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# Reductions and symmetries for a generalized Fisher equation with a diffusion term dependent on density and space

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ABSTRACT

In this work, a generalized Fisher equation with a space–density diffusion term is analyzed by applying the theory of symmetry reductions for partial differential equations. The study of this equation is relevant in terms of its applicability in cell dynamics and tumor invasion. Therefore, classical Lie symmetries admitted by the equation are determined. In addition, by using the multipliers method, we derive some nontrivial conservation laws for this equation. Finally we obtain a direct reduction of order of the ordinary differential equations associated and a particular solution.



## 1. Introduction

Reaction–diffusion equations are a fundamental part in modeling the spread of biological populations. The Fisher equation and its extensions are a family of reaction–diffusion models arising in population dynamics problems [1,2], most prominently in cancer modeling [3,4], applications to brain tumor dynamics [5], in the description of propagating crystallization/polymerization fronts [6], chemical kinetics [7], geochemistry [8] and many others fields. These equations have already been deeply analyzed in the literature [1,9,10], in relation to their solutions and traveling waves for the case of the Fisher equation:

$$u_t = D\,u_{xx} + \mu u(1-u). \tag{1}$$

A plausible generalization for this model has been studied in [1], where growth factor and diffusion were density-dependent:

$$u_t = \left(D(u)u_x\right)_x + f(u). \tag{2}$$

In fact, this equation is not only interesting for cancer models and mutating cells, but also in biochemical reaction kinetics such as the effect of haemoglobin and myoglobin in blood [1].

The study of this kind of equations is interesting in terms of finding exact solutions. In [11], for example, a numerical analysis was performed. To do so, the Lie classical method is useful to obtain reductions to ODEs and if it is possible, families of exact solutions.

Lie symmetries of the density dependent reaction–diffusion equation (2) were calculated in [12], as well as the optimal system of one-dimensional subalgebras of the invariant equation. Several reductions and exacts solutions were also

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obtained. In [13], some nontrivial conservation laws were constructed for the generalized Fisher equation (2) associated with symmetries of the differential equations. A non-linear multidimensional reaction–diffusion system with variables diffusivities was also considered in [14]. In this paper, the classical Lie symmetry of this system is calculated.

Over the last decades a lot of attention has been paid on using Lie point symmetry methods to exploit the invariance of the generalized equation

$$u_t = (A(u)u_x)_x + B(u)u_x + C(u). \tag{3}$$

In the case $A = 1, B = C = 0$, the classical heat equation was firstly studied by S. Lie in [15] in terms of maximal invariance algebra. A complete Lie symmetry classification for the non-linear heat equation (3) with $B = C = 0$ was described in [16]. Moreover, for the case $B = 0$ in Eq. (3) the Lie symmetry was completely described in [17]. Later, the Lie symmetries of Eq. (3) were fully described in [18].

In [19], a class of variable coefficient nonlinear diffusion–convection equations of the form

$$f(x)u_t = (c(x)g(u)u_x)_x + K(u)u_x \tag{4}$$

was considered. The authors performed group classification and constructed exact solutions of such equations. In other work, a group classification of a class of variable coefficient reaction–diffusion equations with exponential nonlinearities is obtained [20].

A generalization of the prior equation,

$$u_t = \frac{1}{c(x)}\left(c(x)\cdot g(u)u_x\right)_x + f(u), \tag{5}$$

has also been intensively studied: in [21] some nontrivial conservation laws associated to the symmetries were obtained for $g = k\cdot f_u$ and $f$, $c$ arbitrary functions; in [22] the classical Lie method was applied to derive some nontrivial conservation laws for this equation. Symmetry reductions and exact solutions for (5) were obtained using classical and potential symmetries in [23].

When $c(x) = x$, the equation in (5) turns into the generalized Fisher equation

$$u_t = \frac{1}{x}\left(x\cdot g(u)u_x\right)_x + f(u), \tag{6}$$

which was studied in [24], where some nontrivial conservation laws were obtained.

Generalizations of the Fisher equation are necessary to accurately model diffusion and reaction effects. Therefore, we consider a generalized Fisher equation with density–space-dependent diffusion in the present manuscript as

$$u_t = \left(g(u)c(x)u_x\right)_x + f(u) \tag{7}$$

which arises in a broad range of biological processes [10] and specifically in cancer modeling problems [11]. To illustrate the latter, a particular case of this mathematical model (7) was introduced by [3] to study the complex geometry of the brain and to allow diffusion (or cell motility). Furthermore, Eq. (7) has also been studied in [5] for a space-dependent diffusion term, in order to describe malignancy of gliomas as an invasion of gray matter.

The structure of this work goes as follows: firstly, we apply the Lie classical method to Eq. (7) in order to obtain a group classification. This is done in Section 2. In Section 3, conservations laws for the generalized Fisher equation (7) are obtained, which are useful in case numerical methods were applied to it. Finally, we have focused on a case with a special biological meaning, and then obtained some exact solutions for (7). The family of one-parametric solutions found illustrate this process.

