Implication operators generating pairs of weak negations and their algebraic structure★

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Abstract

Negations operators have been developed and applied in many fields such as image processing, decision making, mathematical morphology, fuzzy logic, etc. One of the most effective non-monotonic operators are weak negations.

This paper studies the algebraic structure and the characterization of the adjoint triples and Galois implication pairs which provides a fixed pair of weak negations. The obtained results allow the user to select the best conjunctor and implications associated with the most suitable negation to be used in the computations of the problem to be solved.

Keywords: Fuzzy sets, adjoint triples, negation operators, pair of weak negations.

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1. Introduction

One important goal for obtaining useful information from (big) data sets is to select the most suitable operators to be considered in the computations. The more versatile and tractable the operators are, the better can be adapted to the data and so, more knowledge can be extracted. For example, in image processing, the noise is very notable and different general operators such as, pre-aggregations, ordered directionally monotone functions, etc., have been introduced in order to obtain better results [47]. Adjoint triples [15, 17, 18] are other general operators which have been used to introduce flexible tools for defining fuzzy versatile frameworks in logic programming [41, 42], formal concept analysis [40],

★Partially supported by the State Research Agency (AEI) and the European Regional Development Fund (ERDF) project TIN2016-76653-P.

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rough set theory [21], fuzzy relation equations [27] and mathematical morphology [2, 37]. These operators are based on the notion of Galois connection, which have attracted the interest of many authors [3, 6, 7, 8, 22, 33, 35, 45].

Non-monotonicity operators are also fundamental in image processing and other important applications [11, 13, 36, 38, 43] and they have been developed and adapted to the current challenges [4, 9, 12, 19, 24, 43, 44]. One of the most useful negation operators are weak negations [29, 30, 32, 49]. These operators were generalized later by Georgescu and Popescu in [34], allowing the consideration of a couple of negations defined on different domains. Although these negations are not residuated negations [10, 46], in general, recently Cornejo et al. [19] have proven that (pairs of) weak negations are a particular case of adjoint negations, that is, these negations can be obtained from the implications of adjoint triples or pairs.

Since different adjoint triples provide the same (pair of) weak negations, as we will show later, the study of the relationship among these triples is interesting in order to discover hierarchies among them. Hence, the study of the whole set of implications and conjunctors that provide (pairs of) weak negations will be the main goal of this paper. Specifically, given a pair of weak negations, this paper will prove that the set of Galois implications pairs associated with these negations forms a complete lattice, with a minimum and a maximum element, and the corresponding set of adjoint triples forms a join-semilattice with a maximum element.

As a consequence, when a specific negation (or pair of negations) is required in the applications, the results given in this paper are helpful to select the most suitable implications and conjunctors associated with this negation. For example, if the user needs a conjunctor providing high values, (s)he will consider the maximum, if (s)he demands a conservative conjunctor (s)he can use one of the minimum conjunctors of the semilattice. On the contrary, if (s)he does not need a conjunctor but only implications, (s)he can choose between the maximum and the minimum Galois implication pairs, or any other in the middle, depending on the requirements and the problem to be solved.

The paper is organized as follows: Section 2 recalls the basic definitions and results used in the rest of the paper. Section 3 studies the algebraic structure of Galois implications pairs generating a given pair of weak negations. Moreover, the definition of these implications are characterized from a family of antitone Galois connections. This study is extended to adjoint triples in Section 4, proving that the algebraic structure is a join-semilatice and characterizing the definition of the operators through a family of operators where only the supremum of non-empty sets is required. This property is very important since the number of possible conjunctors to be considered increases greatly. Finally, the paper finishes with some conclusions and prospects for future work.

2. Preliminaries

We firstly provide some necessary definitions and properties in order to make the paper self-contained.

2.1. Adjoint triples and Galois implications pairs

Adjoint triples generalize triangular norms and their residuated implications, since they preserve their main properties and increase the flexibility of the operators usually used for computation in different frameworks [15, 17]. Taking into account that the conjunctor of an adjoint triple does not need to be commutative, we obtain an interesting generalization of the well-known adjoint property between a t-norm and its residuated implication, which is given in the following definition.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ mappings. We say that $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if the following double equivalence is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called *adjoint property*.

Interesting properties of adjoint triples are shown in the next proposition, which have been extracted from [18]. Since the pair of weak negations are defined on lattices, the following properties are given when the carriers are lattices.

Proposition 2. *Given the complete lattices* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, *an arbitrary operator* $\&\colon L_1 \times L_2 \to L_3$ *and the mappings* $\swarrow\colon L_3 \times L_2 \to L_1$, $\nwarrow\colon L_3 \times L_1 \to L_2$, *defined as* $z \swarrow y = \sup\{x \in L_1 \mid x \,\&\, y \preceq_3 z\}$ *and* $z \nwarrow x = \sup\{y \in L_2 \mid x \,\&\, y \preceq_3 z\}$, *respectively, for all* $x \in L_1$, $y \in L_2$ *and* $z \in L_3$, *the next statements are equivalent:*

1. $(\&, \swarrow, \nwarrow)$ *is an adjoint triple with respect to* L_1, L_2, L_3.

2. $\left(\bigvee_{x_i\in X} x_i\right) \& y = \bigvee_{x_i\in X}(x_i \& y)$, *for any* $X \subseteq L_1$ *and* $y \in L_2$.

$x \& \left(\bigvee_{y_i\in Y} y_i\right) = \bigvee_{y_i\in Y}(x \& y_i)$, *for any* $Y \subseteq L_2$ *and* $x \in L_1$.

3. $z \swarrow y = \max\{x \in L_1 \mid x \& y \preceq_3 z\}$ *and* $z \nwarrow x = \max\{y \in L_2 \mid x \& y \preceq_3 z\}$ *for all* $x \in L_1$, $y \in L_2$ *and* $z \in L_3$, *being* $\&$ *an order-preserving operator in both arguments.*

Proposition 3. *Given three complete lattices* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, *the arbitrary operators* $\swarrow\colon L_3 \times L_2 \to L_1$, $\nwarrow\colon L_3 \times L_1 \to L_2$ *and the mapping* $\&\colon L_1 \times L_2 \to L_3$ *defined as* $x \& y = \inf\{z \in L_3 \mid x \preceq_1 z \swarrow y\} = \inf\{z \in L_3 \mid y \preceq_2 z \nwarrow x\}$, *for all* $x \in L_1$ *and* $y \in L_2$, *the next statements are equivalent:*

1. $(\&, \swarrow, \nwarrow)$ *is an adjoint triple with respect to* L_1, L_2, L_3.
2. $\left(\bigwedge_{z_i\in Z} z_i\right) \swarrow y = \bigwedge_{z_i\in Z}(z_i \swarrow y)$, *for all* $Z \subseteq P_3$ *and* $y \in P_2$.

$\left(\bigwedge_{z_i\in Z} z_i\right) \nwarrow x = \bigwedge_{z_i\in Z}(z_i \nwarrow x)$, *for all* $Z \subseteq P_3$ *and* $x \in P_1$.

