)$.

The *distance* between two vertices u and v, denoted by $d_G(u,v)$, is the length of a shortest path between u and v in G. The *diameter*, $D(G)$, of G is the largest distance between any two vertices of G and two vertices $u, v \in V(G)$ such that $d_G(u,v) = D(G)$ are called *diametral*. If G is not connected, then we assume that the distance between any two vertices belonging to different components of G is infinity and, thus, its diameter is $D(G) = \infty$. A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x,y) = D(G)$. For instance, even cycles and hypercubes are 2-antipodal graphs.

We recall that the *complement* of G is the graph G^c with the same vertex set as G and $uv \in E(G^c)$ if and only if $uv \notin E(G)$. The *subgraph induced by a set* X is denoted by $\langle X \rangle$. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G.

A *clique* in G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. Two distinct vertices u, v are called *true twins* if $N_G[u] = N_G[v]$. In this sense, a vertex x is a *twin* if there exists $y \neq x$ such that they are true twins. We say that $X \subset V(G)$ is a *twin-free clique* in G if the subgraph induced by X is a clique and for every $u, v \in X$ it follows $N_G[u] \neq N_G[v]$, *i.e.*, the subgraph induced by X is a clique and it contains no true twins. The *twin-free clique number* of G, denoted by $\varpi(G)$, is the maximum cardinality among all twin-free cliques in G. So, $\omega(G) \geq \varpi(G)$. We refer to a $\varpi(G)$-set in a graph G as a twin-free clique of cardinality $\varpi(G)$. Figure 1 shows examples of basic concepts such as true twins and twin-free clique.
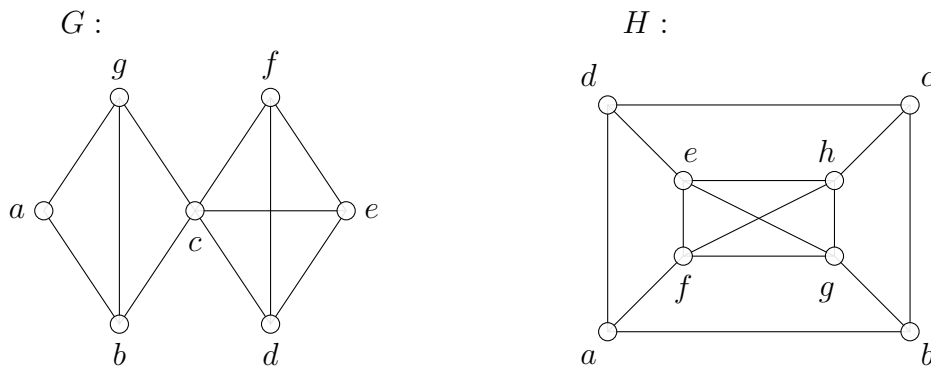

Figure 1: The set $\{d, e, f\} \subset V(G)$ is composed by true twin vertices in G. Notice that b and g are true twin vertices in G which are not simplicial, while f and d are true twin and simplicial vertices. The set $\{e, f, g, h\} \subset V(H)$ is a twin-free clique in H.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

1.2. Strong Metric Dimension of Graphs

A vertex $w \in V(G)$ *strongly resolves* two different vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, *i.e.*, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path

containing u. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex in S. The minimum cardinality among all strong metric generators for G is called the *strong metric dimension* and is denoted by $dim_s(G)$. A *strong metric basis* of G is a strong metric generator for G of cardinality $dim_s(G)$.

Several researches on the strong metric dimension of graphs have recently been developed. For instance, the trivial bounds $1 \leq dim_s(G) \leq n - 1$ are known from the first works as well as characterizations on whether they are achieved. Moreover, it has been noticed that the strong metric dimension of several graphs can be straightforwardly computed for some basic examples which we next remark.

Observation 1.

(a) $dim_s(G) = 1$ *if and only if G is isomorphic to the path P_n on $n \geq 2$ vertices.*

(b) $dim_s(G) = n - 1$ *if and only if G is isomorphic to the complete graph K_n on $n \geq 2$ vertices.*

(c) *For any cycle C_n of order n, $dim_s(C_n) = \lceil n/2 \rceil$.*

(d) *For any tree T with $l(T)$ leaves, $dim_s(T) = l(T) - 1$.*

(e) *For any complete bipartite graph $K_{r,t}$, $dim_s(K_{r,t}) = r + t - 2$.*

The strong metric dimension is a relatively new parameter (defined in 2004). Since then, this parameter has been investigated for several classes of graphs. For instance, we cite the works on Cartesian product graphs Kratica et al. (2012b); Oellermann and Peters-Fransen (2007); Rodríguez-Velázquez et al. (2014), Cartesian sum graphs Kuziak et al. (2015a), corona graphs Kuziak et al. (2013), direct product graphs Kuziak et al. (2017); Rodríguez-Velázquez et al. (2014), strong product graphs Kuziak et al. (2015b), lexicographic product graphs Kuziak et al. (2016), Cayley graphs Oellermann and Peters-Fransen (2007), Sierpiński graphs Estaji and Rodríguez-Velázquez (2017), distance-hereditary graphs May and Oellermann (2011), and convex polytopes Kratica et al. (2012a). Also, some Nordhaus-Gaddum type results for the strong metric dimension of a graph and its complement are known Yi (2013). Besides the theoretical results related to the strong metric dimension,

a mathematical programming model Kratica et al. (2012a) and metaheuristic approaches Kratica et al. (2008); Mladenović et al. (2012) for finding this parameter have been developed. Some complexity and approximation results are also known from the works Oellermann and Peters-Fransen (2007) and DasGupta and Mobasheri (2017), respectively. On the other hand, a fractional version of the strong metric dimension has been studied in Kang (2016); Kang et al. (2016); Kang and Yi (2013). In these three works the strong resolving graph is also used as an important tool. For more information we refer the reader to the survey Kratica et al. (2014) and the Ph.D. thesis Kuziak (2014).

1.3. The Strong Resolving Graph

In Oellermann and Peters-Fransen (2007), the authors have developed an approach which transforms the problem of finding the strong metric dimension of a graph to the problem of computing the vertex cover number of some other related graph. This relationship arises in connection with the following definitions.

A vertex u of G is *maximally distant* from v if for every vertex $w \in N_G(u)$, $d_G(v,w) \leq d_G(u,v)$. We denote by $M_G(v)$ the set of vertices of G which are maximally distant from v. The collection of all vertices of G that are maximally distant from some vertex of the graph is called the *boundary* of the graph, see Brešar et al. (2008); Cáceres et al. (2005), and is denoted by $\partial(G)$.[2] If u is maximally distant from v and v is maximally distant from u, then u and v are *mutually maximally distant* (from now on MMD for short).

Remark 2. $\partial(G) = \{u \in V(G) :$ *there exists* $v \in V(G)$ *such that* u, v *are MMD*$\}$.

Proof. On the one hand, if u is maximally distant from v, and v is not maximally distant from u, then v has a neighbor v_1, such that $d_G(v_1, u) > d_G(v, u)$, *i.e.*, $d_G(v_1, u) = d_G(v, u) + 1$. It is easily seen that u is maximally distant from v_1. If v_1 is not maximally distant from u, then v_1 has a neighbor v_2, such that $d_G(v_2, u) > d_G(v_1, u)$. Continuing in this manner we construct a sequence of vertices $v_1, v_2, \ldots$ such that $d_G(v_{i+1}, u) > d_G(v_i, u)$ for every i.

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[2]In fact, the boundary $\partial(G)$ of a graph was defined first in Chartrand et al. (2003) as the subgraph of G induced by the set mentioned in our work with the same notation. We follow the approach of Brešar et al. (2008); Cáceres et al. (2005) where the boundary of the graph is just the subset of the boundary vertices defined in this article.

Since G is finite this sequence terminates with some v_k. Thus for all neighbors x of v_k we have $d_G(v_k, u) \geq d_G(x, u)$, and so v_k is maximally distant from u and u is maximally distant from v_k. Hence every boundary vertex belongs to the set $S = \{u \in V(G) : \text{there exists } v \in V(G) \text{ such that } u, v \text{ are MMD}\}$. On the other hand, certainly every vertex of S is a boundary vertex. $\qquad\square$

For some basic graph classes the boundary is simply the whole vertex set. For instance, this happens for all 2-antipodal graphs and for all vertex transitive graphs (complete graphs K_n, complete bipartite graphs $K_{r,s}$, cycles C_n and hypercube graphs Q_k as some examples). The boundary of a tree consists of its leaves. Also, it is readily seen that every simplicial vertex is a boundary vertex, that is $\sigma(G) \subseteq \partial(G)$.

Figure 2 shows examples of basic concepts such as maximally distant vertices, MMD vertices and boundary. As a direct consequence of the definition of MMD vertices, we have the following.
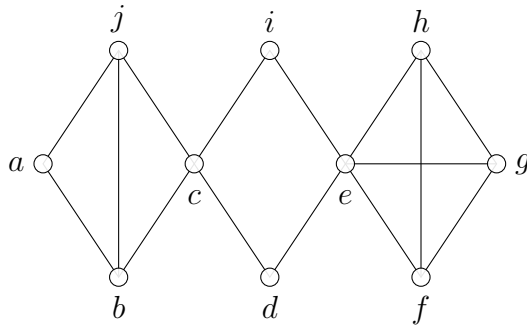

Figure 2: The set $\{a, f, g, h\}$ is composed by simplicial vertices and its elements are MMD between them. Also, b and j (d and i) are MMD. Thus, the boundary of G is $\partial(G) = \{a, b, d, f, g, h, i, j\}$. Now, $M_G(d) = \{a, f, g, h, i\}$ is the set of vertices which are maximally distant from d. Nevertheless, the vertex d is maximally distant only from the vertex i.

Remark 3. *For every pair of MMD vertices x, y of a connected graph G and for every strong metric basis S of G, it follows that $x \in S$ or $y \in S$.*

By using the concepts of boundary of a graph and MMD vertices, the notion of strong resolving graph was introduced in Oellermann and Peters-Fransen (2007) in the following way. The *strong resolving graph* of G has vertex set of $V(G)$ and two vertices u, v are adjacent if and only if u and v are MMD in G. Observe that the vertices belonging to $V(G) - \partial(G)$ are

isolated vertices in the strong resolving graph. According to this fact, in this work we use two slightly different versions of it, which are next stated.