## 2. Lie symmetries and reductions

Lie classical method is based on the determination of the point symmetry group of a differential equation, i.e., the largest group of transformations acting on dependent and independent variables of the equation so that it maps solutions of the equation into other solutions.

In order to apply Lie classical method to Eq. (7) we consider the one-parameter Lie group of infinitesimal transformations in $(x, t, u)$, given by

$$x^* = x + \epsilon\,\xi(x, t, u) + O(\epsilon^2), \tag{8}$$
$$t^* = t + \epsilon\,\tau(x, t, u) + O(\epsilon^2), \tag{9}$$
$$u^* = u + \epsilon\,\eta(x, t, u) + O(\epsilon^2), \tag{10}$$

where $\epsilon$ is the group parameter. The point symmetry group of Eq. (7) will be given by the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \tag{11}$$

Eq. (7) admits a Lie point symmetry provided that

$$pr^{(2)}\mathbf{v}(\Delta) = 0 \qquad \text{when} \qquad \Delta = 0,$$

where $\Delta = u_t - f(u) - (g(u)c(x)u_x)_x$ and $pr^{(2)}\mathbf{v}$ is the second prolongation of the vector field (11). We obtain a set of determining equations for the infinitesimals $\xi = \xi(x,t,u)$, $\tau = \tau(x,t,u)$ and $\eta = \eta(x,t,u)$. From the determining system, we get that $\xi = \xi(x,t)$, $\tau = \tau(t)$, where $\eta, \tau, \xi, g, f$ and $c$ must satisfy the following equations:

$$\begin{aligned}
&c\, g_u\, \eta + c\, \tau_t\, g - 2\, c\, \xi_x\, g = 0,\\
&c\, g\, \eta_{u\,u} + c\, g_u\, \eta_u + c\, g_{u\,u}\, \eta + c\, \tau_t\, g_u - 2\, c\, \xi_x\, g_u = 0,\\
&2\, c\, g_u\, \eta_x + 2\, c\, g\, \eta_{u\,x} + c_x\, g_u\, \eta + c_x\, \tau_t\, g - c\, \xi_{x\,x}\, g - c_x\, \xi_x\, g + \xi_t = 0,\\
&-c\, g\, \eta_{x\,x} - c_x\, g\, \eta_x + f\, \eta_u + \eta_t - f_u\, \eta - \tau_t\, f = 0.
\end{aligned} \tag{12}$$

After solving the determining equations, we can distinguish different cases in which the symmetries are admitted by Eq. (7) for functional forms of $c(x)$, $f(u)$ and $g(u)$, where $c' \neq 0$, $f' \neq 0$, $g' \neq 0$. We distinguish as well the corresponding generators and group transformations, which are given below:

**Case 1**. For $c = c(x)$, $f = f(u)$ and $g = g(u)$ arbitrary functions we get the generator

$$\mathbf{X}_1 = \partial_t. \tag{13a}$$

$$(x^*, t^*, u^*) = (x, t+\epsilon, u) \quad \textit{time translation}. \tag{13b}$$

**Case 2**. For $f = f(u)$, $g = g(u)$ arbitrary functions and $c(x) = \dfrac{1}{4}\,(c_1\, x + c_2)^2$ we get the generator $\mathbf{X}_1$ and besides

$$\mathbf{X}_2 = (c_1 x + c_2)\partial_x. \tag{14a}$$

$$(x^*, t^*, u^*) = \left(e^{c_1\,\epsilon}\left(x + \frac{c_2}{c_1}\right) - \frac{c_2}{c_1}, t, u\right) \quad \textit{scaling and shift}. \tag{14b}$$

**Case 3**. For $f(u) = f_2(g_2-u)^{-f_1}$, $g(u) = g_3(g_2-u)^{g_1}$ and $c(x) = c_3(c_2-x)^{c_1}$ with arbitrary values of the constants $f_1, g_1$, such that $f_1 + g_1 + 1 \neq 0$, Eq. (7) admits the generator $\mathbf{X}_1$ and the following:

$$\mathbf{X}_3 = (c_2 - x)\partial_x + \frac{(c_1-2)(f_1+1)t}{f_1+g_1+1}\partial_t + \frac{(c_1-2)(u-g_2)}{f_1+g_1+1}\partial_u. \tag{15a}$$

$$\begin{aligned}(x^*, t^*, u^*) = \Bigg(& e^{-\epsilon}\,(x-c_2) + c_2,\ \exp\left(\frac{(c_1-2)(f_1+1)}{f_1+g_1+1}\epsilon\right)t,\\
& g_2 + \exp\left(\frac{c_1-2}{f_1+g_1+1}\epsilon\right)(u-g_2)\Bigg) \quad \textit{scaling and shift}.\end{aligned} \tag{15b}$$