3. $x \& y = \min\{z \in L_3 \mid x \preceq_1 z \swarrow y\} = \min\{z \in L_3 \mid y \preceq_2 z \nwarrow x\}$, *for all* $x \in L_1$ *and* $y \in L_2$, *being* $\swarrow$ *and* $\nwarrow$ *order-preserving operators in the first argument.*

Galois implications pairs will also play an important role in this paper. In the following, we will introduce the notion of Galois implications pairs which are the basic operators used in frameworks as Formal Concept Analysis within multi-adjoint paradigm [16, 40].

Definition 4. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ mappings. We say that $(\swarrow, \nwarrow)$ is a *Galois implications pair* with respect to P_1, P_2, P_3 if the next equivalence is verified:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{2}$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$

Now, we will present some properties deduced from Equivalence (2).

Proposition 5. *Let* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *be two complete lattices,* $(P_3, \leq_3)$ *a poset, and* $\swarrow\colon P_3 \times L_2 \to L_1$, $\nwarrow\colon P_3 \times L_1 \to L_2$ *two mappings. The following statements are equivalent:*

1. $(\swarrow, \nwarrow)$ *is a Galois implications pair with respect to* L_1, L_2, P_3.
2. $z \swarrow \left(\bigvee_{y_i \in Y} y_i\right) = \bigwedge_{y_i \in Y} (z \swarrow y_i)$*, for all* $Y \subseteq L_2$ *and* $z \in P_3$.
3. $z \nwarrow \left(\bigvee_{x_i \in X} x_i\right) = \bigwedge_{x_i \in X} (z \nwarrow x_i)$*, for any* $X \subseteq P_1$ *and* $z \in P_3$.

A wide theoretical study including more illustrative examples related to adjoint triples and Galois implications pairs can be found in [15, 18].

Next, we recall the formal definition of antitone Galois connections and some properties which will be helpful throughout the paper.

Definition 6. Given the posets $(P, \leq_P)$ and $(Q, \leq_Q)$, the pair $(^\uparrow, ^\downarrow)$ of mappings $^\downarrow\colon P \to Q$, $^\uparrow\colon Q \to P$ is an *antitone Galois connection* between P and Q if the following equivalence holds:

$$p \leq_P q^\uparrow \quad \text{if and only if} \quad q \leq_Q p^\downarrow$$

for all $p \in P$ and $q \in Q$.

The following properties are straightfordwarly obtained from the definition of antitone Galois connection.

Proposition 7. *Let* $^\downarrow\colon P \to Q$ *and* $^\uparrow\colon Q \to P$ *be two mappings between the posets* $(P, \leq_P)$ *and* $(Q, \leq_Q)$*. If* $(^\uparrow, ^\downarrow)$ *is an antitone Galois connection then the following properties are satisfied:*

1. $^\uparrow$ *and* $^\downarrow$ *are order-reversing;*
2. $p \leq_P p^{\downarrow\uparrow}$ *and* $q \leq_Q q^{\uparrow\downarrow}$*, for all* $p \in P$, $q \in Q$.
3. $\bot_P{}^\downarrow = \top_Q$ *and* $\bot_Q{}^\uparrow = \top_P$ *when* $(P, \leq_P, \bot_P, \top_P)$ *and* $(Q, \leq_Q, \bot_Q, \top_Q)$ *are bounded posets.*
4. *When the supremum and the infimum exist, for all* $X \subseteq P$ *and* $Y \subseteq Q$*:*

$$\left(\bigvee_{p \in X} p\right)^\downarrow = \bigwedge_{p \in X} p^\downarrow \qquad \textit{and} \qquad \left(\bigvee_{q \in Y} q\right)^\uparrow = \bigwedge_{q \in Y} q^\uparrow$$

It is important to mention that, when we consider complete lattices instead of posets, the definition of antitone Galois connection is equivalent to items (1) and (2) of Proposition 7. Indeed, item (4) of Proposition 7 is also equivalent to the definition of antitone Galois connection. More properties and examples associated with Galois connections can be found in [23].

2.2. Adjoint negations and pairs of weak negations

Adjoint negations were introduced in [19] as a generalization of residuated negations [10, 31, 46]. These negation operators are defined on two different posets since they are built from the implications of an adjoint triple with respect to three different posets. In the following, we will include the definition of adjoint negations and some interesting properties satisfied by them.

Definition 8. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets, $(P_3, \leq_3, \perp_3)$ a lower bounded poset and $(\&, \swarrow, \nwarrow)$ an adjoint triple with respect to P_1, P_2 and P_3. The mappings $n_n \colon P_1 \to P_2$ and $n_s \colon P_2 \to P_1$ defined, for all $x \in P_1$, $y \in P_2$ as

$$n_n(x) = \perp_3 \nwarrow x \qquad n_s(y) = \perp_3 \swarrow y$$

are called *adjoint negations* with respect to P_1 and P_2.

The operators n_s and n_n satisfying that $x = n_s(n_n(x))$ and $y = n_n(n_s(y))$, for all $x \in P_1$ and $y \in P_2$, are called *strong adjoint negations*.

Corollary 9 is straightforwardly obtained taking into account that the pair formed by adjoint negations (n_s, n_n) is an antitone Galois connection [25, 28, 48].

Corollary 9. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be posets, $(P_3, \leq_3, \perp_3)$ a lower bounded poset and n_s, n_n adjoint negations. The following statements hold:*

1. *If $(P_1, \leq_1, \perp_1, \top_1)$ and $(P_2, \leq_2, \perp_2, \top_2)$ are bounded partially ordered sets, then $n_s(\perp_2) = \top_1$ and $n_n(\perp_1) = \top_2$;*
2. *n_n and n_s are order-reversing;*
3. *$x \leq_1 n_s n_n(x)$ and $y \leq_2 n_n n_s(y)$;*
4. *$n_s n_n n_s = n_s \quad$ and $\quad n_n n_s n_n = n_n$;*
5. *$n_s n_n \quad$ and $\quad n_n n_s$ are closure operators;*
6. *$x \leq_1 n_s(y) \quad$ iff $\quad y \leq_2 n_n(x)$, for all $x \in P_1$, $y \in P_2$;*
7. *When the supremum and the infimum exist, for any $X \subseteq P_1$, $Y \subseteq P_2$,*
   (a) $n_s\left(\bigvee_{y \in Y} y\right) = \bigwedge_{y \in Y} n_s(y)$;
   (b) $n_n\left(\bigvee_{x \in X} x\right) = \bigwedge_{x \in X} n_n(x)$.

On the other hand, we will show the notion of pair of weak negations given by Georgescu and Popescu in [34] and the relationship to adjoint negations introduced in [19].

Definition 10. Let $(P, \leq, \bot, \top)$ be a bounded partially ordered set and two mappings $n_1 : P \to P$, $n_2 : P \to P$, the pair (n_1, n_2) is said to be a *pair of weak negations* on P, if the following conditions hold, for all $x \in P$:

1. $n_1(\top) = n_2(\top) = \bot$;
2. n_1 and n_2 are order-reversing;
3. $x \leq n_2 n_1(x)$ and $x \leq n_1 n_2(x)$.

The following result shows that every pair of weak negations can be derived from the implications of an adjoint triple [19].

Theorem 11. *Given a pair of weak negations (n_1, n_2) on $(P, \leq, \bot, \top)$, there exists an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to P satisfying that $n_1 = n_s$ and $n_2 = n_n$.*

The following example shows that different adjoint triples can generate the same pair of weak negations. Hence, the unicity of adjoint triples generating adjoint negations, which coincide with a given pair of weak negations, is not guaranteed.