The first version is denoted as G_{SR} while the second one is denoted by G_{SR+I}. The graph G_{SR} has vertex set $\partial(G)$, while G_{SR+I} has vertex set $V(G)$. Clearly, the difference between G_{SR} and G_{SR+I} is the existence of isolated vertices in G_{SR+I}, when $V(G) - \partial(G) \neq \emptyset$ and notice that the graph G_{SR+I} coincides with the original definition presented in Oellermann and Peters-Fransen (2007). The concept of the strong resolving graph G_{SR} is used in this work rather than that of G_{SR+I}. The main reason of this fact is to have a simpler notation and more clarity while proving the results. Figure 3 shows the strong resolving graphs G_{SR} and G_{SR+I} of the graph G illustrated in Figure 2.
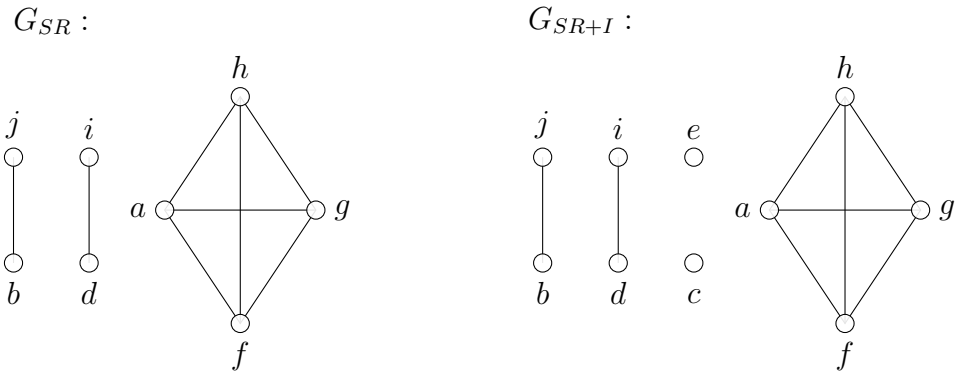

Figure 3: G_{SR} and G_{SR+I} of the graph G illustrated in Figure 2.

There are several families of graphs for which the strong resolving graph can be relatively easily described. We next state some of these cases.

Observation 4.

(a) *If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{|\partial(G)|}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T, $T_{SR} \cong K_{l(T)}$.*

(b) *For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.*

(c) *For odd cycles $(C_{2k+1})_{SR} \cong C_{2k+1}$.*

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(d) *For any complete k-partite graph $G = K_{p_1, p_2, \ldots, p_k}$ such that $p_i \geq 2$, $i \in \{1, 2, \ldots, k\}$, $G_{SR} \cong \bigcup_{i=1}^{k} K_{p_i}$.*

Recall that a set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\beta(G)$, is the minimum cardinality among all vertex covers of G. We refer to a $\beta(G)$-set in a graph G as a vertex cover of cardinality $\beta(G)$.

As shown in Oellermann and Peters-Fransen (2007), the problem of finding the strong metric dimension of a connected graph G can be transformed to the problem of finding the vertex cover number of G_{SR+I}.

Theorem 5. (Oellermann and Peters-Fransen (2007)) *For any connected graph G, $dim_s(G) = \beta(G_{SR+I})$.*

Now, it is readily seen that $\beta(G_{SR+I}) = \beta(G_{SR})$. Therefore, an analogous theorem to the one above can be stated by using G_{SR} instead of G_{SR+I}.

Theorem 6. *For any connected graph G, $dim_s(G) = \beta(G_{SR})$.*

Figure 4 illustrates this theorem, which has proved its high usefulness in several situations.
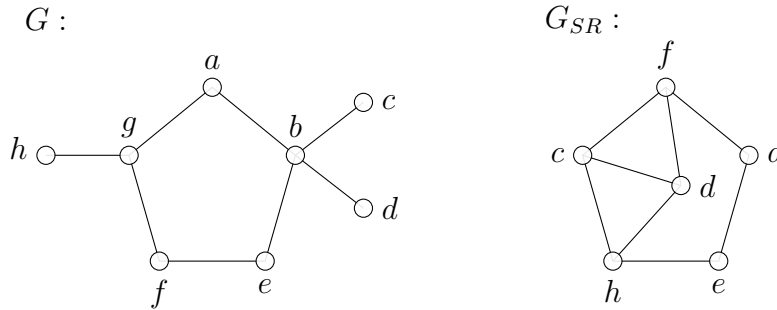

Figure 4: The set $\{a, c, d, h\} \subset V(G)$ forms a strong metric basis of G. Also, the set $\{a, c, d, h\} \subset V(G_{SR})$ is a vertex cover of G_{SR}. Thus, $dim_s(G) = \beta(G_{SR}) = 4$.

Recall that an *independent set* of a graph G is a subset $S \subseteq V(G)$ such that no two vertices in S represent an edge of G. The cardinality of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. We refer to an $\alpha(G)$-set in a graph G as an independent set of cardinality $\alpha(G)$. The following well-known result, from Gallai (1959), states the relationship between the independence number and the vertex cover number of a graph.

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Lemma 7. (Gallai (1959)) *For any graph G of order n,*

$$\alpha(G) + \beta(G) = n.$$

Thus, by using Theorem 6 and Lemma 7 we immediately obtain that for any graph G,

$$dim_s(G) = |\partial(G)| - \alpha(G_{SR}).$$

2. Realization Problem

In this section we study the realization problem for some specific families of graphs, *i.e.*, we study the graph equation $G_{SR} \cong H$ where H is isomorphic to K_n, $K_{1,r}$, C_n, P_n and G^c. In addition, the characterization problem of graphs of diameter two is considered. We begin with the characterization of graphs whose strong resolving graph is complete. To this end, we need the following two lemmas.

Lemma 8 (Müller et al. (2008); Müller et al. (2011)). *Each shortest path in a graph G extends to a shortest path between two boundary vertices.*

Lemma 9. *Let G be a graph and let $v \in \partial(G) \setminus \sigma(G)$. Then there exist $a, b \in \partial(G) \setminus \{v\}$ such that va, vb are not edges of G_{SR}.*

Proof. Let $v_1, v_2 \in N(v)$ be such that $d(v_1, v_2) = 2$. Then $P = v_1 v v_2$ is a shortest path and, by Lemma 8, there exist $a, b \in \partial(G) \setminus \{v\}$ and a shortest path between them that extends P. So v lies on a shortest path between a and b and, in particular, v is not maximally distance from any of them. This means that v is not a neighbor of a nor of b in G_{SR}. $\square$

With these tools we obtain the following characterization.

Theorem 10. *Let G be a connected graph. Then $G_{SR} \cong K_{|\partial(G)|}$ if and only if $\partial(G) = \sigma(G)$.*

Proof. If $\partial(G) = \sigma(G)$, it is clear that $G_{SR} \cong K_{|\partial(G)|}$. Conversely, assume now that $\sigma(G) \subsetneq \partial(G)$ and let $v \in \partial(G) \setminus \sigma(G)$. By Lemma 9, there exist $a \in \partial(G) \setminus \{v\}$ such that v is not a neighbor of a in G_{SR}, so G_{SR} is not a complete graph. $\square$

If G is a connected graph of order n, then $\sigma(G) = V(G)$ if and only if $G \cong K_n$. Hence, the following result is a direct consequence of Theorem 10.

Corollary 11. *Let G be a connected graph of order $n \geq 2$. Then $G_{SR} \cong K_n$ if and only if $G \cong K_n$.*

Another particular case of Theorem 10 is Proposition 13, which can be deduced from the next lemma. We recall that a *cut vertex* in a graph G is a vertex when removed (together with its adjacent edges) from G results in a new graph with increased number of connected components.

Lemma 12. (Estaji and Rodríguez-Velázquez (2017)) *Let G be a connected graph. If v is a cut vertex of G, then $v \notin \partial(G)$.*

Proposition 13. *Let G be a connected graph and let $\varepsilon(G)$ be the number of vertices of degree one. If every vertex of degree greater than one is a cut vertex of G, then $G_{SR} \cong K_{\varepsilon(G)}$.*

In order to present the next result we need to introduce some more terminology. Given a graph G, we define G^* as the graph with vertex set $V(G^*) = V(G)$ such that two vertices u, v are adjacent in G^* if and only if either $d_G(u, v) \geq 2$ or u, v are true twins. If a graph G has at least one isolated vertex, then we denote by G_- the graph obtained from G by removing all its isolated vertices. In this sense, G_-^* is obtained from G^* by removing all its isolated vertices. Notice that if G is true twin-free, then $G^* \cong G^c$.

Proposition 14. *For any graph G of diameter two, $G_{SR} \cong G_-^*$.*

Proof. Assume that G has diameter two and let u, v be two different vertices of G. If $u \nsim v$ or $N_G[u] = N_G[v]$, then u and v are MMD in G. Now, if $u \sim v$ and $N_G[u] \neq N_G[v]$, then there exists, $w \in V(G) \setminus \{u, v\}$ such that either $(w \sim u$ and $w \nsim v)$ or $(w \nsim u$ and $w \sim v)$, which implies that u and v are not MMD. Therefore, the result follows. $\square$

Theorem 15. (Estrada-Moreno et al. (2016)) *Let G be a connected graph. Then $G_{SR} \cong G^c$ if and only if $D(G) = 2$ and G is a true twin-free graph.*

We next show that star graphs and complete bipartite graphs $K_{2,r}$ are not realizable as the strong resolving graph of any graph.

Proposition 16. *Let G be a connected graph of order $n \geq 2$ and let $r \geq 1$ be an integer. Then the following statements hold.*

- $G_{SR} \cong K_{1,r}$ *if and only if $G \cong P_n$ and $r = 1$.*

- *The graph equation $G_{SR} \cong K_{2,r}$ has no solution.*

Proof. Obviously, $(P_n)_{SR} \cong K_2 \cong K_{1,1}$. Now, if $G_{SR} \cong K_{1,r}$, then $\dim_s(G) = \beta(G_{SR}) = 1$, which implies that $G \cong P_n$, by Observation 1 (a), and so $r = 1$. Therefore, the first statement holds.