**3.1.** If $f_1 + g_1 + 1 = 0$ then Eq. (7) admits $\mathbf{X}_1$ as a generator for $c(x)$ an arbitrary function.

**Case 4**. For $f(u) = f_1\ (u-g_2) + f_2\ (g_2-u)^{g_1+1}$, $g(u) = g_3(g_2-u)^{g_1}$ and $c = c(x)$ an arbitrary function with arbitrary values of the constants $f_1, g_1, g_2$ such that $g_1, f_1 \neq 0$, Eq. (7) admits the generator $\mathbf{X}_1$ and the following:

$$\mathbf{X}_4 = e^{-f_1\, g_1\, t}\partial_t + e^{-f_1\, g_1\, t} f_1\ (u-g_2)\ \partial_u. \tag{16a}$$

$$\begin{aligned}(x^*, t^*, u^*) = \Bigg(& \frac{1}{f_1 g_1}\ln\left|e^{f_1\, g_1\, t} + f_1 g_1 \epsilon\right|, x,\\
& g_2 + \exp\left(\frac{f_1\,\epsilon}{f_1\, g_1\,\epsilon + e^{f_1\, g_1\, t}}\right)(u-g_2)\Bigg) \quad \textit{time dilation and shift}.\end{aligned} \tag{16b}$$

**4.1.** If $f_1 = 0$, with $f(u) = f_2\ (g_2-u)^{g_1+1}$, $g(u) = g_3(g_2-u)^{g_1}$ and arbitrary $c(x)$, then Eq. (7) admits the generator $\mathbf{X}_1$ and as $g_1 \neq 0$,

$$\mathbf{X}_{4a} = t\,\partial_t + \frac{(u-g_2)}{g_1}\partial_u \tag{17a}$$

$$(x^*, t^*, u^*) = \left(e^{\epsilon}\, t, x, \exp\left(\frac{\epsilon}{g_1}\right)(u-g_2) + g_2\right) \quad \textit{scaling and shift}. \tag{17b}$$

**Case 5**. For $f(u) = f_2\,(g_2-u)^{-f_1}$, $g(u) = g_3(g_2-u)^{g_1}$, with arbitrary values of the constants $f_1, f_2, g_1, g_2, g_3$, we consider the following subcases:

**5.1.** For $c(x)=\left(\frac{(2g_1+3)(c_1x+c_2)}{3g_1+4}\right)^{\frac{3g_1+4}{2g_1+3}}$ with arbitrary values of the constants $c_1, c_2$ such that $f_1+g_1+1\neq 0,\ g_1\neq 0, -\frac{4}{3}, -\frac{3}{2}$, Eq. (7) admits the following generator $\mathbf{X}_1$, and besides:

$$\mathbf{X}_5 =(c_1\,x+c_2)\partial_x+\frac{(f_1+1)\,(g_1+2)c_1\,t}{(2\,g_1+3)\,(g_1+f_1+1)}\partial_t+ -\frac{(g_2-u)\,(g_1+2)c_1}{(2\,g_1+3)\,(g_1+f_1+1)}\partial_u. \tag{18a}$$

$$(x^*,t^*,u^*)=\left(e^{c_1\epsilon}\left(x+\frac{c_2}{c_1}\right)-\frac{c_2}{c_1},\exp\left(\frac{c_1(g_1+2)(f_1+1)}{(2g_1+3)(g_1+f_1+1)}\epsilon\right)t,\right. \left. g_2+\exp\left(\frac{c_1(g_1+2)}{(2g_1+3)(g_1+f_1+1)}\epsilon\right)(u-g_2)\right)\quad \textit{scaling and shift}. \tag{18b}$$

**5.2.** We consider $c\,(x)=c_2\exp(c_1x)$ for arbitrary values of the constants $c_1, c_2$. If $g_1=-\frac{3}{2}$, and $f_1\neq\frac{1}{2}$ then Eq. (7) admits the generator $\mathbf{X}_1$ and

$$\mathbf{X}_{5b}=\frac{-2\,t\,(f_1+1)\,c_1}{2\,f_1-1}\partial_t+\partial_x+2\,\frac{(g_2-u)\,c_1}{2\,f_1-1}\partial_u. \tag{19a}$$

$$(x^*,t^*,u^*)=\left(t\ \exp\left(\frac{-2\,(f_1+1)\,c_1\epsilon}{2\,f_1-1}\right),x+\epsilon,\right. \left.\exp\left(-\frac{2\,c_1\epsilon}{2\,f_1-1}\right)(u-g_2)+g_2\right)\quad \textit{scaling and shift}. \tag{19b}$$

**5.3.** We consider $c\,(x)=c_2\exp(c_1x)$ for arbitrary values of the constants $c_1, c_2$. If $g_1=-\frac{3}{2}$, and $f_1=\frac{1}{2}$ then Eq. (7) admits the generator $\mathbf{X}_1$ and

$$\mathbf{X}_{5c}=\frac{c_1\,t}{2}\partial_t+\partial_x+c_1\,(u-g_2)\partial_u. \tag{20a}$$

$$(x^*,t^*,u^*)=\left(t\,e^{\frac{c_1\epsilon}{2}},x+\epsilon,e^{c_1\epsilon}(u-g_2)+g_2\right)\quad \textit{scaling and shift}. \tag{20b}$$

**5.4.** If $g_1=-\frac{4}{3}$ or $f_1+g_1+1=0$ , then Eq. (7) admits the generator $\mathbf{X}_1$ for $c(x)$ an arbitrary function.