Example 12. Consider the complete lattice $(L, \preceq)$ and the pair of weak negations (n_1, n_2) defined on $(L, \preceq)$, both given by Figure 1. Given the adjoint triples $(\&, \swarrow, \nwarrow)$ and $(\&_*, \swarrow^*, \nwarrow_*)$ defined on $(L, \preceq)$, as it is shown in Table 1, it is easy to see that these adjoint triples are different and they satisfy Theorem 11, for all $x, y \in L$, that is:

$$n_s(y) = \bot \swarrow y = n_1(y) \qquad n_n(x) = \bot \nwarrow x = n_2(x)$$
$$n_{s_*}(y) = \bot \swarrow^* y = n_1(y) \qquad n_{n_*}(x) = \bot \nwarrow_* x = n_2(x)$$

Hence, we can conclude that the adjoint implications giving rise to adjoint negations, which coincide with a given pair of weak negations, are not unique. □

Figure 1: Pair of weak negations (n_1, n_2) and lattice $(L, \preceq)$ of Example 12.

	$\bot$	a	b	c	$\top$
n_1	$\top$	a	a	a	$\bot$
n_2	$\top$	c	$\bot$	$\bot$	$\bot$

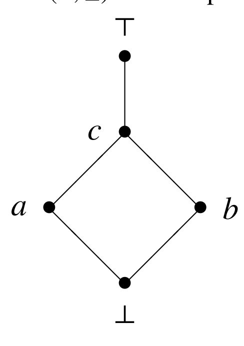

Table 1: Definition of $(\&, \swarrow, \nwarrow)$ and $(\&_*, \swarrow^*, \nwarrow_*)$ in Example 12.

$\&$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	b
b	$\bot$	b	b	b	b
c	$\bot$	b	b	b	b
$\top$	$\bot$	c	c	c	c

$\swarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	a	a	a	$\bot$
a	$\top$	a	a	a	$\bot$
b	$\top$	c	c	c	c
c	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	c	$\bot$	$\bot$	$\bot$
a	$\top$	c	$\bot$	$\bot$	$\bot$
b	$\top$	$\top$	$\top$	$\top$	$\bot$
c	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\&_*$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	b
b	$\bot$	c	c	c	c
c	$\bot$	c	c	c	c
$\top$	$\bot$	c	c	c	c

$\swarrow^*$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	a	a	a	$\bot$
a	$\top$	a	a	a	$\bot$
b	$\top$	a	a	a	a
c	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow_*$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	c	$\bot$	$\bot$	$\bot$
a	$\top$	c	$\bot$	$\bot$	$\bot$
b	$\top$	$\top$	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

It is important to emphasize that, according to Definition 8, it seems that the implications included in an adjoint triple to define adjoint negations are needed, which is asserted in the following result.

Theorem 13. *Given a pair of weak negations (n_1, n_2) on $(P, \leq, \bot, \top)$, there exists a Galois implications pair $(\swarrow, \nwarrow)$ with respect to P satisfying that $n_1 = n_s$ and $n_2 = n_n$.*

Proof. First of all, we will define the operators $\swarrow$ and $\nwarrow$, for each $x, y \in P$, as follows:

$$z \swarrow y = \begin{cases} n_1(y) & \text{if} \quad z \neq \top \\ \top & \text{if} \quad z = \top \end{cases} \qquad z \nwarrow x = \begin{cases} n_2(x) & \text{if} \quad z \neq \top \\ \top & \text{if} \quad z = \top \end{cases}$$

We will see that $(\swarrow, \nwarrow)$ is a Galois implications pair proving that Equivalence (2) holds. We suppose that $z \neq \top$, then we can ensure that the inequality $x \leq z \swarrow y$ is equivalent to $y \leq z \nwarrow x$, since by conditions (2) and (3) of Definition 10, the inequality $x \leq n_1(y)$ is equivalent to $y \leq n_2(x)$. Otherwise, the equivalence $x \leq z \swarrow y$ if and only if $y \leq z \nwarrow x$ is trivially obtained. Therefore, we conclude that $\swarrow$ and $\nwarrow$ form a Galois implications pair.

Finally, from the definition of the operators $\swarrow$ and $\nwarrow$, we obtain that the following equalities hold:

$$\begin{aligned} n_s(y) &= \bot \swarrow y = n_1(y) \\ n_n(x) &= \bot \nwarrow x = n_2(x) \end{aligned}$$

$\square$

It is convenient to mention that the proof of the previous result is obtained without using the properties of the adjoint conjunctor unlike what happens in the proof of Theorem 11 introduced in [19]. Once again, we cannot guarantee the unicity of the Galois implications pair which allow us to ensure that each pair of weak negation is actually an adjoint negation.

As we just show, we can define adjoint negations from the implications of either a Galois implications pair or an adjoint triple. Now, we are interested in: (1) characterizing Galois implications pairs, without associated conjunctor, whose adjoint negations coincide with a given pair of weak negations; and (2) studying the algebraic structure formed by these Galois implications pairs. A similar study will be carry out with respect to adjoint triples generating a given pair of weak negations.

3. Galois implications pairs generating a given pair of weak negations

This section is devoted to introduce mechanisms to define Galois implications pairs whose adjoint negations coincide with a given pair of weak negations. In addition, a hierarchy among these Galois implications pairs will be established and the obtained algebraic structure will be studied.

Given a pair of weak negations and an arbitrary Galois implications pair, the following result proposes a first procedure to define Galois implications pairs whose adjoint negations coincide with the given pair of weak negations.

Proposition 14. *Given a pair of weak negations* (n_1, n_2) *on a complete lattice* $(L, \preceq)$ *and a Galois implications pair* $(\swarrow, \nwarrow)$ *with respect to* $(L, \preceq)$*, the mappings defined, for all* $x, y, z \in L$*, as:*

$$z \swarrow^{n_1} y = \begin{cases} n_1(y) & \text{if} \quad z = \bot \\ z \swarrow y & \text{if} \quad z \neq \bot \end{cases} \qquad z \nwarrow_{n_2} x = \begin{cases} n_2(x) & \text{if} \quad z = \bot \\ z \nwarrow x & \text{if} \quad z \neq \bot \end{cases}$$

form a Galois implications pair with respect to $(L, \preceq)$*. Moreover, the equalities* $n_1 = n_{s_{n_1}}$ *and* $n_2 = n_{n_{n_2}}$ *hold, being* $n_{s_{n_1}}$ *and* $n_{n_{n_2}}$ *the adjoint negations associated with the implications* $\swarrow^{n_1}$ *and* $\nwarrow_{n_2}$*, respectively.*

Proof. First of all, we will prove that $(\swarrow^{n_1}, \nwarrow_{n_2})$ is a Galois implications pair with respect to $(L, \preceq)$. According to Proposition 5, it is equivalent to demonstrate that, for all $Y \subseteq L$ and $z \in L$, the following equality holds:

$$z \swarrow^{n_1} \left(\bigvee_{y \in Y} y \right) = \bigwedge_{y \in Y} (z \swarrow^{n_1} y)$$

We will distinguish two cases:

1. Case $z \neq \bot$: Taking into account that $(\swarrow, \nwarrow)$ is a Galois implications pair with respect to $(L, \preceq)$, the next chain of equalities is obtained from Proposition 5.

$$z \swarrow^{n_1} \left(\bigvee_{y \in Y} y \right) = z \swarrow \left(\bigvee_{y \in Y} y \right) = \bigwedge_{y \in Y} (z \swarrow y) = \bigwedge_{y \in Y} (z \swarrow^{n_1} y)$$

2. Case $z = \bot$: Applying the definition of the implication $\swarrow^{n_1}$, we have

$$\bot \swarrow^{n_1} \left(\bigvee_{y \in Y} y \right) = n_1 \left(\bigvee_{y \in Y} y \right) \overset{(*)}{=} \bigwedge_{y \in Y} n_1(y) = \bigwedge_{y \in Y} (\bot \swarrow^{n_1} y)$$

   The equality $(*)$ is straightforwardly obtained by Theorem 11 and Corollary 9.