Now, assume that G_{SR} is a complete bipartite graph $(U_1 \cup U_2, E)$, where $|U_1|, |U_2| \geq 2$. Since the subgraph of G_{SR} induced by $\sigma(G)$ is a clique, $|U_1 \cap \sigma(G)| \leq 1$ and $|U_2 \cap \sigma(G)| \leq 1$. Hence, Lemma 9 immediately leads to $|U_1| \geq 3$ and $|U_2| \geq 3$, which implies that the graph equation $G_{SR} \cong K_{2,r}$ has no solution. $\qquad\square$

By Proposition 16 we learned that the graph equations $G_{SR} \cong K_{1,r}$ and $G_{SR} \cong K_{2,r}$, for $r \geq 2$, have no solution. Based on these facts, we propose the following conjecture.

Conjecture 17. *The graph equation $G_{SR} \cong K_{r,s}$ has no solution for any $r, s \geq 2$.*

It is worth mentioning that, concerning Proposition 16, although no star graph $K_{1,r}$, $r \geq 2$, is a strong resolving graph, there are graphs G for which G_{SR} contains a component isomorphic to a star graph $K_{1,r}$ for any $r \geq 2$. To see this, consider the family $\mathcal{F}$ of graphs G_r constructed in Kang et al. (2016) as follows.

- Consider $r + 1$ paths $a_i b_i c_i$ with $i \in \{0, \ldots, r\}$.

- Add the edges $a_i a_0$, $b_i b_0$ and $c_i c_0$ for every $i \in \{1, \ldots, r\}$.

- Add a vertex x and the edges $x a_0$ and $x c_0$.

An example of a graph in $\mathcal{F}$ and its strong resolving graph is given in Figure 5.

We can easily notice the following.

- The vertex a_i is MMD only with the vertices c_j such that $j \neq 0, i$.

- Similarly, the vertex c_i is MMD only with the vertices a_j such that $j \neq 0, i$.

- The vertex b_i is MMD only with the vertex x and viceversa.

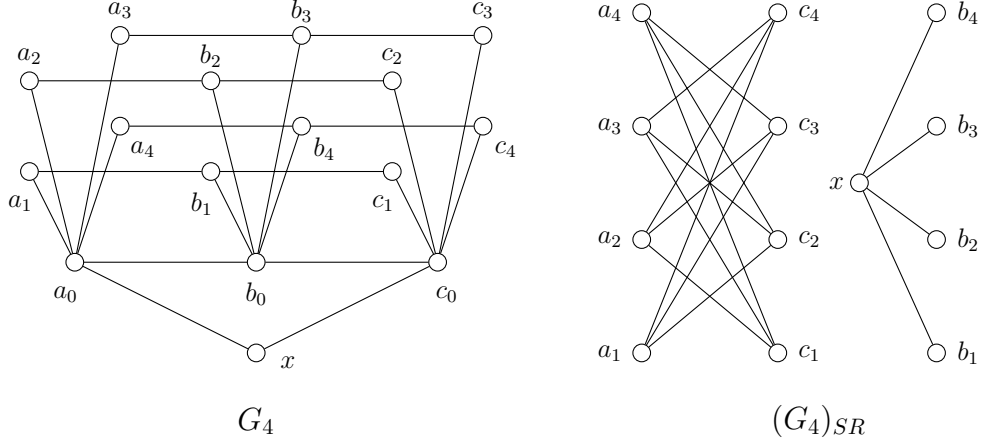- The vertices a_0, b_0, c_0 are not MMD with any vertex in G_r.

Figure 5: The graph $G_4 \in \mathcal{F}$ and its strong resolving graph.

As a consequence of the facts above it clearly happens that $(G_r)_{SR}$ contains two connected components. One of them isomorphic to a star graph $S_{1,r}$ with r leaves, and the second one isomorphic to a complete bipartite graph $K_{r,r}$ minus a perfect matching.

Our next result concerns the equation $G_{SR} \cong P_n$, with $n \neq 3$. To this end, we consider the family $\mathcal{F}_P$ of graphs G_P^n with $n \geq 5$ given as follows.

- We begin with a path on $n-1$ vertices $v_1 v_2 \dots v_{n-1}$.

- If n is even, then

  – add $\frac{n-2}{2}$ vertices $a_1, a_2, \dots, a_{(n-2)/2}$ and $\frac{n-2}{2}$ vertices $b_1, b_2, \dots, b_{(n-2)/2}$,

  – add the edges $a_i v_{2i-1}$, $a_i v_{2i+1}$ with $i \in \{1, \dots, (n-2)/2\}$, the edges $b_i v_{2i}$, $b_i v_{2i+2}$ with $i \in \{1, \dots, (n-4)/2\}$ and the edges $b_{(n-2)/2} v_{n-2}$, $b_{(n-2)/2} v_{n-1}$.

- If n is odd, then

  – add $\frac{n-1}{2}$ vertices $a_1, a_2, \dots, a_{(n-1)/2}$ and $\frac{n-3}{2}$ vertices $b_1, b_2, \dots, b_{(n-3)/2}$,

  – add the edges $a_i v_{2i-1}$, $a_i v_{2i+1}$ with $i \in \{1, \dots, (n-3)/2\}$, the edges $a_{(n-1)/2} v_{n-2}$, $a_{(n-1)/2} v_{n-1}$ and the edges $b_i v_{2i}$, $b_i v_{2i+2}$ with $i \in \{1, \dots, (n-3)/2\}$.

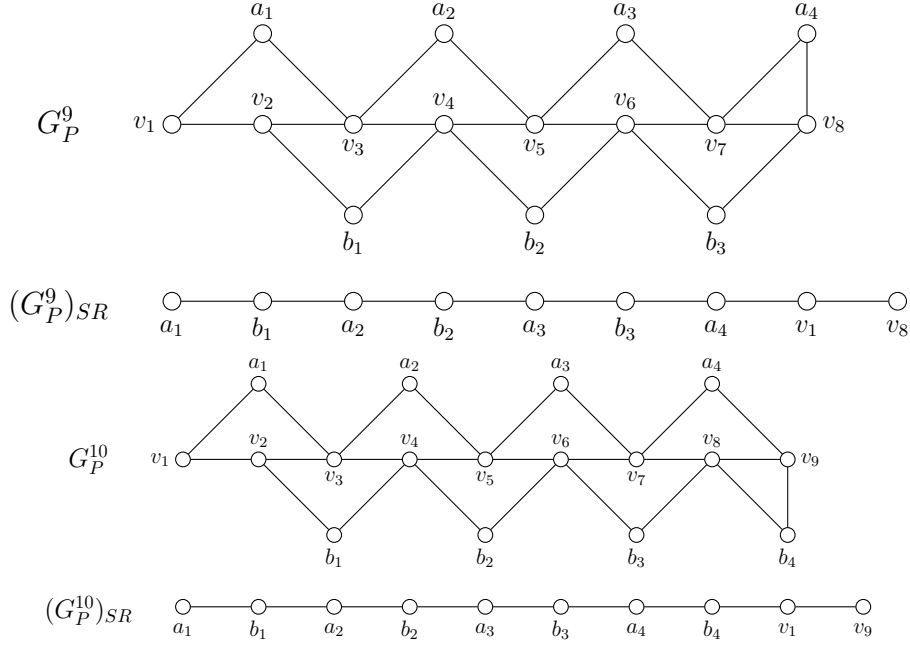Examples of the graphs of family $\mathcal{F}_P$ are given in Figure 6.

Figure 6: The graphs G_P^9 and G_P^{10} in $\mathcal{F}_P$ and their strong resolving graphs.

Proposition 18. *For any integer $n \geq 2$ and $n \neq 3$, there exists a graph G such that $G_{SR} \cong P_n$.*

Proof. If $n = 2$, then any path P_t satisfies that $(P_t)_{SR} \cong P_2$. If $n = 3$, then by Proposition 16 we know there is no graph G such $G_{SR} \cong G$. If $n = 4$, then consider the join graph $K_1 + P_4$, for which it is not difficult to see that $(K_1 + P_4)_{SR} \cong P_4$. If $n \geq 5$, then we consider a graph $G_P^n \in \mathcal{F}_P$, where the following facts are observed. Assume n is even.

- Every vertex a_i, with $i \in \{2, \ldots, (n-2)/2\}$, is only MMD with the vertices b_i and b_{i-1}.

- The vertex a_1 is only MMD with the vertex b_1.

- Every vertex b_i, with $i \in \{1, \ldots, (n-4)/2\}$, is only MMD with the vertices a_i and a_{i+1}.

- The vertex $b_{(n-2)/2}$ is only MMD with the vertices $a_{(n-2)/2}$ and v_1.

- The vertices v_1 and v_{n-1} are MMD between them.

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- No vertex v_i, with $i \in \{2, \ldots, n-2\}$, belongs to the boundary of G_P^n.

According to the items above it clearly follows that $(G_P^n)_{SR}$ is isomorphic to the path $P_n = a_1 b_1 a_2 b_2 \cdots a_{(n-2)/2} b_{(n-2)/2} v_1 v_{n-1}$. A similar procedure can be used for the case n odd, which completes the proof. $\square$

Our next result concerns the realization of cycles C_n as strong resolving graphs. From Observation 4 (iii) we know that for any odd cycle C_{2k+1}, it follows $(C_{2k+1})_{SR} \cong C_{2k+1}$. Also, from Proposition 16, the cycle $C_4 \cong K_{2,2}$ is not realizable as the strong resolving graph of any graph. In general, the following can be stated.

Proposition 19. *For any integer $n \geq 3$ and $n \neq 4$, there exists a graph G such that $G_{SR} \cong C_n$.*

Proof. If $n = 3$, then clearly $(C_3)_{SR} \cong C_3$. Consider a cycle graph of order $n \geq 5$. Since C_n^c is a twin-free graph and has diameter two, by Theorem 15, $(C_n^c)_{SR} \cong (C_n^c)^c \cong C_n$. That is, the complement of a cycle of order n gives a strong resolving graph isomorphic to the cycle C_n, which completes the realization. $\square$

More in general, since $D(G) \geq 4$ leads to $D(G^c) = 2$, the following result is a direct consequence of Theorem 15.

Corollary 20. *Any false twin-free graph of diameter greater than or equal to four is the strong resolving graph of a true twin-free graph of diameter two.*

A summary of the results we obtained related to the Realization Problem can be found in Table 1.