**Case 6**. For $f(u)=f_2\,(g_1\,u+g_2)^{\frac{f_1}{g_1}}$, $g(u)=\left(-\frac{4}{3}\,(g_1\,u+g_2)^{-1}\right)^{\frac{4}{3}}$ and $c(x)=c_3\,(c_2-x)^{c_1}$ with arbitrary values of the constants $f_1, f_2, g_1, g_2, c_1, c_2, c_3$ such that $3f_1+g_1\neq 0,\ g_1\neq 0,\ c_1\neq 0$, Eq. (7) admits the generator $\mathbf{X}_1$ and the following:

$$\mathbf{X}_6 =(c_2-x)\partial_x+\frac{3\,(c_1-2)\,(f_1-g_1)\,t}{3\,f_1+g_1}\partial_t -\frac{3\,(u+g_2)\,(c_1-2)\,g_1}{3\,f_1+g_1}\partial_u. \tag{21a}$$

$$(x^*,t^*,u^*)=\left(e^{-\epsilon}\,(x-c_2)+c_2,\exp\left(\frac{3\,(c_1-2)\,(f_1-g_1)}{3\,f_1+g_1}\epsilon\right)t,\right. \left.\exp\left(\frac{-3\,(c_1-2)g_1}{3\,f_1+g_1}\epsilon\right)(u+g_2)-g_2\right)\quad \textit{scaling and shift}. \tag{21b}$$

**6.1.** If $3f_1+g_1=0$ then Eq. (7) admits the generator $\mathbf{X}_1$ and

$$\mathbf{X}_{6*}=t\,\partial_t+3\frac{(u-g_2)}{4}\partial_u \tag{22a}$$

$$(x^*,t^*,u^*)=\left(e^{\epsilon}\,t,x,\exp\left(\frac{3\epsilon}{4}\right)(u-g_2)+g_2\right)\quad \textit{scaling and shift}. \tag{22b}$$

**Case 7**. For $c = c(x)$ an arbitrary function, $f(u) = f_1 \left(u + \frac{g_2}{g_1}\right) + f_2 \left(u + \frac{g_2}{g_1}\right)^{-\frac{1}{3}}$ and $g(u) = \left(-\frac{4}{3}\,(g_1\,u + g_2)^{-1}\right)^{\frac{4}{3}}$, with arbitrary values of the constants $f_1, f_2, g_1, g_2$ such that $g_1 \neq 0$, Eq. (7) admits the generator $\mathbf{X}_1$ and the following:

$$\mathbf{X}_7 = e^{-\frac{4}{3}f_1\,t}\partial_t - \frac{f_1\,(g_1\,u + g_2)\,e^{-\frac{4}{3}f_1\,t}}{g_1}\partial_u. \tag{23a}$$

$$(x^*, t^*, u^*) = \Bigg(\frac{-3}{4f_1}\ln\left|e^{-\frac{4}{3}f_1\,t} - \frac{4}{3}f_1\epsilon\right|, x, \tag{23b}$$

$$\exp\left(f_1\left(\frac{4}{3}f_1\epsilon - e^{-\frac{4}{3}f_1\,t}\right)\epsilon\right)\left(u + \frac{g_2}{g_1}\right) - \frac{g_2}{g_1}\Bigg)$$

$$\textit{exponential dilation and shift}. \tag{23c}$$

**Case 8**. For $f(u) = f_1\,(u - g_2) + f_2(u - g_2)^{g_1+1}$ and $g(u) = g_3(g_2 - u)^{g_1}$, we consider arbitrary values of the constants $f_1, f_2, g_1, g_2$, and $g_3$. We distinguish the following subcases:

**8.1.** We consider $c(x) = \left(\frac{(2\,g_1+3)(c_1\,x+c_2)}{3\,g_1+4}\right)^{\frac{3\,g_1+4}{2\,g_1+3}}$ for arbitrary values of the constants $c_1, c_2$. If $f_2 = 0$ and $g_1 \neq -2, -\frac{4}{3}, -\frac{3}{2}, -1$, then Eq. (7) admits the generators $\mathbf{X}_1$, $\mathbf{X}_4$ and also

$$\mathbf{X}_{8a} = \frac{g_1\,(2\,g_1 + 3)\,(c_1\,x + c_2)}{c_1\,(g_1 + 2)}\partial_x + (u - g_2)\partial_u, \tag{24a}$$

$$(x^*, t^*, u^*) = \Bigg(\left(x + \frac{c_2}{c_1}\right)\exp\left(\frac{g_1(2\,g_1 + 3)}{g_1 + 2}\epsilon\right) - \frac{c_2}{c_1}, \tag{24b}$$

$$t, e^{\epsilon}(u - g_2) - g_2\Bigg) \qquad \textit{scaling and shift}.$$

and

$$\mathbf{X}_{8b} = \frac{2}{g_1\,c_1}(2\,g_1 + 3)(c_1\,x + c_2)\left(x + \frac{c_2}{c_1}\right)^{-\frac{g_1+1}{2\,g_1+3}}\partial_x \tag{25a}$$