As a consequence, we can ensure that $(\swarrow^{n_1}, \nwarrow_{n_2})$ is a Galois implications pair with respect to $(L, \preceq)$. Moreover, taking into account the definitions of adjoint negations and the operators $\swarrow^{n_1}, \nwarrow_{n_2}$, we can conclude that:

$$n_{s_{n_1}}(y) = \bot \swarrow^{n_1} y = n_1(y)$$

$$n_{n_{n_2}}(x) = \bot \nwarrow_{n_2} x = n_2(x)$$

for all $x, y \in L$. $\square$

As it happened with respect to adjoint triples, we can find different Galois implications pair whose adjoint negations also coincide with the given pair of weak negations, for example, exchanging the Galois implications pair in the definition of the operators $\swarrow^{n_1}, \nwarrow_{n_2}$ defined in Proposition 14.

The next result shows what conditions should be satisfied in order to guarantee that the Galois implications pairs considered in Proposition 14, $(\swarrow, \nwarrow)$ and $(\swarrow^{n_1}, \nwarrow_{n_2})$, coincide.

Proposition 15. *Let (n_1, n_2) be a pair of weak negations defined on a complete lattice, $(\swarrow, \nwarrow)$ a Galois implications pair with respect to $(L, \preceq)$ and $(\swarrow^{n_1}, \nwarrow_{n_2})$ the Galois implications pair defined as in Proposition 14. If both Galois implications pairs $(\swarrow^{n_1}, \nwarrow_{n_2})$, $(\swarrow, \nwarrow)$ have adjoint conjunctors and, for some $Z \subseteq L$, the equality $\bigwedge_{z \in Z} z = \bot$ holds, being $z \neq \bot$ for all $z \in Z$, then $\swarrow^{n_1} = \swarrow$ and $\nwarrow_{n_2} = \nwarrow$.*

Proof. Given $Z \subseteq L$, such that $\bigwedge_{z\in Z} z = \bot$ and $z \neq \bot$, for all $z \in Z$, we obtain the following chain of equalities:

$$\begin{aligned} n_1(y) = \bot \swarrow^{n_1} y &= \left(\bigwedge_{z\in Z} z\right) \swarrow^{n_1} y \\ &\overset{(1)}{=} \bigwedge_{z\in Z} (z \swarrow^{n_1} y) \\ &\overset{(2)}{=} \bigwedge_{z\in Z} (z \swarrow y) \\ &\overset{(3)}{=} \left(\bigwedge_{z\in Z} z\right) \swarrow y \\ &= \bot \swarrow y \end{aligned}$$

where (1) and (3) is obtained by Proposition 3 and (2) holds because $z \neq \bot$ and then $z \swarrow^{n_1} y = z \swarrow y$. Consequently, we obtain that $\swarrow^{n_1} = \swarrow$. The other equality follows similarly. □

The next example evinces that Proposition 15 is not true when we consider a complete lattice $(L, \preceq)$ which does not satisfy the existence of a set $Z \subseteq L$ such that $\bigwedge_{z\in Z} z = \bot$ and $z \neq \bot$, for all $z \in Z$.

Example 16. Let (n_1, n_2) be the pair of weak negations defined on the complete lattice $(L = \{\bot, a, b, \top\}, \preceq)$, with $\bot \preceq a \preceq b \preceq \top$, as Table 2 shows. Following

Table 2: Pair of weak negations (n_1, n_2) of Example 16.

	$\bot$	a	b	$\top$
n_1	$\top$	a	a	$\bot$
n_2	$\top$	b	$\bot$	$\bot$

the procedure given in Proposition 14, from this pair of weak negations and the Galois implications pair $(\swarrow, \nwarrow)$ defined as $z \swarrow y = z \nwarrow y = \top$, for all $y, z \in L$, we obtain the Galois implications pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ depicted in Table 3. By using Proposition 3 and making simple computations, it is easy to see that $(\swarrow, \nwarrow)$ and $(\swarrow^{n_1}, \nwarrow_{n_2})$ have adjoint conjunctors, which are displayed in Table 4.

Clearly, the adjoint triples $(\&, \swarrow, \nwarrow)$ and $(\&_{n_1 n_2}, \swarrow^{n_1}, \nwarrow_{n_2})$ are different and therefore, we can conclude that Proposition 15 is not true when we consider a

Table 3: Galois implications pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ of Example 16.

$\swarrow^{n_1}$	$\perp$	a	b	$\top$
$\perp$	$\top$	a	a	$\perp$
a	$\top$	$\top$	$\top$	$\top$
b	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow_{n_2}$	$\perp$	a	b	$\top$
$\perp$	$\top$	b	$\perp$	$\perp$
a	$\top$	$\top$	$\top$	$\top$
b	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Table 4: Adjoint conjunctors $\&$ and $\&_{n_1 n_2}$ of Example 16.

$$x \,\&\, y = \perp, \quad \text{for all } x, y \in L$$

$\&_{n_1 n_2}$	$\perp$	a	b	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	a
b	$\perp$	a	a	a
$\top$	$\perp$	a	a	a

complete lattice $(L, \preceq)$ which does not satisfy the existence of a set $Z \subseteq L$ such that $\bigwedge_{z \in Z} z = \perp$ and $z \neq \perp$, for all $z \in Z$. This fact is due to the condition required in Proposition 15 is not verified by $(L = \{\perp, a, b, \top\}, \preceq)$. Although this condition can seem restrictive, it is important to mention that it is satisfied by a large number of lattices such as the diamond lattice M_2 and the non-distributive lattices M_3, N_5, among others. □

The following result shows that Galois implications pairs whose adjoint negations coincide with a given pair of weak negations can also be defined from more general operators than Galois implications.

Proposition 17. *Given a pair of weak negations (n_1, n_2) on a complete lattice $(L, \preceq)$, the pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ defined as:*

$$z \swarrow^{n_1} y = \begin{cases} n_1(y) & \text{if} \quad z = \perp \\ f_z(y) & \text{if} \quad z \neq \perp \end{cases} \qquad z \nwarrow_{n_2} x = \begin{cases} n_2(x) & \text{if} \quad z = \perp \\ g_z(x) & \text{if} \quad z \neq \perp \end{cases}$$

for all $x, y, z \in L$, is a Galois implications pair with respect to $(L, \preceq)$ verifying that $n_1 = n_{s_{n_1}}$ and $n_2 = n_{n_{n_2}}$, if and only if the family of mappings $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z \in L \setminus \{\perp\}}$ are antitone Galois connections.