	K_n $(n \geq 2)$	$K_{1,r}$ $(r \geq 1)$	$K_{2,r}$ $(r \geq 1)$	$K_{s,r}$ $(r \geq 3)$	P_n $(n \geq 2)$	C_n $(n \geq 3)$
$= G_{SR}$	$n \geq 2$	$r = 1$	none	unknown	$n \neq 3$	$n \neq 4$
$\neq G_{SR}$	none	$r \geq 2$	$r \geq 1$	unknown	$n = 3$	$n = 4$

Table 1: Notable graphs families and the Realization Problem

3. Strong Resolving Graphs of Product Graphs

We begin this section with a brief overview on those products of graphs that will be further considered. According to the two books Hammack et al. (2011); Imrich and Klavžar (2000), a graph product of the graphs G and H means a graph whose vertex set is defined on the cartesian product $V(G) \times V(H)$ of the vertex sets of G and H, and edges are determined by a function on the edges of G and H. The graphs G and H are called the *factor graphs*. Considering such mentioned rules, there are exactly 256 possible products. However, according to several their properties such as associativity, commutativity, complementarity, etc., the most common and well investigated are the Cartesian product, the direct product, the strong product, and the lexicographic product, which are also known as the *standard products* Hammack et al. (2011); Imrich and Klavžar (2000). Nonetheless, there exist other less known operations with graphs which are interesting for some investigations, for instance we could mention the Cartesian sum graph and the corona product graphs, among other ones.

Studies on finding relationships between properties of product graphs and properties of the factors have attracted several researchers in the last recent year. The case of strong metric generators has not escaped to this and several investigations have been published concerning this. In such researches a powerful tool has been deducing the structure of the strong resolving graph of a product from that of its factors. In this section we precisely survey some results concerning the structure of the strong resolving graphs of product graphs. To this end, we first gives some background on the definitions and some basic properties of product graphs.

The *direct product* of two graphs G and H is the graph $G \times H$, such that $V(G \times H) = V(G) \times V(H)$ and two vertices $(g, h), (g', h')$ are adjacent in $G \times H$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$.

The direct product is also known as the *Kronecker product*, the *tensor product*, the *categorical product*, the *cardinal product*, the *cross product*, the *conjunction*, the *relational product* or the *weak direct product*. This product is commutative and associative in a natural way Hammack et al. (2011); Imrich and Klavžar (2000). The distance and connectedness in the direct product are more subtle than for other products. The formula on the vertex distances in the direct product is the following.

Remark 21. (Kim (1991)) *For any graphs G and H and any two vertices*

(g, h), (g', h') of $G \times H$,

$$d_{G \times H}((g, h), (g', h')) = \min\{\max\{d_G^e(g, g'), d_H^e(h, h')\}, \max\{d_G^o(g, g'), d_H^o(h, h')\}\},$$

where $d_G^e(g, g')$ means the length of a shortest walk of even length between g and g' in G, and $d_G^o(g, g')$ the length of a shortest odd walk between g and g' in G. If such a walk does not exist, we set $d_G^e(g, g')$ or $d_G^o(g, g')$ to be infinite.

On the other hand, the connectedness in the direct product of two graphs relies on the bipartite properties of the factor graphs, namely, the result presented at next.

Theorem 22. (Weichsel (1962)) *A direct product of nontrivial graphs is connected if and only if both factors are connected and at least one factor is nonbipartite.*

In contrast to distances, the direct product is the most natural product for open neighborhoods:

$$N_{G \times H}(g, h) = N_G(g) \times N_H(h). \tag{2}$$

The *Cartesian product* of two graphs G and H is the graph $G \square H$, such that $V(G \square H) = V(G) \times V(H)$ and two vertices $(g, h), (g', h') \in V(G \square H)$ are adjacent in $G \square H$ if and only if either ($g = g'$ and $hh' \in E(H)$), or ($gg' \in E(G)$ and $h = h'$). The Cartesian product is a straightforward and natural construction, and is in many respects the simplest graph product Hammack et al. (2011); Imrich and Klavžar (2000).

The Cartesian product is a commutative and associative operation. Moreover, it is connected whenever the factors are both connected. The distance between any two of its vertices is given by

$$d_{G \square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')$$

while the neighborhood of a vertex $(g, h) \in V(G \square H)$ is

$$N_{G \square H}(g, h) = (N_G(g) \times \{h\}) \cup (\{g\} \times N_H(h)).$$

The *strong product* of two graphs G and H is the graph $G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$, and two vertices $(g, h), (g', h') \in V(G \boxtimes H)$ are adjacent in $G \boxtimes H$ if and only if either ($g = g'$ and $hh' \in E(H)$), or ($gg' \in E(G)$ and $h = h'$), or ($gg' \in E(G)$ and $hh' \in E(H)$).

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Similarly to the Cartesian product, the strong product is a commutative and associative operation and, it is connected if and only if the factors are both connected. The distance between any two of its vertices is computed by using the following formula

$$d_{G \boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}.$$

On the other hand, the neighborhood of a vertex $(g,h) \in V(G \boxtimes H)$ is given by

$$N_{G \boxtimes H}(g,h) = N_G[g] \times N_H[h].$$

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ with vertex set $V(G \circ H) = V(G) \times V(H)$ and two vertices $(g,h) \in V(G \circ H)$ and $(g',h') \in V(G \circ H)$ are adjacent in $G \circ H$ if and only if either $gg' \in E(G)$, or $(g = g'$ and $hh' \in E(H))$.

Note that the lexicographic product of two graphs is the only not commutative operation among the four standard products. Moreover, $G \circ H$ is a connected graph if and only if G is connected. The distances and neighborhoods in the lexicographic product are obtained as the following known results show.

Theorem 23. (Hammack et al. (2011)) *Let G and H be two nontrivial graphs such that G is connected. Then the following assertions hold for any $g, g' \in V(G)$ and $h, h' \in V(H)$ such that $g \neq g'$.*

(i) $N_{G \circ H}(g,h) = (\{g\} \times N_H(h)) \cup (N_G(g) \times V(H))$.

(ii) $d_{G \circ H}((g,h),(g',h')) = d_G(g,g')$.

(iii) $d_{G \circ H}((g,h),(g,h')) = \min\{d_H(h,h'), 2\}$.

The *Cartesian sum* of two graphs G and H, denoted by $G \oplus H$, is the graph with vertex set $V(G \circ H) = V(G) \times V(H)$, where $(g,h)(g',h') \in E(G \oplus H)$ if and only if $gg' \in E(G)$, or $hh' \in E(H)$.

This notion of graph product was introduced in Ore (1962). Nevertheless it has passed almost unnoticed and just few results (for instance Čížek and Klavžar (1994); Scheinerman and Ullman (1997)) have been presented about this. The Cartesian sum is also known as the *disjunctive product* Scheinerman and Ullman (1997) and it is a commutative and associative operation Hammack et al. (2011).

Next result summarizes some properties about the diameter of the Cartesian sum graph.

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Proposition 24. (Kuziak et al. (2015a)) *Let G and H be two nontrivial graphs such that at least one of them is noncomplete and let $n \geq 2$ be an integer. Then the following assertions hold.*

(i) $D(G \oplus N_n) = \max\{2, D(G)\}$.

(ii) *If G and H have isolated vertices, then $D(G \oplus H) = \infty$.*

(iii) *If neither G nor H has isolated vertices, then $D(G \oplus H) = 2$.*

(iv) *If $D(H) \leq 2$, then $D(G \oplus H) = 2$.*

(v) *If $D(H) > 2$, H has no isolated vertices and G is a nonempty graph having at least one isolated vertex, then $D(G \oplus H) = 3$.*

The neighborhood of a vertex $(g, h) \in V(G \oplus H)$ is

$$N_{G \oplus H}(g, h) = (N_G(g) \times V(H)) \cup (V(G) \times N_H(h)).$$

The *corona product* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and $n = |V(G)|$ copies of H and joining by an edge each vertex from the ith-copy of H with the ith-vertex of G. We denote by $V = \{g_1, g_2, \ldots, g_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $g_i \sim v$ for every $v \in V_i$. Observe that $G \odot H$ is connected if and only if G is connected. The concept of corona product of two graphs was first introduced in Frucht and Harary (1970).

The following expression for the distance between two vertices x and y of $G \odot H$ is a direct consequence of the definition of corona product graph.

$$d_{G \odot H}(x, y) = \begin{cases} d_G(x, y), & x, y \in V; \\ d_G(g_i, g_j) + 1, & x = g_i, \ y \in V_j; \\ d_G(g_i, g_j) + 2, & x \in V_i, \ y \in V_j, \ i \neq j; \\ \min\{d_{H_i}(x, y), 2\}, & x, y \in V_i. \end{cases} \tag{3}$$

3.1. Cartesian Product and Direct Product of Graphs

The next result establishes an interesting connection between the strong resolving graph of the Cartesian product of two graphs and the direct product

of the strong resolving graphs of its factors. Such result was a powerful tool used in Rodríguez-Velázquez et al. (2014) while studying the strong metric dimension of Cartesian product graphs.

Theorem 25. (Rodríguez-Velázquez et al. (2014)) *Let G and H be two connected graphs. Then*

$$(G\Box H)_{SR} \cong G_{SR} \times H_{SR}.$$

Figure 7 illustrates the Cartesian product of two cycles of order three and its strong resolving graph. Since the strong resolving graph of C_3 is isomorphic to C_3, we can easily observe that $(C_3\Box C_3)_{SR}$ is isomorphic to $(C_3)_{SR} \times (C_3)_{SR}$.
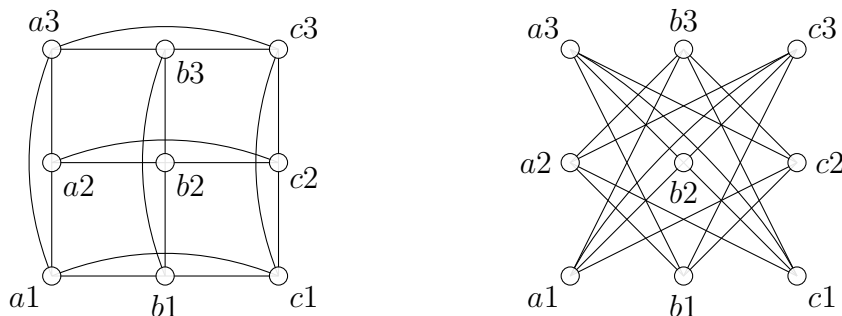

Figure 7: Cartesian product graph $C_3\Box C_3$ and its strong resolving graph $(C_3\Box C_3)_{SR}$.