$$+ \frac{2}{c_1}(g_1 + 1)(u - g_2)\left(x + \frac{c_2}{c_1}\right)^{-\frac{g_1+1}{2\,g_1+3}}\partial_u.$$

$$(x^*, t^*, u^*) = \left(\left(\left(x + \frac{c_2}{c_1}\right)^{\frac{g_1+1}{2\,g_1+3}} + 2\epsilon + \frac{2\epsilon}{g_1}\right)^{\frac{2\,g_1+3}{g_1+1}} - \frac{c_2}{c_1}, t, \right. \tag{25b}$$

$$\left. e^{\left(\frac{2}{c_1}(g_1+1)\,g1\,\epsilon\left(\left(x+\frac{c_2}{c_1}\right)^{\frac{g_1+1}{2\,g_1+3}}g_1 + 2\epsilon\,(g_1+1)\right)^{-1}\right)}(u - g_2) + g_2\right)$$

*exponential dilation and shift*.

**8.2.** We consider $c\,(x) = -\frac{f_1\,g_2\,x^2}{2\,g_3} + c_1\,x + c_2$ for arbitrary values of the constants $c_1, c_2$. If $f_2 = 0$ and $g_1 = -1$, then Eq. (7) admits the generators $\mathbf{X}_1$, $\mathbf{X}_4$ and $\mathbf{X}_{8c}$ for $K = \sqrt{g_3\,\left({c_1}^2 g_3 + 2\,c_2\,f_1\,g_2\right)}$ with

$$r(x) = \operatorname{arctanh}\left(\frac{c_1\,g_3 - f_1\,g_2\,x}{K}\right) \tag{26}$$

and

$$\mathbf{X}_{8c} = \frac{1}{2}\,r\,(x)\,\left(f_1\,g_2\,x^2 - 2\,g_3\,(c_1\,x + c_2)\right)\partial_x \tag{27a}$$

$$+ (g_2 - u)\,((f_1\,g_2\,x - c_1\,g_3)\,r\,(x) + K)\,\partial_u.$$

$$(x^*, t^*, u^*) = \frac{1}{f_1\,g_2}\Bigg(\left(-\tanh\left(e^{1/2\,K\epsilon}r(x)\right)K + c_1\,g_3\right), t, \tag{27b}$$

$$(u-g_2)\,\mathrm{e}^{-\left(-\tanh\left(e^{\frac{K\epsilon}{2}}r(x)\right)K\operatorname{arctanh}\left(\tanh\left(e^{\frac{K\epsilon}{2}}r(x)\right)\right)+K\right)\epsilon}+g_2\Bigg)$$

*oscillatory dilation and shift*.

**8.3.** If $f_2=0$ and $g_1=-2$, then Eq. (7) admits the generators $\mathbf{X}_1$, $\mathbf{X}_{4a}$ and $\mathbf{X}_{8d}$ and $\mathbf{X}_{8e}$ for any $c(x)$ that verifies

$$c''(x)=\frac{c'(x)^2\,g_3-4\,c(x)f_1}{2\,g_3\,c(x)} \tag{28}$$

with

$$\mathbf{X}_{8d}=\partial_x+\frac{(u-g_2)c'(x)}{2\,c(x)}\partial_u. \tag{29a}$$

$$(x^*,t^*,u^*)=\left(x+\epsilon,\,t,\,\exp\left(\frac{\epsilon\,c'(x+\epsilon)}{2\,c(x+\epsilon)}\right)(u-g_2)+g_2\right) \tag{29b}$$

*scaling and shift*.

and

$$\mathbf{X}_{8e}=x\,\partial_x+(u-g_2)\left(\frac{x\,c'(x)}{2\,c(x)}-1\right)\partial_u. \tag{30a}$$

$$(x^*,t^*,u^*)=\left(e^{\epsilon}x,\,t,\,\exp\left(\frac{\epsilon\,(x+\epsilon)\,c'(x+\epsilon)}{2\,c(x+\epsilon)}-\epsilon\right)(u-g_2)+g_2\right) \tag{30b}$$

*scaling and shift*.

**8.4.** We consider $c\,(x)=c_2\exp(c_1x)$ for arbitrary values of the constants $c_1, c_2$. If $f_2=0$ and $g_1=-\frac{3}{2}$, then Eq. (7) admits the generators $\mathbf{X}_1$, $\mathbf{X}_4$ and

$$\mathbf{X}_{8f}=\frac{3}{2\,c_1}\partial_x+(u-g_2)\partial_u. \tag{31a}$$

$$(x^*,t^*,u^*)=\left(x+\frac{3}{2\,c_1}\epsilon,\,t,\,e^{\epsilon}(u-g_2)+g_2\right)\quad \textit{scaling and shift}. \tag{31b}$$

**8.5.** If $f_2=0$ and $g_1=-\frac{4}{3}$ then Eq. (7) admits $\mathbf{X}_1$ and $\mathbf{X}_4$ as generators with $c(x)$ an arbitrary function.

**8.6.** If $f_2\neq 0$ then Eq. (7) admits $\mathbf{X}_1$ and $\mathbf{X}_4$ as generators with $c(x)$ an arbitrary function.