Proof. Assuming that $(\swarrow^{n_1}, \nwarrow_{n_2})$ is a Galois implications pair, we will prove that $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z \in L \setminus \{\perp\}}$ is a family of antitone Galois connections. In order to achieve this goal, by Proposition 7, we need to prove that:

$$f_z\left(\bigvee_{y\in Y} y\right) = \bigwedge_{y\in Y} f_z(y) \qquad \text{and} \qquad g_z\left(\bigvee_{x\in X} x\right) = \bigwedge_{x\in X} g_z(x)$$

for all $X, Y \subseteq L$ and $z \in L \setminus \{\bot\}$. Notice that, the following chains of equalities hold:

$$f_z\left(\bigvee_{y\in Y} y\right) = z \swarrow^{n_1} \left(\bigvee_{y\in Y} y\right) \stackrel{(*)}{=} \bigwedge_{y\in Y}(z \swarrow^{n_1} y) = \bigwedge_{y\in Y} f_z(y)$$

$$g_z\left(\bigvee_{x\in X} x\right) = z \nwarrow_{n_2} \left(\bigvee_{x\in X} x\right) \stackrel{(*)}{=} \bigwedge_{x\in X}(z \nwarrow_{n_2} x) = \bigwedge_{x\in X} g_z(x)$$

taking into account the definition of the operators $\swarrow^{n_1}, \nwarrow_{n_2}$ and being $(*)$ obtained by Proposition 5.

Conversely, supposing that $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z\in L\setminus\{\bot\}}$ is a family of antitone Galois connections, we prove that $(\swarrow^{n_1}, \nwarrow_{n_2})$ is a Galois implications pair, verifying that $n_1 = n_{s_{n_1}}$ and $n_2 = n_{n_{n_2}}$, following an analogous reasoning to the one given in Proposition 14. $\square$

Now, we will show that the pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ defined as in Proposition 17 cannot be associated with an adjoint conjunctor $\&$, in general.

Example 18. We will consider the complete lattice $(L, \preceq)$ and the pair of weak negations (n_1, n_2) defined on L which are depicted in Figure 2. First of all, we will present a Galois implications pair, whose adjoint negations coincide with n_1 and n_2, and we will see that they can be defined as in Proposition 17. For instance, the implications $\swarrow^{n_1}$ and $\nwarrow_{n_2}$ displayed in Table 5 can be defined, for all $x, y, z \in L$, as follows:

$$z \swarrow^{n_1} y = \begin{cases} n_1(y) & \text{if} \quad z = \bot \\ f_z(y) & \text{if} \quad z \neq \bot \end{cases} \qquad z \nwarrow_{n_2} x = \begin{cases} n_2(x) & \text{if} \quad z = \bot \\ g_z(x) & \text{if} \quad z \neq \bot \end{cases} \tag{3}$$

where $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z\in L\setminus\{\bot\}}$ is a family of mappings defined as follows:

$$f_z(y) = \begin{cases} b & \text{if} \quad y \neq \bot \\ \top & \text{if} \quad y = \bot \end{cases} \qquad g_z(x) = \begin{cases} \bot & \text{if} \quad x = a \quad \text{or} \quad x = \top \\ \top & \text{if} \quad x = \bot \quad \text{or} \quad x = b \end{cases} \tag{4}$$

for all $x, y \in L$ and $z \in L \setminus \{\bot\}$. Making simple computations, we obtain that the set $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z\in L\setminus\{\bot\}}$, whose mappings are defined as in Equation (4), is a family of antitone Galois connections. Then, by Proposition 17, we can ensure that the pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ displayed in Table 5 is a Galois implications pair with respect to $(L, \preceq)$. Clearly, we have that the equalities:

$$\begin{aligned} n_{s_{n_1}}(y) &= \bot \swarrow^{n_1} y = n_1(y) \\ n_{n_{n_2}}(x) &= \bot \nwarrow_{n_2} x = n_2(x) \end{aligned}$$

Figure 2: Pair of weak negations (n_1, n_2) and lattice $(L, \preceq)$ of Example 18.

	$\bot$	a	b	$\top$
n_1	$\top$	b	$\bot$	$\bot$
n_2	$\top$	$\bot$	a	$\bot$

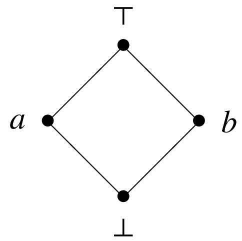

Table 5: Galois implications pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ of Example 18.

$\swarrow^{n_1}$	$\bot$	a	b	$\top$
$\bot$	$\top$	b	$\bot$	$\bot$
a	$\top$	b	b	b
b	$\top$	b	b	b
$\top$	$\top$	b	b	b

$\nwarrow_{n_2}$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\bot$	a	$\bot$
a	$\top$	$\bot$	$\top$	$\bot$
b	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\bot$

Finally, we will see that the pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ is not associated with an adjoint conjunctor $\&$. To reach this goal, we prove that the equality $x \,\&\, y = \min\{z \in L \mid x \preceq z \swarrow^{n_1} y\}$ is not verified, for all $x, y \in L$. Specifically, considering the elements $x = b$ and $y = \top$, we obtain that $\inf\{z \in L \mid b \preceq z \swarrow^{n_1} \top\} = \inf\{a, b, \top\} = \bot$. Clearly, $\bot \notin \{z \in L \mid b \preceq z \swarrow^{n_1} \top\}$. As a consequence, we obtain that the infimum is not a minimum. □

Notice that, the operators $\swarrow^{n_1}$ and $\nwarrow_{n_2}$ defined in Proposition 17 satisfy the following boundary conditions corresponding to the classical implications:

$$\begin{aligned} \bot \swarrow^{n_1} \bot &= n_1(\bot) = \top & \qquad \bot \nwarrow_{n_2} \bot &= n_2(\bot) = \top \\ \bot \swarrow^{n_1} \top &= n_1(\top) = \bot & \qquad \bot \nwarrow_{n_2} \top &= n_2(\top) = \bot \\ \top \swarrow^{n_1} \bot &= f_\top(\bot) = \top & \qquad \top \nwarrow_{n_2} \bot &= g_\top(\bot) = \top \end{aligned}$$

The boundary condition $\top \swarrow^{n_1} \top = \top \nwarrow_{n_2} \top = \top$ will be satisfied by $\swarrow^{n_1}$ and $\nwarrow_{n_2}$ when it is assumed that $f_\top(y) = \top$ and $g_\top(x) = \top$, for all $x, y \in L$.

The following theorem introduces the structure formed by all Galois implications pairs generating a given pair of weak negations. Henceforth, given a pair of weak negations (n_1, n_2) defined on a complete lattice $(L, \preceq)$, we denote the set of all Galois implications pairs generating (n_1, n_2) as $\mathcal{I}_{n_1 n_2}$.