A *matching* of a graph G is a set of edges of G such that no two edges share a vertex in common. A matching is *maximum* if it has the maximum possible cardinality. Moreover, if every vertex of the graph is incident to exactly one edge of the matching, then it is called a *perfect matching*.

The next result, implicitly deduced in (Rodríguez-Velázquez et al., 2014, Proof of Theorem 6), deals with graphs whose strong resolving graphs are bipartite with a perfect matching.

Theorem 26. *Let G and H be two connected graphs such that H_{SR} is bipartite with a perfect matching. Let G_i, $i \in \{1, \ldots, k\}$, be the connected components of G_{SR}. If for each $i \in \{1, \ldots, k\}$, G_i is Hamiltonian or G_i has a perfect matching, then $(G\Box H)_{SR}$ is bipartite and has a perfect matching.*

Proof. Since H_{SR} is bipartite, $G_{SR} \times H_{SR}$ is bipartite. We show next that $G_{SR} \times H_{SR}$ has a perfect matching. Let n_i be the order of G_i, $i \in \{1, \ldots, k\}$,

and let $\{x_1y_1, x_2y_2, \ldots, x_{|\partial(H)|/2}y_{|\partial(H)|/2}\} \subset E(H_{SR})$ be a perfect matching of H_{SR}. We distinguish two cases.

Case 1: G_i has a perfect matching. If $\{g_1g_1', g_2g_2', \ldots, g_{n_i/2}g_{n_i/2}'\} \subset E(G_i)$ is a perfect matching of G_i, then the set of edges

$$\{(g_1, y_1)(g_1', x_1),\ (g_1', y_1)(g_1, x_1), \ldots, (g_{n_i/2}, y_1)(g_{n_i/2}', x_1),$$

$$(g_{n_i/2}', y_1)(g_{n_i/2}, x_1), (g_1, y_2)(g_1', x_2),\ (g_1', y_2)(g_1, x_2), \ldots,$$

$$(g_{n_i/2}, y_2)(g_{n_i/2}', x_2),\ (g_{n_i/2}', y_2)(g_{n_i/2}, x_2), \ldots,$$

$$(g_1, y_{|\partial(H)|/2})(g_1', x_{|\partial(H)|/2}),\ (g_1', y_{|\partial(H)|/2})(g_1, x_{|\partial(H)|/2}), \ldots,$$

$$(g_{n_i/2}, y_{|\partial(H)|/2})(g_{n_i/2}', x_{|\partial(H)|/2}),\ (g_{n_i/2}', y_{|\partial(H)|/2})(g_{n_i/2}, x_{|\partial(H)|/2})\}$$

is a perfect matching of $G_i \times H_{SR}$.

Case 2: G_i is Hamiltonian. Let $g_1, g_2, \ldots, g_{n_i}, g_1$ be a Hamiltonian cycle of G_i. If n_i is even, then G_i has a perfect matching and this case coincides with Case 1. So we suppose that n_i is odd. In this case, the set of edges

$$\{(g_1, x_1)(g_2, y_1),\ (g_2, x_1)(g_3, y_1), \ldots, (g_{n_i-1}, x_1)(g_{n_i}, y_1),\ (g_{n_i}, x_1)(g_1, y_1),$$

$$(g_1, x_2)(g_2, y_2),\ (g_2, x_2)(g_3, y_2), \ldots, (g_{n_i-1}, x_2)(g_{n_i}, y_2),\ (g_{n_i}, x_2)(g_1, y_2), \ldots,$$

$$(g_1, x_{|\partial(H)|/2})(g_2, y_{|\partial(H)|/2}),\ (g_2, x_{|\partial(H)|/2})(g_3, y_{|\partial(H)|/2}), \ldots,$$

$$(g_{n_i-1}, x_{|\partial(H)|/2})(g_{n_i}, y_{|\partial(H)|/2}),\ (g_{n_i}, x_{|\partial(H)|/2})(g_1, y_{|\partial(H)|/2})\}$$

is a perfect matching of $G_i \times H_{SR}$.

According to Cases 1 and 2 the graph $\bigcup_{i=1}^{k} G_i \times H_{SR} \cong G_{SR} \times H_{SR}$ has a perfect matching. $\qquad\square$

Since 2-antipodal graphs have strong resolving graphs that are bipartite with a perfect matching, the next result follows from the previous theorem and Observation 4.

Corollary 27. *Let G be a 2-antipodal graph. If H is a 2-antipodal graph or it is connected and $\partial(H) = \sigma(H)$, then $(G \square H)_{SR}$ is bipartite and has a perfect matching.*

The next well known result characterizing whether Cartesian product graphs are direct product graphs give also an interesting consequence for describing some strong resolving graphs.

Lemma 28. (Miller (1968)) *Let G and H be two connected graphs. Then, $G \Box H \cong G \times H$ if and only if $G \cong H \cong C_{2k+1}$ for some positive integer k.*

The characterization above, Theorem 25 and Observation 4, allow us to immediately determine the strong resolving graph of $C_{2k+1} \times C_{2k+1}$.

Remark 29. *For any positive integer k, $(C_{2k+1} \times C_{2k+1})_{SR} \cong C_{2k+1} \times C_{2k+1}$.*

It is not difficult to check that $(K_r \times K_t)^c \cong K_r \Box K_t$ and $K_r \times K_t$ is a true twin-free graph of diameter two for $r, t \geq 3$. Hence, Theorem 15 leads to the following result.

Proposition 30. (Rodríguez-Velázquez et al. (2014)) *For any positive integers $r, t \geq 3$,*
$$(K_r \times K_t)_{SR} \cong K_r \Box K_t.$$

From Theorem 25 and Proposition 30 we obtain the following.

Proposition 31. *For any positive integers $r, t \geq 3$,*
$$((K_r \times K_t)_{SR})_{SR} \cong K_r \times K_t.$$

We now describe the structure of the strong resolving graphs of some particular cases of direct product graphs, which in contrast to Cartesian product graphs, becomes more challenging and tedious. Moreover, the results are not stated for general direct product graphs, since it is quite frequently not a connected graphs. From now on, we say that a graph G is 2-*MMD free*, or 2MMF for short, if there exists no pair of MMD vertices u and v with $d_G(u, v) = 2$. Clearly diameter two graphs are not 2MMF graphs.

We start the first particular case while describing the structure of the strong resolving graph of $G \times K_n$ for any connected graph G. From now on we use the following notation. Consider a set of vertices V and two graphs G and H defined over the sets of vertices $U_1 \subseteq V$ and $U_2 \subseteq V$, respectively. The graph $G \sqcup H$ is defined over the set of vertices $V(G \sqcup H) = U_1 \cup U_2$ and $E(G \sqcup H) = E(G) \cup E(H)$. Note that U_1 and U_2 are not necessarily disjoint, as well as $E(G)$ and $E(H)$. For example, consider a set of seven vertices $v_1, v_2, \ldots, v_7$, the cycle $C_6 = v_1 v_2 \ldots v_6 v_1$ and the star $S_{1,6}$ with

central vertex in v_7 and $v_1, v_2, \ldots, v_6$. Thus, the wheel graph $W_{1,6}$ can be obtained as the graph $C_6 \sqcup S_{1,6}$. Another interesting example is for instance the strong product graph $G \boxtimes H$ which can be obtained as $(G \square H) \sqcup (G \times H)$ (notice that in this case the set of vertices of $G \square H$ and $G \times H$ coincide).

The following result was recently presented in Kuziak et al. (2017). However, the published version of the result has a mistake, as G has to be a noncomplete graph. Here we present the correct version of the result.

Theorem 32. (Kuziak et al. (2017)) *Let G be a connected noncomplete 2MMF graph of order at least three and let the integer $n \geq 3$. If W is the subset of $V(G)$ which contains all vertices belonging to a triangle in G, $N_{|W|}$ is the empty graph with vertex set W and the graphs K_n, N_n are defined over the same set of vertices, then*

$$(G \times K_n)_{SR} \cong (G \square N_n) \sqcup (G_{SR} \circ N_n) \sqcup (N_{|W|} \square K_n).$$

In Figure 8 we exemplify the theorem above. There we give a direct product graph and its strong resolving graph, drawn in such way we can see all the three subgraphs appearing in the union given in Theorem 32.

The following result was also implicitly deduced in (Rodríguez-Velázquez et al., 2014, Proof of Theorem 37), although we now present part of it by using the ideas of Theorem 32. To this end, given an odd cycle $C_n = v_0 v_1 \ldots v_n v_0$, by C_n^* we mean the cycle $v_0 v_{\lfloor n/2 \rfloor} v_{2 \cdot \lfloor n/2 \rfloor} v_{3 \cdot \lfloor n/2 \rfloor} \ldots v_{(n-1) \cdot \lfloor n/2 \rfloor} v_0$ where the multiplication operation $x \cdot \lfloor n/2 \rfloor$ with $x \in \{1, \ldots, n-1\}$ is done modulo n.