**Case 9**. For $f(u)=f_1\;(u-g_2)+f_2(u-g_2)^{g_1+1}$, $g(u)=g_3(g_2-u)^{g_1}$ we distinguish the following subcases:

**9.1.** For $c(x)=-\frac{(c_1-x)^2}{2(2+g_1)}$, with $g_1\neq -2,-\frac{4}{3},-1$, and $f_1\neq 0$, we obtain the generators $\mathbf{X}_1$, $\frac{1}{f_1}\mathbf{X}_4$ and the following:

$$\mathbf{X}_9=\frac{x-c_1}{\sqrt{-2(2+g_1)}}\partial_x. \tag{32a}$$

$$(x^*,t^*,u^*)=\left(\exp\left(\ln|x-c_1|+\frac{\epsilon}{\sqrt{-2(2+g_1)}}\right)+c_1,\,t,\,u\right) \tag{32b}$$

*exponential dilation and shift*.

**9.2.** If $f_1\neq 0$ and $g_1=-2,-\frac{4}{3},-1$, we obtain the generators $\mathbf{X}_1$, and $\mathbf{X}_4$ for arbitrary $c(x)$.

**9.3.** If $f_1=0$ we obtain the generators $\mathbf{X}_1$ and $\mathbf{X}_{4a}$ for arbitrary $c(x)$.

**Case 10**. For $f(u)=f_1\left(u+\frac{g_2}{g_3}\right)+f_2\left(u+\frac{g_2}{g_3}\right)^{-\frac{1}{3}}$, $g(u)=\left(-\frac{4}{3}\,(g_3\,u+g_2)^{-1}\right)^{\frac{4}{3}}$ and $c(x)=\dfrac{1}{4}\,(c_1\,x+c_2)^2$ with arbitrary values of the constants $f_1, f_2, g_3, g_2, c_1, c_2$ we distinguish the following subcases:

**10.1.** If $g_3=-1$ then we obtain for Eq. (7) the generators $\mathbf{X}_1$, $\mathbf{X}_4$ for $g_1=-\frac{4}{3}$, and

$$\mathbf{X}_{10a}=\frac{1}{2}\left(x+\frac{c_2}{c_1}\right)\partial_x. \tag{33a}$$

$$(x^*,t^*,u^*)=\left(\exp\left(\frac{\epsilon}{2}\right)\left(x+\frac{c_2}{c_1}\right)-\frac{c_2}{c_1},\,t,\,u\right)\quad \textit{scaling and shift}. \tag{33b}$$

**10.2.** If $g_3 \neq -1$ then we obtain the generators $\mathbf{X}_1$ and $\mathbf{X}_7$

**Case 11**. For $f(u) = f_2\, e^{f_1\, u}$, $g(u) = g_2\, e^{g_1\, u}$, $c(x) = c_2\, e^{c_1\, x}$ with arbitrary values of the constants $f_1, f_2, g_3, g_2, c_1, c_2$ we distinguish the following subcases:

**11.1.** If $f_1 \neq g_1$, then we obtain the generator $\mathbf{X}_1$ and the following

$$\mathbf{X}_{11a} = \frac{c_1 f_1\, t}{f_1 - g_1}\partial_t + \partial_x - \frac{c_1}{f_1 - g_1}\partial_u. \tag{34a}$$

$$(x^*, t^*, u^*) = \left(\exp\left(\frac{c_1 f_1}{f_1 - g_1}\epsilon\right) t, x + \epsilon, u - \frac{c_1}{f_1 - g_1}\epsilon\right) \quad \textit{scaling and shift}. \tag{34b}$$

**11.2.** If $f_1 = g_1$, then we obtain the generator $\mathbf{X}_1$ and the following

$$\mathbf{X}_{11b} = t\,\partial_t - \frac{1}{f_1}\partial_u \tag{35a}$$

$$(x^*, t^*, u^*) = \left(e^{\epsilon}\, t, x, u - \frac{1}{f_1}\epsilon\right) \quad \textit{scaling and shift}. \tag{35b}$$

## 3. Conservation laws and some exacts solutions

We will obtain conservation laws for the generalized Fisher equation (7) by applying the general multiplier method [25–28]. Conservation laws are also of basic importance in the study of evolution equations because they provide physical, conserved quantities for all solutions $u(x, t)$, and they can be used to check the accuracy of numerical solution methods [29,30].

A local conservation law for Eq. (7)is a continuity equation

$$D_t C^1 + D_x C^2 = 0, \tag{36}$$

that holds for the whole set of solutions $u(x, t)$, where the conserved density $C^1$ and the spatial flux $C^2$ are functions of $x, t$, $u$, and derivatives of $u$ [30]. Here $D_t$, $D_x$ denote total derivatives with respect to $t$ and $x$ respectively. The pair of expressions $(C^1, C^2)$ is called a conserved current.