Theorem 19. *Let (n_1, n_2) be a pair of weak negations defined on a complete lattice $(L, \preceq)$. Then, the pair $(\mathcal{I}_{n_1 n_2}, \sqsubseteq_{\mathcal{I}_{n_1 n_2}})$ forms a complete lattice with respect to $(L, \preceq)$, where $\sqsubseteq_{\mathcal{I}_{n_1 n_2}}$ is the ordering relation defined, for all $y, z \in L$, as:*

$$(\swarrow^{n_{1j}}, \nwarrow_{n_{2j}}) \sqsubseteq_{\mathcal{I}_{n_1 n_2}} (\swarrow^{n_{1k}}, \nwarrow_{n_{2k}}) \qquad \textit{iff} \qquad z \swarrow^{n_{1j}} y \preceq_1 z \swarrow^{n_{1k}} y$$

and $(\swarrow^{n_{1j}}, \nwarrow_{n_{2j}}), (\swarrow^{n_{1k}}, \nwarrow_{n_{2k}}) \in \mathcal{I}_{n_1 n_2}$. Moreover, the greatest and least elements of the set $\mathcal{I}_{n_1 n_2}$, denoted by $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ and $(\swarrow^{n_{1l}}, \nwarrow_{n_{2l}})$, respectively, are defined as:

$$z \swarrow^{n_{1g}} y = \begin{cases} n_1(y) & \textit{if} \quad z = \bot \\ \top & \textit{if} \quad z \neq \bot \end{cases} \qquad z \swarrow^{n_{1l}} y = \begin{cases} n_1(y) & \textit{if} \quad z = \bot \\ \bot & \textit{if} \quad z \neq \bot \textit{ and } y \neq \bot \\ \top & \textit{if} \quad z \neq \bot \textit{ and } y = \bot \end{cases}$$

$$z \nwarrow_{n_{2g}} x = \begin{cases} n_2(x) & \textit{if} \quad z = \bot \\ \top & \textit{if} \quad z \neq \bot \end{cases} \qquad z \nwarrow_{n_{2l}} x = \begin{cases} n_2(x) & \textit{if} \quad z = \bot \\ \bot & \textit{if} \quad z \neq \bot \textit{ and } x \neq \bot \\ \top & \textit{if} \quad z \neq \bot \textit{ and } x = \bot \end{cases}$$

for all $x, y, z \in L$.

Proof. First of all, given a family $\{(\swarrow^{n_{1i}}, \nwarrow_{n_{2i}})\}_{i \in I} \subseteq \mathcal{I}_{n_1 n_2}$ where I is a non-empty index set, we will prove the mappings $\swarrow^{\inf}, \nwarrow_{\inf} \colon L \times L \to L$, defined as:

$$z \swarrow^{\inf} y = \bigwedge_{i \in I} \{z \swarrow^{n_{1i}} y\} \qquad z \nwarrow_{\inf} x = \bigwedge_{i \in I} \{z \nwarrow_{n_{2i}} x\}$$

for all $x, y, z \in L$, form a Galois implications pair with respect to $(L, \preceq)$ by means of Equivalence (2). We will suppose that the inequality $x \preceq z \swarrow^{\inf} y$ is verified, being $x, y, z \in L$. As $x \preceq \bigwedge_{i \in I} \{z \swarrow^{n_{1i}} y\}$ then we have that $x \preceq z \swarrow^{n_{1i}} y$, for all $i \in I$. Taking into account that $(\swarrow^{n_{1i}}, \nwarrow_{n_{2i}})$ is a Galois implications pair, the inequality $x \preceq z \swarrow^{n_{1i}} y$ is equivalent to $y \preceq z \nwarrow_{n_{2i}} x$, for all $i \in I$. From the infimum property, we obtain that $y \preceq \bigwedge_{i \in I} \{z \nwarrow_{n_{2i}} x\}$ and therefore, $y \preceq z \nwarrow_{\inf} x$.

Following a similar reasoning to the previous one, we can prove the another implication and we can conclude that $(\swarrow^{\inf}, \nwarrow_{\inf})$ is a Galois implications pair with respect to $(L, \preceq)$.

Notice that, considering the point-wise ordering between the implications, we have that $\swarrow^{\text{inf}}$ is the infimum of $\{\swarrow^{n_{1i}}\}_{i\in I}$ and $\nwarrow_{\text{inf}}$ is the infimum of $\{\nwarrow_{n_{2i}}\}_{i\in I}$. Therefore, given a non-empty index set I and the family $\{(\swarrow^{n_{1i}}, \nwarrow_{n_{2i}})\}_{i\in I} \subseteq \mathcal{I}_{n_1n_2}$, we have that the Galois implications pair $(\swarrow^{\text{inf}}, \nwarrow_{\text{inf}})$ is the infimum of the family in $\mathcal{I}_{n_1n_2}$.

Now, we need to prove that $(\swarrow^{\text{inf}}, \nwarrow_{\text{inf}})$ is in $\mathcal{I}_{n_1n_2}$. Since $(\swarrow^{n_{1i}}, \nwarrow_{n_{2i}}) \in \mathcal{I}_{n_1n_2}$, for every $i \in I$, we can ensure that the equalities $n_{s_{n_{1i}}}(y) = \bot \swarrow^{n_{1i}} y = n_1(y)$ and $n_{n_{n_{2i}}}(x) = \bot \nwarrow_{n_{2i}} y = n_2(x)$ hold, for all $x, y \in L$ and $i \in I$. Hence, taking into account the definition of $\swarrow^{\text{inf}}$ and $\nwarrow_{\text{inf}}$, we obtain:

$$n_{s_{\text{inf}}}(y) = \bot \swarrow^{\text{inf}} y = \bigwedge_{i\in I}\{\bot \swarrow^{n_{1i}} y\} = n_1(y)$$

$$n_{n_{\text{inf}}}(x) = \bot \nwarrow_{\text{inf}} x = \bigwedge_{i\in I}\{\bot \nwarrow_{n_{2i}} x\} = n_2(x)$$

Therefore, we have proven that $(\swarrow^{\text{inf}}, \nwarrow_{\text{inf}})$ is a Galois implications pair in $\mathcal{I}_{n_1n_2}$. As a consequence, we can ensure that $(\mathcal{I}_{n_1n_2}, \sqsubseteq_{\mathcal{I}_{n_1n_2}})$ is a complete meet-semilattice.

Now, we will prove that $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is an element of the set $\mathcal{I}_{n_1n_2}$. Given the pair of weak negations (n_1, n_2) defined on a complete lattice $(L, \preceq)$, we can rewrite the pair $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$, for all $x, y \in L$, as follows:

$$z \swarrow^{n_{1g}} y = \begin{cases} n_1(y) & \text{if} \quad z = \bot \\ f_z(y) & \text{if} \quad z \neq \bot \end{cases} \qquad z \nwarrow_{n_{2g}} x = \begin{cases} n_2(x) & \text{if} \quad z = \bot \\ g_z(x) & \text{if} \quad z \neq \bot \end{cases}$$

where $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z\in L\setminus\{\bot\}}$ is a family of mappings defined, for each $z \in L \setminus \{\bot\}$, as $f_z(y) = g_z(x) = \top$, with $x, y \in L$. By Proposition 17, we only need to prove that $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z\in L\setminus\{\bot\}}$ is a family of antitone Galois connections in order to ensure that $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is a Galois implications pair. This fact is equivalent to prove that, for each $z \in L \setminus \{\bot\}$, the mappings f_z and g_z form an antitone Galois connection. Clearly, fixed $z \in L \setminus \{\bot\}$, the mappings f_z and g_z defined as $f_z(y) = g_z(x) = \top$, for all $x, y \in L$, satisfy the equivalence $x \preceq f_z(y)$ if and only if $y \preceq g_z(x)$, with $x, y \in L$. Indeed, (f_z, g_z) is trivially the greatest antitone Galois connection defined on $(L, \preceq)$. Moreover, from the definition of $\swarrow^{n_{1g}}$ and $\nwarrow_{n_{2g}}$, we obtain the following chains of equalities, for all $x, y \in L$:

$$\begin{aligned} n_{s_{n_{1g}}}(y) &= \bot \swarrow^{n_{1g}} y = n_1(y) \\ n_{n_{n_{2g}}}(x) &= \bot \nwarrow_{n_{2g}} x = n_2(x) \end{aligned}$$

As a consequence, we conclude that $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}}) \in \mathcal{I}_{n_1n_2}$.