Proposition 33. *Let $r \geq 4$ and $t \geq 3$ be positive integers. Let $V(K_t) = V(N_t) = \{g_1, g_2, \ldots, g_t\}$ and $C_r = h_0 h_1 \ldots h_{r-1} h_0$ Then the following assertions hold.*

(i) *If $r \in \{4, 5\}$, then $(C_r \times K_t)_{SR} \cong \bigcup_{i=1}^{t} K_r$.*

(ii) *If $r \geq 6$ is even and $K_2^{(i)}$ is a complete graph on the two vertices $u_i, u_{i+r/2}$ with $i \in \{0, \ldots, r/2 - 1\}$, then*

$$(C_r \times K_t)_{SR} \cong (C_r \square N_t) \sqcup \left(\bigsqcup_{i=0}^{r/2-1} (K_2^{(i)} \circ N_t) \right).$$

(iii) *If $r \geq 7$ is odd, then*

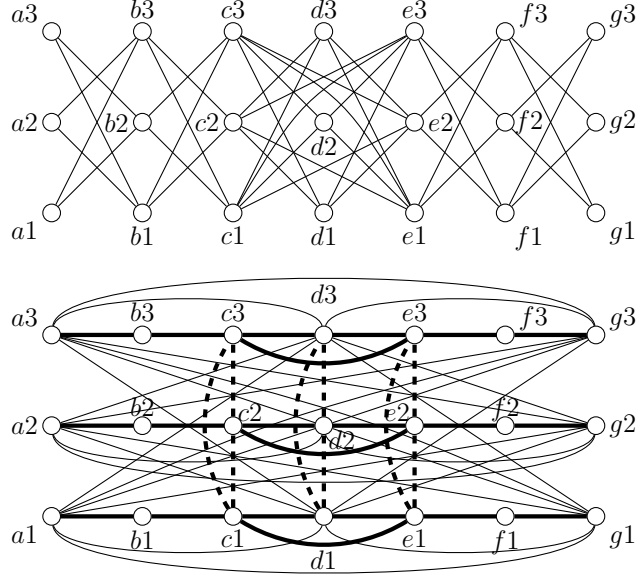$$(C_r \times K_t)_{SR} \cong (C_r \square N_t) \sqcup (C_r^* \circ N_t).$$

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Figure 8: The direct product $H_7 \times K_3$ and its strong resolving graph, where H_7 is obtained from a path $P_7 = abcdefg$ by adding the edge ce. According to Theorem 32, notice that $W = \{c, d, e\}$. In the strong resolving graph $(H_7 \times K_3)_{SR}$: the edges in bold correspond to the subgraph $H_7 \square N_3$ (N_3 has vertex set $\{1, 2, 3\}$); the dashed edges to the subgraph $N_3 \square K_3$ (N_3 has vertex set $W = \{c, d, e\}$); and the remaining edges to the subgraph $((H_7)_{SR} \circ N_3) \cong (K_3 \circ N_3)$ (K_3 has vertex set $\{a, d, g\}$).

Proof. Let $V(K_t) = \{h_1, h_2, \ldots, h_t\}$ and $V(C_r) = \{g_0, g_1, \ldots, g_{r-1}\}$, where $g_i \sim g_{i+1}$ for every $i \in \{0, \ldots, r-1\}$ and $g_{r-1} \sim g_0$. From now on all the operations with the subscript of a vertex g_i of C_r are expressed modulo r. Let $(g_i, h_j), (g_l, h_k)$ be two distinct vertices of $C_r \times K_t$.

(i) Let $r = 4$ or 5. We differentiate four cases.

Case 1: $g_i = g_l$. Hence, $d_{C_r \times K_t}((g_i, h_j), (g_l, h_k)) = 2$. Since $(g_i, h_j) \sim (g_{i-1}, h_k)$, if $k \neq j$ and $d_{C_r \times K_t}((g_{i-1}, h_k), (g_l, h_k)) = 3$, then it follows that (g_i, h_j) and (g_l, h_k) are not MMD in $C_r \times K_t$.

Case 2: $h_j = h_k$. If $l = i+1$ or $i = l+1$, then without loss of generality we suppose $l = i+1$ and we have that $d_{C_r \times K_t}((g_i, h_j), (g_l, h_k)) = 3 = D(C_r \times K_t)$. Thus, (g_i, h_j) and (g_l, h_k) are MMD in $C_r \times K_t$. On the other hand, if $l \neq i+1$ and $i \neq l+1$, then $d_{C_r \times K_t}((g_i, h_j), (g_l, h_k)) = 2$. Since for every vertex $(g, h) \in N_{C_r \times K_t}(g_i, h_j)$ we have that $d_{C_r \times K_t}((g, h), (g_l, h_k)) \leq 2$ and also for every vertex $(g, h) \in N_{C_r \times K_t}(g_l, h_k)$ we have that $d_{C_r \times K_t}((g, h), (g_i, h_j)) \leq 2$, we obtain that (g_i, h_j) and (g_l, h_k) are MMD in $C_r \times K_t$.

Case 3: $g_i \neq g_l$, $h_j \neq h_k$ and $(g_i, h_j) \sim (g_l, h_k)$. So, there exists a vertex $(g, h) \in N_{C_r \times K_t}(g_l, h_k)$ such that $d_{C_r \times K_t}((g, h), (g_i, h_j)) = 2$ and, as a consequence, (g_i, h_j) and (g_l, h_k) are not MMD in $C_r \times K_t$.

Case 4: $g_i \neq g_l$, $h_j \neq h_k$ and $(g_i, h_j) \not\sim (g_l, h_k)$. Hence, we have that $d_{C_r \times K_t}((g_i, h_j), (g_l, h_k)) = 2$. We can suppose, without loss of generality, that $l = i + 2$. Since

- $(g_i, h_j) \sim (g_{l-1}, h_k)$ and $(g_l, h_k) \sim (g_{l-1}, h_j)$ and also,

- $d_{C_r \times K_t}((g_i, h_j), (g_{l-1}, h_j)) = 3$ and $d_{C_r \times K_t}((g_l, h_k), (g_{l-1}, h_k)) = 3$,

we obtain that (g_i, h_j) and (g_l, h_k) are not MMD in $C_r \times K_t$. Hence the strong resolving graph $(C_r \times K_t)_{SR}$ is isomorphic to $\bigcup_{i=1}^{t} K_r$.

The cases (ii) and (iii) are direct consequences of Theorem 32. $\qquad \square$

The first item of next result was implicitly deduced in (Rodríguez-Velázquez et al., 2014, Proof of Theorem 38).

Proposition 34. *Let $r \geq 2$ and $t \geq 3$ be positive integers. Then the following assertions hold.*

(i) *If $r \in \{2, 3\}$, then $(P_r \times K_t)_{SR} \cong \bigcup_{i=1}^{t} K_r$.*

(ii) *If $r \geq 4$, $P_r = g_0 g_1 \ldots g_{r-1}$ and $P_2 = g_0 g_{r-1}$, then $(P_r \times K_t)_{SR} \cong (P_r \square N_t) \sqcup (P_2 \circ N_t)$.*

Proof. Let $V(K_t) = \{h_1, h_2, \ldots, h_t\}$ and $V(P_r) = \{g_0, g_1, \ldots, g_{r-1}\}$, where $g_i \sim g_{i+1}$ for every $i \in \{0, \ldots, r-1\}$.

If $r = 2$, then a vertex (g_i, h_j) in $P_2 \times K_t$ is MMD only with the vertex (g_l, h_j), where $i \neq l$. So, $(P_2 \times K_t)_{SR} \cong \bigcup_{m=1}^{t} K_2$.

If $r = 3$, then a vertex (g_i, h_j) in $P_3 \times K_t$ is MMD only with those vertices (g_l, h_j), where $i \neq l$. Thus, $(P_3 \times K_t)_{SR} \cong \bigcup_{m=1}^{t} K_3$.

If $r \geq 4$, then the result is a particular case of Theorem 32. $\qquad \square$

We next deal with the direct product of a complete bipartite graph and a complete graph. In contrast with Theorem 32, in this case all the MMD vertices of the complete bipartite graph are at distance two.

Theorem 35. (Kuziak et al. (2017)) *For any $r, t \geq 1$ and any $n \geq 3$,*

$$(K_{r,t} \times K_n)_{SR} \cong \bigcup_{i=1}^{n} K_{r+t}.$$

Now we present some results for graphs of diameter two as factors of a direct product. Since it is necessary to be careful with connectedness of the direct product, the results are separated with respect to whether one factor is bipartite or not. It is not hard to see that the only bipartite graphs of diameter two are the complete bipartite graphs $K_{k,\ell}$, where $\max\{k,\ell\} \geq 2$.

Another important measure for the strong resolving graphs of a direct product of two graphs of diameter two is when the factors are triangle free and moreover, when every pair of vertices is on a five-cycle. Hence, we call a graph in which every pair of vertices is on a common five-cycle, a C_5-*connected* graph. Clearly, a C_5-connected graph has diameter at most two. Moreover, if G is a triangle free C_5-connected graph, then its diameter equals two. The Petersen graph is C_5-connected triangle free graph. The graph G of Figure 9 is an example of a triangle free graph of diameter two in which u and v are not on a common five-cycle and G is not C_5-connected. The graph H of the same figure is a triangle free C_5-connected graph of diameter two.
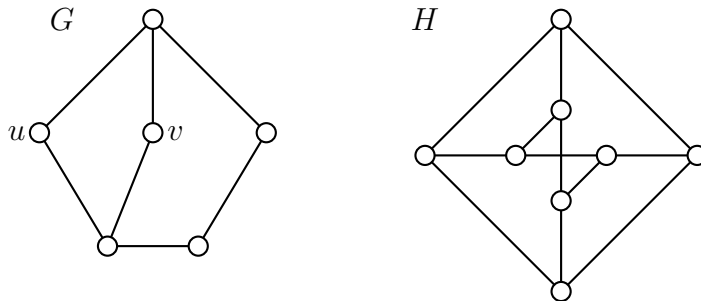

Figure 9: Two triangle free graphs of diameter two.

The next two results were implicitly deduced in (Kuziak et al., 2017, Proofs of Theorem 13 and 14).