Two local conservation laws are considered to be equivalent [28] if they differ by a trivial conservation law $C^1 = D_x\Theta$, $C^2 = -D_t\Theta$, where $C^1$ and $C^2$ are evaluated on the set of solutions of Eq. (7), and $\Theta$ is some function of $x, t, u$, and derivatives of $u$.

We begin by observing that Eq. (7) has a Cauchy–Kovalevskaya form. Consequently, the results in [25,27] show that all non-trivial conservation laws arise from multipliers. Specifically, when we move off of the set of solutions of Eq. (7), every non-trivial local conservation law (36) is equivalent to one that can be expressed in the characteristic form

$$D_t\tilde{C}^1 + D_x\tilde{C}^2 = \left(u_t - f(u) - \left(g(u)c(x)u_x\right)_x\right) Q, \tag{37}$$

where $Q(x, t, u, u_x, u_t, \ldots)$ is a multiplier, and where $(\tilde{C}^1, \tilde{C}^2)$ differs from $(C^1, C^2)$ by a trivial conserved current. On the set of solutions $u(x, t)$ of Eq. (7), the characteristic form (37) reduces to the conservation law (36).

In general, a function $Q(x, t, u, u_x, u_t, \ldots)$ is a multiplier if it is non-singular on the set of solutions $u(x, t)$ of Eq. (7), and if its product with Eq. (7) is a divergence expression with respect to $t$ and $x$.

The determining equation to obtain all multipliers is

$$\frac{\delta}{\delta u}\Big(\left(u_t - f(u) - \left(g(u)c(x)u_x\right)_x\right) Q\Big) = 0. \tag{38}$$

This equation must hold off of the set of solutions of Eq. (7). Once the multipliers are found, the corresponding non-trivial conservation laws are obtained either by using a homotopy formula [25–27] or by integrating the characteristic equation (37) [30].

In order to obtain local conservation laws of physical interest for nonlinear diffusion–reaction equations, we typically focus on low-order multipliers [28,31]. The general form of a low-order multiplier $Q$ in terms of $u$ and derivatives of $u$ is given by variables which can be differentiated to obtain a leading derivative of the equation. The leading derivatives of Eq. (7) consist of $u_t$ and $u_{xx}$. Clearly, $u_t$ can be obtained by differentiation of $u$, while $u_{xx}$ can be obtained by differentiation of $u$ and $u_{xx}$. We determine

$$Q(x, t, u, u_x), \tag{39}$$

as the general form of a low-order multiplier for the diffusion–reaction equation (7). The determining Eq. (38) splits with respect to the variables $u_t, u_{tx}, u_{xx}$.

This yields a linear determining system for $Q(x, t, u, u_x)$ which can be solved by the same algorithmic method used to solve the determining equation for infinitesimal symmetries. By using Maple we solve this determining system subject to the classification conditions $f' \neq 0, g' \neq 0, c' \neq 0$.

We obtain the following results:

**Case 1**: For $f(u)$ nonlinear and $g = k_2 f' + k_1$ for $k_1, k_2$ arbitrary constants we get the following multiplier:

$$Q_1(t,x) = F'(x) e^{\frac{k_1 t}{k_2}} \tag{40}$$

with $F(x)$ satisfying

$$k_2 F''(x) c(x) + F(x) + k_3 = 0, \tag{41}$$

for $k_3$ an arbitrary constant. We obtain the corresponding conserved density and flux:

$$C^1 = F'(x) e^{\frac{k_1 t}{k_2}} u, \tag{42}$$

$$C^2 = \Big( (k_1 u + k_2 f(u)) F''(x) - (k_1 + f'(u) k_2) F'(x) u_x \Big) e^{\frac{k_1 t}{k_2}} c(x). \tag{43}$$

**Case 2**: For $f(u) = k\,u$ with $k$ a constant and $g(u)$ arbitrary, we get the following multiplier:

$$Q_2(t,x) = F(x) e^{-kt}, \tag{44}$$

with

$$F(x) = k_1 + k_2 \int \frac{1}{c(x)} \, \mathrm{d}x, \qquad k_1, k_2 \in \mathbb{R}. \tag{45}$$

We obtain the corresponding conserved density and flux:

$$C^1 = F(x) e^{-kt} u, \tag{46}$$

$$C^2 = e^{-kt} c(x) \left( -g(u) F(x) u_2 + F'(x) \int g(u) \, \mathrm{d}u \right). \tag{47}$$

For both cases, at $t = 0$, (42) and (46) are simply the mass density weighted by a factor that compensates for the non-homogeneity $c(x)$ in (7).