Finally, we can ensure that $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is the greatest element of the set $\mathcal{I}_{n_1n_2}$, due to $(\swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is defined from the greatest antitone Galois connection defined on $(L, \preceq)$. Thus, $(\mathcal{I}_{n_1n_2}, \sqsubseteq_{\mathcal{I}_{n_1n_2}})$ is a meet-semilattice with maximum element and consequently, it is a complete lattice.

In the following, we will prove that $(\swarrow^{n_{1l}}, \nwarrow_{n_{2l}})$ is the minimum element of the set $\mathcal{I}_{n_1n_2}$. It is easy to see that $(\swarrow^{n_{1l}}, \nwarrow_{n_{2l}})$ can be rewritten, for all $x, y, z \in L$, as follows:

$$z \swarrow^{n_1} y = \begin{cases} n_1(y) & \text{if} \quad z = \bot \\ f_z(y) & \text{if} \quad z \neq \bot \end{cases} \qquad z \nwarrow_{n_2} x = \begin{cases} n_2(x) & \text{if} \quad z = \bot \\ g_z(x) & \text{if} \quad z \neq \bot \end{cases}$$

where $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z \in L \setminus \{\bot\}}$ is a family of mappings defined as in Equation (5):

$$f_z(y) = \begin{cases} \bot & \text{if} \quad y \neq \bot \\ \top & \text{if} \quad y = \bot \end{cases} \qquad g_z(x) = \begin{cases} \bot & \text{if} \quad x \neq \bot \\ \top & \text{if} \quad x = \bot \end{cases} \tag{5}$$

for all $x, y \in L$ and $z \in L \setminus \{\bot\}$. Therefore, the same pair is considered for every $z \in L \setminus \{\bot\}$. Once again, by Proposition 17, we only need to prove that $\{(f_z, g_z) \mid f_z, g_z \colon L \to L\}_{z \in L \setminus \{\bot\}}$ is a family of antitone Galois connections to ensure that $(\swarrow^{n_{1l}}, \nwarrow_{n_{2l}})$ is a Galois implications pair. Now, given $z \in L \setminus \{\bot\}$, we will see that the mappings f_z and g_z satisfy that if $x \preceq f_z(y)$ then $y \preceq g_z(x)$, with $x, y \in L$. Suppose that $x \preceq f_z(y)$, with $x, y \in L$. We will distinguish the following cases:

- Case $y = \bot$: as $x \preceq f_z(y)$ and $f_z(y) = f_z(\bot) = \top$ then $x \preceq \top$. Clearly, $y = \bot \preceq g_z(x)$, for all $x \in L$.

- Case $y \neq \bot$: as $x \preceq f_z(y)$ and $f_z(y) = \bot$ then $x = \bot$, and so, $y \preceq \top = g_z(\bot) = g_z(x)$.

The another implication, if $y \preceq g_z(x)$ then $x \preceq f_z(y)$ being $x, y \in L$, is obtained analogously. Hence, we can ensure that the pair (f_z, g_z) is an antitone Galois connection.

In addition, from the definition of $\swarrow^{n_{1l}}$ and $\nwarrow_{n_{2l}}$, we obtain the following chains of equalities, for all $x, y \in L$:

$$\begin{aligned} n_{s_{n_{1_l}}}(y) &= \bot \swarrow^{n_{1l}} y = n_1(y) \\ n_{n_{n_{2_l}}}(x) &= \bot \nwarrow_{n_{2l}} x = n_2(x) \end{aligned}$$

Consequently, we conclude that $(\swarrow^{n_{1l}}, \nwarrow_{n_{2l}}) \in \mathcal{I}_{n_1n_2}$.

Finally, we will demonstrate that $(\swarrow^{n_{1_l}}, \nwarrow_{n_{2_l}})$ is the least element in the set $\mathcal{I}_{n_1n_2}$, since it is defined from the least antitone Galois connection defined on $(L, \preceq)$. Therefore, we will prove that (f_z, g_z) is actually the least antitone Galois connection. Given two mappings $f_z^*, g_z^* \colon L \to L$ such that (f_z^*, g_z^*) is an antitone Galois connection, with $z \in L \setminus \{\bot\}$, we will prove that $f_z(y) \preceq f_z^*(y)$ and $g_z(x) \preceq g_z^*(x)$, for all $x, y \in L$. For that, we distinguish the following cases:

- Case $y = \bot$: by using Equation (5) and Proposition 7(3), we obtain that $f_z(\bot) = \top = f_z^*(\bot)$.
- Case $y \neq \bot$: clearly, by Equation (5), we have that $f_z(y) = \bot \preceq f_z^*(y)$.

Analogously, we prove that $g_z(x) \preceq g_z^*(x)$, for all $x \in L$. Therefore, we have proven that given $z \in L \setminus \{\bot\}$ the mappings f_z and g_z defined as in Equation (5) form the least antitone Galois connection defined on $(L, \preceq)$. □

To finish this section, we will continue with Example 18 in order to clarify the previous results related to Galois implications pairs generating a given pair of weak negations. In particular, we will show that $(\mathcal{I}_{n_1n_2}, \sqsubseteq_{\mathcal{I}_{n_1n_2}})$ is not a linear complete lattice, in general.

Example 20. From the framework given in Example 18 and following a similar reasoning to the one introduced in this example, we can check that the pairs given in Table 6 are also Galois implications pairs defined as in Proposition 17. As a straightforward consequence of Theorem 19, we obtain that the pairs $(\swarrow^{n_{1*}}, \nwarrow_{n_{2*}})$ and $(\swarrow^{n_{1+}}, \nwarrow_{n_{2+}})$ belong to $\mathcal{I}_{n_1n_2}$. Obviously, we can define more Galois implications pairs generating the pair of weak negations (n_1, n_2) being either greater or lesser than the ones given in this example. Specifically, the greatest and the least Galois implication pairs in $\mathcal{I}_{n_1n_2}$ are the pairs $(\swarrow^{n_{1_g}}, \nwarrow_{n_{2_g}})$ and $(\swarrow^{n_{1_l}}, \nwarrow_{n_{2_l}})$, respectively, which are given in Theorem 19.