Proposition 36. *Let G be a nonbipartite triangle free graph of order $n \geq 2$ and let $\max\{k,\ell\} \geq 2$. If G is C_5-connected, then*

$$(G \times K_{k,\ell})_{SR} \cong N_n \square K_{k+\ell}.$$

Proof. Assume that $V(G) = \{g_1, \ldots, g_n\}$ and $U(K_{k,\ell}) = U_1 \cup U_2$ where $U_1 = \{u_1, \ldots, u_k\}$ and $U_2 = \{v_1, \ldots, v_\ell\}$. Clearly, $d^e_{K_{k,\ell}}(u_i, v_j) = \infty$, $d^o_{K_{k,\ell}}(u_i, v_j) = 1$, $d^o_{K_{k,\ell}}(v_i, v_j) = \infty$ and $d^o_{K_{k,\ell}}(u_i, u_j) = \infty$ for any i and j. Also, $d^e_{K_{k,\ell}}(u_i, u_j) = $

2 and $d^e_{K_{k,\ell}}(v_i, v_j) = 2$ for every $i \neq j$. Conversely, by C_5-connectedness of G, $d^e_G(g_i, g_j)$ and $d^o_G(g_i, g_j)$ always exist. Moreover, $d^e_G(g_i, g_j)$ is between 0 and 4, while $d^o_G(g_i, g_j)$ is between 1 and 5. Hence, by the distance formula presented in Remark 21 we can have the distances between 0 and 5 in $G \times K_{k,\ell}$. Again, by this distance formula, it is easy to see that $d_{G \times K_{k,\ell}}((g_1, u_1), (g_1, v_j)) = 5$ for any $j \in \{1, \ldots, \ell\}$ and that $d_{G \times K_{k,\ell}}((g_1, u_1), (g_1, u_j)) = 2$ for any $j \in \{2, \ldots, k\}$. We show that vertices satisfying these equalities above are the only neighbors of (g_1, u_1) in the strong resolving graph $(G \times K_{k,\ell})_{SR}$. Clearly, (g_1, u_1) and (g_1, v_j) are MMD, since they are diametral vertices for any $j \in \{1, \ldots, \ell\}$. Since $N_{K_{k,\ell}}(u_1) = N_{K_{k,\ell}}(u_j)$, for any $j \in \{2, \ldots, k\}$, by equation (2) that describes neighborhoods in the direct product, we see that (g_1, u_1) and (g_1, u_j) have the same neighborhood and therefore, they are MMD.

Next we show that no other vertex of $G \times K_{k,\ell}$ is MMD with (g_1, u_1). In this case, we reduce it to a five-cycle, since G is C_5-connected. We may assume that $g_1 g_2 g_3 g_4 g_5 g_1$ is a five-cycle. By the symmetry of a five-cycle we need to present the arguments only for g_2 and g_3. For every $j \in \{1, \ldots, \ell\}$ and $i \in \{2, \ldots, \ell\}$ they are as follows:

- $(g_2, v_j) \sim (g_3, u_1)$ and (g_2, v_j) is closer to (g_1, u_1) than (g_3, u_1);

- $(g_2, u_i) \sim (g_1, v_1)$ and (g_2, u_i) is closer to (g_1, u_1) than (g_1, v_1);

- $(g_3, v_j) \sim (g_2, u_1)$ and (g_3, v_j) is closer to (g_1, u_1) than (g_2, u_1);

- $(g_3, u_i) \sim (g_4, v_1)$ and (g_3, u_i) is closer to (g_1, u_1) than (g_4, v_1).

See the graph $C_5 \times K_{1,2} \cong C_5 \times P_3$ on the left part of Figure 10, where the distances from (g_1, u_1) are marked. Thus, the vertex (g_1, u_1) is adjacent to all vertices of $\{g_1\} \times (V(K_{k,\ell}) - \{u_1\})$ in $(G \times K_{k,\ell})_{SR}$. Notice that the same argument also holds when $\min\{k, \ell\} = 1$. We can use the same arguments for any vertex of $G \times K_{k,\ell}$ and therefore, we have $(G \times K_{k,\ell})_{SR} \cong N_n \square K_{k+\ell}$. $\square$

Theorem 37. *For any nonbipartite triangle free C_5-connected graphs G and H of diameter two,*

$$(G \times H)_{SR} \cong G \square H.$$

Proof. Let $V(G) = \{g_1, \ldots, g_n\}$ and $V(H) = \{h_1, \ldots, h_k\}$. Note that G and H are C_5-connected graphs, which implies that their even and odd distances
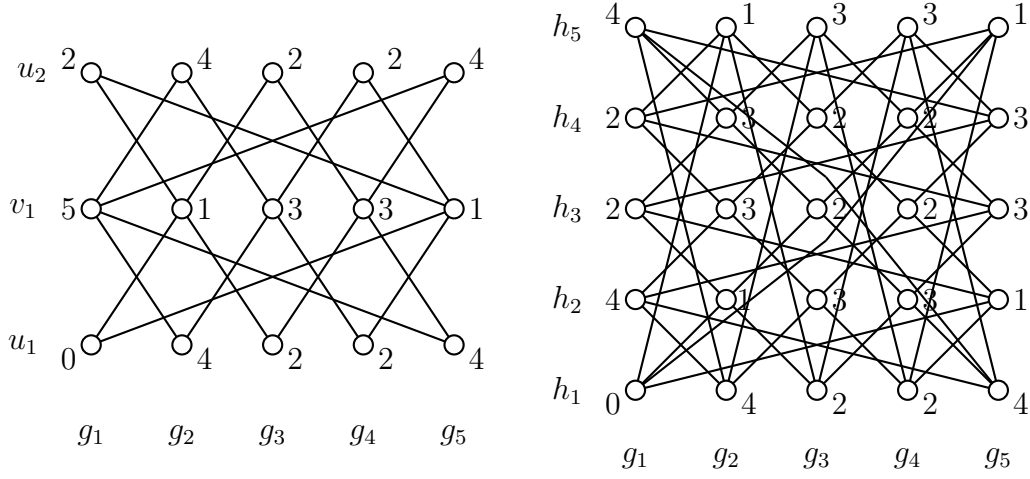
Figure 10: Situations from the proofs of Theorems 36 and 37. The label of each vertex corresponds to the distance of each vertex to the vertex (g_1, u_1) in the left side, and to (g_1, h_1) in the right side.

between arbitrary vertices always exist. Moreover, the even distances are between 0 and 4, while the odd distances are between 1 and 5. Now, according to Remark 21, the distances in $G \times H$ are between 0 and 4. We may assume that $g_1 g_2 g_3 g_4 g_5 g_1$ and $h_1 h_2 h_3 h_4 h_5 h_1$ are induced five-cycles of triangle free C_5-connected graphs G and H, respectively. Again, by this distance formula, it is easy to see that $d_{G \times H}((g_1, h_1), (g_1, h_j)) = 4$ for $j \in \{2, 5\}$ and that $d_{G \times H}((g_1, h_1), (g_j, h_1)) = 4$ for $j \in \{2, 5\}$. We show that these are the only neighbors of (g_1, u_1) in $(G \times H)_{SR}$. Clearly, these pairs are mutually maximally distant since they are diametral vertices.

We now show that (g_1, u_1) is not MMD with any other vertex of $G \times K_{k,\ell}$. By the symmetry of a five-cycle and the commutativity of the direct product we need to present the arguments only for g_1, g_2 and g_3 and for h_1, h_2 and h_3. They are as follows:

- $(g_1, h_3) \sim (g_2, h_4)$ and (g_1, h_3) is closer to (g_1, h_1) than (g_2, h_4);

- $(g_2, h_2) \sim (g_3, h_1)$ and (g_2, h_2) is closer to (g_1, h_1) than (g_3, h_1);

- $(g_2, h_3) \sim (g_1, h_2)$ and (g_2, h_3) is closer to (g_1, h_1) than (g_1, h_2);

- $(g_3, h_1) \sim (g_4, h_2)$ and (g_3, h_1) is closer to (g_1, h_1) than (g_4, h_2);

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- $(g_3, h_2) \sim (g_2, h_1)$ and (g_3, h_2) is closer to (g_1, h_1) than (g_2, h_1);

- $(g_3, h_3) \sim (g_2, h_4)$ and (g_3, h_3) is closer to (g_1, h_1) than (g_2, h_4).

See the graph $C_5 \times C_5$ on the right part of Figure 10, where the distances from (g_1, h_1) are marked. So, the vertex (g_1, h_1) is adjacent to the vertices (g_1, h_2), (g_1, h_5), (g_2, h_1) and (g_5, h_1) in $(G \times K_{k,\ell})_{SR}$. Continuing with the same arguments, we obtain that (g_1, u_1) is adjacent to all vertices of $(\{g_1\} \times N_H(h_1)) \cup (N_G(g_1) \times \{h_1\})$ in $(G \times H)_{SR}$. By using the same arguments for any vertex of $G \times H$ we obtain $(G \times H)_{SR} \cong G \square H$, which completes the proof. $\qquad \square$

3.2. Cartesian Sum and Strong Product Graphs

The description of the strong resolving graphs for some cases of $G \oplus H$ can easily be obtained from Proposition 14 and Proposition 24.

Proposition 38. (Kuziak et al. (2015a)) *Let G and H be two nontrivial graphs such that at least one of them is noncomplete. If $D(G) \leq 2$ or neither G nor H has isolated vertices, then*

$$(G \oplus H)_{SR} \cong (G \oplus H)^*_-.$$

Proof. We assume that $D(G) \leq 2$ or neither G nor H has isolated vertices. Then, by Proposition 24 we have $D(G \oplus H) = 2$ and hence, by Proposition 14, $(G \oplus H)_{SR} \cong (G \oplus H)^*_-$. $\qquad \square$

We now describe the structure of the strong resolving graph of $G \boxtimes H$.

Lemma 39. (Kuziak et al. (2015b)) *Let G and H be two connected nontrivial graphs. Let g, g' be two vertices of G and let h, h' be two vertices of H. Then (g, h) and (g', h') are MMD vertices in $G \boxtimes H$ if and only if one of the following conditions holds:*

  (i) *g, g' are MMD in G and h, h' are MMD in H;*

 (ii) *g, g' are MMD in G and $h = h'$;*

(iii) *h, h' are MMD in H and $g = g'$;*

(iv) *g, g' are MMD in G and $d_G(g, g') > d_H(h, h')$;*

 (v) *h, h' are MMD in H and $d_G(g, g') < d_H(h, h')$.*

We need to introduce more notation. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G and we denote that by $G' \sqsubseteq G$. Notice that Lemma 39 leads to the following relationship.

Theorem 40. (Kuziak et al. (2015b)) *For any connected graphs G and H,*

$$G_{SR+I} \boxtimes H_{SR+I} \sqsubseteq (G \boxtimes H)_{SR+I} \sqsubseteq G_{SR+I} \oplus H_{SR+I}.$$

3.3. Lexicographic Product Graphs

From the next lemmas we can describe the structure of the strong resolving graph of $G \circ H$.