## 4. Some reductions and exact solutions

In this section we will focus on case 4 from Section 2, as function $c = c(x)$ is arbitrary, and functions $f(u) = f_1\,(u - g_2) + f_2\,(g_2 - u)^{g_1+1}$ and $g(u) = g_3 (g_2 - u)^{g_1}$ have a biological interest in terms of modeling, respectively, cancer cell proliferation as a Verhulst's law of growth [1,3,32], and the diffusion term as a typical glioma invasion [1,32–34]. By using the generator $\mathbf{X}_4$, we obtain the similarity variable and similarity solution

$$z = x, \quad u = e^{f_1 t} h(z) + g_2, \tag{48}$$

and the $\mathrm{ODE}_4$

$$h_{zz} + \frac{g_1 h_z^2}{h} + \frac{c_z h_z}{c} - \frac{f_2 h}{c\, g_3} = 0. \tag{49}$$

If we set $h(z) = -\sqrt{v(z)}$, we obtain that (49) is equivalent to

$$v_{zz} - \frac{v_z^2}{2\,v}(g_1 - 1) + \frac{c_z v_z}{c} - \frac{2 f_2 v}{g_3 c}. \tag{50}$$

We set $g_1 = 1$ as it provides us a linear density diffusion term, which is a Malthusian rate of growth [35]. Then, Eq. (50) is transformed into

$$v_{zz} + \frac{c_z v_z}{c} - \frac{2 f_2 v}{g_3 c} = 0. \tag{51}$$

It can be easily proved that a first integral of Eq. (51) is the Riccati equation

$$w_z + w^2 + \frac{c_z}{c} w - 2 \frac{f_2}{g_3 c} = 0, \tag{52}$$

with $w = w(z)$ and the change $v(z) = \exp(\alpha(z))$ for $\alpha'(z) = w(z)$.

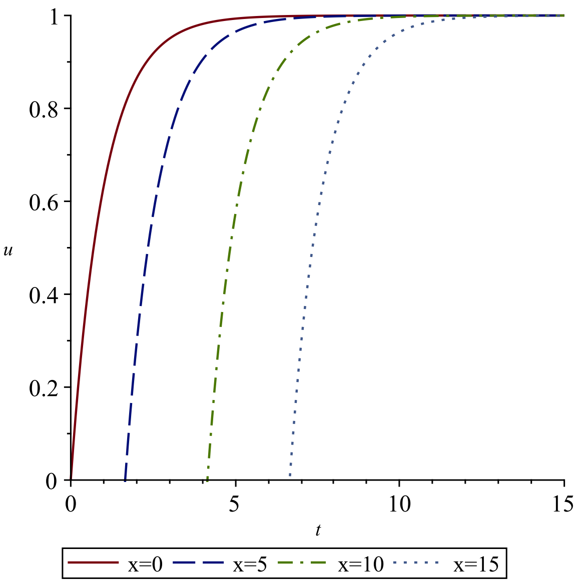


**Fig. 1.** Population density solutions (56) are shown for, $g_2 = 1, K_1 = 1, K_2 = -1$ over different times $t$ and displacement $x$ given. The asymptotic behavior can be observed.

Besides, we want $c = c(x)$ to have an asymptotic behavior (for large $x$) related to $\tanh(x)$, which has biological interest as it models single and multiple sharp transition regions [11]. With $g_1 = 1$, we search a solution of (52) such as

$$w(z) = \frac{1}{K_1} \tanh\left(\frac{z + K_2}{K_1}\right), \quad K_1 \neq 0, \tag{53}$$

so that $c = c(x)$ becomes

$$c(x) = \frac{2 f_2 {K_1}^2}{g_3} + \frac{K_3 \sqrt{1 - \left(\tanh\left(\frac{x+K_2}{K_1}\right)\right)^2}}{\tanh\left(\frac{x+K_2}{K_1}\right)} - K_4, \quad K_1 \neq 0, x \geq 0. \tag{54}$$

where $K_4 = \operatorname{arctanh}\left(\frac{K_3 g_3}{\sqrt{4{f_2}^2{K_1}^4 + {K_3}^2{g_3}^2}}\right) K_1 - K_2$ as the diffusion term cannot be negative [1]. In this case, the asymptotic behavior of $c = c(x)$ is the following:

$$\lim_{x\to\infty} c(x) = \frac{2 f_2 {K_1}^2}{g_3} - K_4. \tag{55}$$

Therefore we have provided a one-parameter family of exact solutions of Eq. (7)

$$u(x, t) = g_2 - \frac{e^{f_1 t}}{\sqrt[4]{1 - \tanh\left(\frac{x+K_2}{K_1}\right)^2}}, \tag{56}$$

for each $K_1 \neq 0$ and $c = c(x)$ as in (54).

The carrying capacity in this equation can be seen as $g_2$, and we obtained for the solution (56) that, for $f_1 < 0$,

$$\lim_{t\to\infty} u = g_2. \tag{57}$$

This implies that in any region of the space the solutions assume the value of the limit concentration of cells. This is shown in Fig. 1. Correspondingly, the density-diffusion and growth function asymptotically disappear, this is

$$\lim_{t\to\infty} f = 0, \quad \lim_{t\to\infty} g = 0. \tag{58}$$

## 5. Conclusions

A Fisher equation (7) with a density–space dependent reaction–diffusion term was presented, which can be considered as an essential part of cancer modeling and cell dynamics. By applying the classical Lie group method, we obtained a symmetry classification for Eq. (7).

We have also constructed some nontrivial conservation laws for this generalized Fisher equation, by considering the Anco and Bluman multipliers method. Finally, we have obtained a reduction of order of the $ODE_4$ derived from (7). In particular, we have found a one-parameter family of solutions with biological meaning.

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