Lemma 41. (Kuziak et al. (2016)) *Let G be a connected nontrivial graph and let H be a nontrivial graph. Let $g, g' \in V(G)$ be such that they are not true twin vertices and let $h, h' \in V(H)$. Then (g, h) and (g', h') are MMD in $G \circ H$ if and only if g and g' are MMD in G.*

Lemma 42. (Kuziak et al. (2016)) *Let G be a connected nontrivial graph, let H be a graph of order $n \geq 2$, let $g, g' \in V(G)$ be two distinct true twin vertices and let $h, h' \in V(H)$. Then (g, h) and (g', h') are MMD in $G \circ H$ if and only if both, h and h', have degree $n - 1$.*

The strong resolving graph of the lexicographic product can be described using graphs G^* and G_-^* already defined in Section 2.

Remark 43. (Kuziak et al. (2016)) *Let G be a connected graph of diameter $D(G)$, order n and maximum degree $\Delta(G)$.*

(i) *If $\Delta(G) \leq n - 2$, then $G^* \cong (K_1 + G)_{SR}$.*

(ii) *If $D(G) \leq 2$, then $G_-^* \cong G_{SR}$.*

(iii) *If G has no true twins, then $G^* \cong G^c$.*

Lemma 44. (Kuziak et al. (2016)) *Let G be a connected nontrivial graph. Let $h, h' \in V(H)$ be two distinct vertices of a graph H and let $g \in V(G)$. Then (g, h) and (g, h') are MMD vertices in $G \circ H$ if and only if g and h are adjacent in H^*.*

Proposition 45. (Kuziak et al. (2016)) *Let G be a connected graph of order $n \geq 2$ and let H be a noncomplete graph. If G has no true twin vertices, then*

$$(G \circ H)_{SR} \cong (G_{SR} \circ H^*) \cup \bigcup_{i=1}^{n-|\partial(G)|} H_-^*.$$

Figure 11 shows the graph $P_4 \circ P_3$ and its strong resolving graph. Notice that $(P_3)_-^* \cong K_2$, $(P_3)^* \cong K_2 \cup K_1$ and $(P_4)_{SR} \cong K_2$. So, $(P_4 \circ P_3)_{SR} \cong K_2 \circ (K_2 \cup K_1) \cup K_2 \cup K_2$.
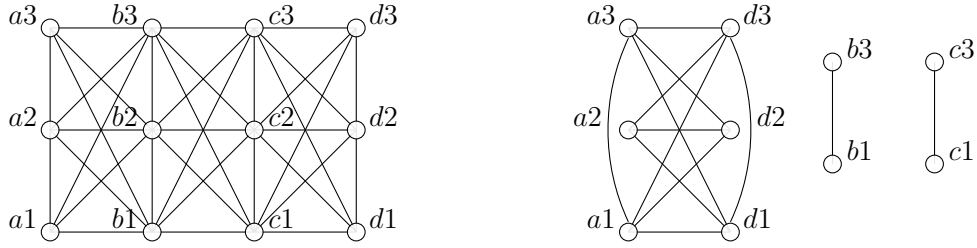

Figure 11: The graph $P_4 \circ P_3$ and its strong resolving graph

Since Lemmas 41, 42 and 44 describe the structure of the strong resolving graph of lexicographic product graphs, they are important tools used in Kuziak et al. (2016) to deduce results like the following one.

Proposition 46. (Kuziak et al. (2016)) *For any connected nontrivial graph G and any integer $n' \geq 2$,*

$$(G \circ K_{n'})_{SR} \cong (G_{SR} \circ K_{n'}) \cup \bigcup_{i=1}^{n-|\partial(G)|} K_{n'}.$$

We have studied the case in which the second factor in the lexicographic product is a complete graph. Since this product is not commutative, we now consider the case in which the first factor is a complete graph.

Proposition 47. (Kuziak et al. (2016)) *Let $n \geq 2$ be an integer and let H be a graph of order $n' \geq 2$. If H has maximum degree $\Delta(H) \leq n' - 2$, then*

$$(K_n \circ H)_{SR} \cong \bigcup_{i=1}^{n} H^*.$$

We define the *TF-boundary* of a noncomplete graph $G = (V, E)$ as a set $\partial_{TF}(G) \subseteq \partial(G)$, where $x \in \partial_{TF}(G)$ whenever there exists $y \in \partial(G)$, such that x and y are MMD in G and $N_G[x] \neq N_G[y]$ (which means that x, y are not true twins). The *strong resolving TF-graph* of G is a graph G_{SRS} with vertex set $V(G_{SRS}) = \partial_{TF}(G)$, where two vertices u, v are adjacent in G_{SRS} if and only if u and v are MMD in G and $N_G[x] \neq N_G[y]$. Since the strong resolving TF-graph is a subgraph of the strong resolving graph, an instance of the problem of transforming a graph into its strong resolving TF-graph forms part of the general problem of transforming a graph into its strong resolving graph. From Oellermann and Peters-Fransen (2007), it is known that this general transformation is polynomial. Thus, the problem of transforming a graph into its strong resolving TF-graph is also polynomial.

Proposition 48. (Kuziak et al. (2016)) *Let G be a connected noncomplete graph of order $n \geq 2$ and let H be a graph of order $n' \geq 2$. If H has maximum degree $\Delta(H) \leq n' - 2$, then*

$$(G \circ H)_{SR} \cong (G_{SRS} \circ H^*) \cup \bigcup_{i=1}^{n - |\partial_{TF}(G)|} H^*.$$

Figure 12 shows the graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph. Notice that $(P_4)^* \cong P_4$ and $(K_1 + (K_1 \cup K_2))_{SRS} \cong P_3$. So, $((K_1 + (K_1 \cup K_2)) \circ P_4)_{SR} \cong (P_3 \circ P_4) \cup P_4$.
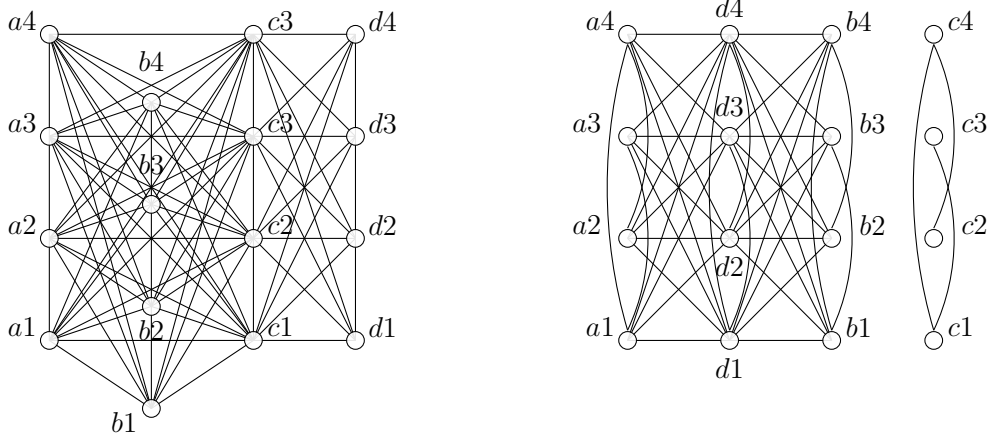

Figure 12: The graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph

3.4. Corona Product Graphs

The structure of the strong resolving graph of the corona product can be easily described. By equation (3), that shows the distance between vertices in the corona product, we deduce that $\partial(G \odot H) = \bigcup_{i=1}^{n} V_i$ and two vertices x, y are adjacent in $(G \odot H)_{SR}$ if and only if either $x \in V_i$ and $y \in V_j$, where $i \neq j$, or $x, y \in V_i$ and they are adjacent in H^*, that is, they are true twins in H or they are adjacent in H^c. Therefore $(G \odot H)_{SR}$ is obtained from the complete graph of vertex set $\partial(G \odot H) = \bigcup_{i=1}^{n} V_i$ by removing the edges of each copy of H connecting two non-true twin vertices. So we can deduce the following result.

Remark 49. *Let G, H be two graphs, then $(G \odot H)_{SR}$ is a complete graph if and only if H is either a complete graph or an empty graph.*

An interesting example of a strong resolving TF-graph defined in Section 3.3 can be obtained from the corona graph $G \odot K_{n'}$, $n' \geq 2$, where G has order $n \geq 2$. Notice that any two distinct vertices belonging to any two copies of the complete graph $K_{n'}$ are MMD, but if they are in the same copy, then they are also true twins. Thus, in this case $\partial_{TF}(G \odot K_{n'}) = \partial(G \odot K_{n'})$, while we have that $(G \odot K_{n'})_{SR} \cong K_{nn'}$ and $(G \odot K_{n'})_{SRS}$ is isomorphic to a complete n-partite graph $K_{n',n',\dots,n'}$.

4. Open Problems

The strong resolving graph G_{SR} of a graph G is still not enough known as an interesting and very useful construction. In this sense, some of the next problems would be worthwhile to be dealt with.

- It is already known that constructing the strong resolving graph G_{SR} of a graph G can be done in polynomial time. However, not much is known on deciding whether a given graph H is the strong resolving graph of a graph G. Some partial results are given in this work, but still much more is required to get a complete characterization.

- Is it possible to describe some properties of the strong resolving graph G_{SR} based on some properties of the graph G? Can we state for instance whether G_{SR} is connected, bipartite or hamiltonian? Can we assert which is the diameter or the girth of G_{SR} based on some properties of G?

- In Proposition 16 we proved that no star graph $K_{1,r}$, $r \geq 2$, represents the strong resolving graph of a graph. However, we also proved that there are graphs G for which G_{SR} contains a component isomorphic to a star graph $K_{1,r}$ for any $r \geq 2$. Moreover, in Proposition 16 we proved that no complete bipartite graph $K_{2,r}$, $r \geq 2$, represents the strong resolving graph of a graph. Could we similarly find a graph G for which G_{SR} contains a component isomorphic to $K_{2,r}$ for any $r \geq 2$?

- Prove or disprove Conjecture 17. In the case Conjecture 17 would be true, could it be possible to find a graph G for which G_{SR} contains a component isomorphic to $K_{r,t}$ for any $r, t \geq 2$?

- Is it possible to find some forbidden subgraphs in a given graph that avoid its realization as the strong resolving graph of other graph?

- Is it possible to characterize the family of graphs G for which $G_{SR} \cong G$?

- Given a graph G, is it possible to find all the graphs H such that $H_{SR} \cong G$?

- Is there any other usefulness of the strong resolving graph distinct from that of computing the strong metric dimension?

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