According to the ordering relation $\sqsubseteq_{\mathcal{I}_{n_1n_2}}$ introduced in Theorem 19 and taking into account that the pair $(\swarrow^{n_1}, \nwarrow_{n_2})$ defined in Equation (3) also belongs to $\mathcal{I}_{n_1n_2}$, we obtain that $(\swarrow^{n_{1*}}, \nwarrow_{n_{2*}}) \sqsubseteq_{\mathcal{I}_{n_1n_2}} (\swarrow^{n_1}, \nwarrow_{n_2})$ and the Galois implications pair $(\swarrow^{n_{1+}}, \nwarrow_{n_{2+}})$ is incomparable to $(\swarrow^{n_1}, \nwarrow_{n_2})$ and $(\swarrow^{n_{1*}}, \nwarrow_{n_{2*}})$. As a consequence, when the complete lattice $(L, \preceq)$ and the pair of weak negations (n_1, n_2) depicted in Figure 2 are considered, the obtained complete lattice $(\mathcal{I}_{n_1n_2}, \sqsubseteq_{\mathcal{I}_{n_1n_2}})$ is not linear. □

It is important to mention that the implications operators need to have an adjoint conjunctor in different frameworks, such as fuzzy relation equations [14,

Table 6: Galois implications pairs $(\swarrow^{n_{1*}}, \nwarrow_{n_{2*}})$ and $(\swarrow^{n_{1+}}, \nwarrow_{n_{2+}})$ of Example 20.

$\swarrow^{n_{1*}}$	$\perp$	a	b	$\top$
$\perp$	$\top$	b	$\perp$	$\perp$
a	$\top$	b	b	b
b	$\top$	b	b	b
$\top$	$\top$	b	b	$\perp$

$\nwarrow_{n_{2*}}$	$\perp$	a	b	$\top$
$\perp$	$\top$	$\perp$	a	$\perp$
a	$\top$	$\perp$	$\top$	$\perp$
b	$\top$	$\perp$	$\top$	$\perp$
$\top$	$\top$	$\perp$	b	$\perp$

$\swarrow^{n_{1+}}$	$\perp$	a	b	$\top$
$\perp$	$\top$	b	$\perp$	$\perp$
a	$\top$	$\perp$	$\perp$	$\perp$
b	$\top$	a	a	a
$\top$	$\top$	$\perp$	$\perp$	$\perp$

$\nwarrow_{n_{2+}}$	$\perp$	a	b	$\top$
$\perp$	$\top$	$\perp$	a	$\perp$
a	$\top$	$\perp$	$\perp$	$\perp$
b	$\top$	$\top$	$\perp$	$\perp$
$\top$	$\top$	$\perp$	$\perp$	$\perp$

26, 27], rough set theory [20, 21, 40] and fuzzy mathematical morphology [1, 2, 37]. Hence, it is also interesting to study adjoint triples generating a given pair of weak negations.

4. Adjoint triples generating a given pair of weak negations

This section is focused on analyzing the algebraic structure formed by all adjoint triples whose adjoint negations coincide with a given pair of weak negations. In addition, we will introduce two different mechanisms to define adjoint triples generating a given pair of weak negations. We will include the notion of compatibility between the implications of an adjoint triple and the operators of a pair of weak negations, which will play an important role throughout this section.

To begin with, we will show a restriction that the conjunctors of adjoint triples generating a given pair of weak negations must satisfy.

Proposition 21. *Let (n_1, n_2) be a pair of weak negations defined on a complete lattice $(L, \preceq)$ and $(\&, \swarrow, \nwarrow)$ an adjoint triple with respect to $(L, \preceq)$ satisfying $n_1 = n_s$ and $n_2 = n_n$. If $x \preceq n_1(y)$ then $x \,\&\, y = \perp$, for all $x, y \in L$.*

Proof. We will suppose that (n_1, n_2) is a pair of weak negations defined on a complete lattice $(L, \preceq)$ and $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(L, \preceq)$ satisfying $n_1 = n_s$ and $n_2 = n_n$. If $x \preceq n_1(y)$, then the inequality $x \preceq \perp \swarrow y$ is verified. By the adjoint property, the last inequality $x \preceq \bot \swarrow y$ is equivalent to $x \,\&\, y \preceq \bot$. Taking into account that $\bot \preceq x \,\&\, y$, for all $x, y \in L$, we can conclude that $x \,\&\, y = \bot$. $\square$

Notice that, we need to require only the inequality $x \preceq n_1(y)$ in the statement of Proposition 21, since it is equivalent to $y \preceq n_2(x)$, by conditions (2) and (3) of Definition 10.

Before introducing the first mechanism to define adjoint triples generating a given pair of weak negations, we need to include the following definition.

Definition 22. Let (n_1, n_2) be a pair of weak negations defined on a complete lattice $(L, \preceq)$ and $(\&, \swarrow, \nwarrow)$ an adjoint triple with respect to $(L, \preceq)$. Then, we will say that:

- $\swarrow$ *is compatible with* n_1 if either $n_1(y) \preceq z \swarrow y$ or $z \swarrow y \preceq n_1(y)$ is satisfied, for all $y, z \in L$.
- $\nwarrow$ *is compatible with* n_2 if either $n_2(x) \preceq z \nwarrow x$ or $z \nwarrow x \preceq n_2(x)$ is satisfied, for all $x, z \in L$.

Once we have introduced the compatibility notion between the implications of an adjoint triple and a given pair of weak negations, we will show the first procedure to obtain adjoint triples generating such a pair of weak negations.

Proposition 23. *Let (n_1, n_2) be a pair of weak negations defined on a complete lattice $(L, \preceq)$ and $(\&, \swarrow, \nwarrow)$ an adjoint triple with respect to $(L, \preceq)$ such that the implication $\swarrow$ is compatible with n_1, the implication $\nwarrow$ is compatible with n_2 and the inequalities $\bot \swarrow y \preceq n_1(y)$, $\bot \nwarrow x \preceq n_2(x)$ hold, for all $x, y \in L$. The mappings $\&_{n_1 n_2}, \swarrow\!\!\swarrow^{n_1}, \nwarrow\!\!\nwarrow_{n_2} \colon L \times L \to L$ defined, for all $x, y, z \in L$, as:*

$$x \,\&_{n_1 n_2}\, y = \begin{cases} x \,\&\, y & \textit{if} \quad x \npreceq n_1(y) \\ \bot & \textit{if} \quad x \preceq n_1(y) \end{cases}$$

$$z \swarrow\!\!\swarrow^{n_1} y = \max\{z \swarrow y,\, n_1(y)\} \qquad z \nwarrow\!\!\nwarrow_{n_2} x = \max\{z \nwarrow x,\, n_2(x)\}$$

form an adjoint triple with respect to $(L, \preceq)$ such that $n_1 = n_{S_{n_1}}$ and $n_2 = n_{N_{n_2}}$, where $n_{S_{n_1}}$ and $n_{N_{n_2}}$ are the adjoint negations associated with the implications $\swarrow\!\!\swarrow^{n_1}$ and $\nwarrow\!\!\nwarrow_{n_2}$, respectively.

Proof. We will use Proposition 2 in order to demonstrate that $(\&_{n_1n_2}, \Swarrow^{n_1}, \Nwarrow_{n_2})$ is an adjoint triple with respect to $(L, \preceq)$. Therefore, we need to prove that the operators defined as $z \Swarrow^{n_1} y = \sup\{x \in L \mid x \&_{n_1n_2} y \preceq z\}$ and $z \Nwarrow_{n_2} x = \sup\{y \in L \mid x \&_{n_1n_2} y \preceq z\}$ for all $x, y, z \in L$, are actually maximums, being $\&_{n_1n_2}$ an order-preserving operator in both arguments.

First of all, given $y, z \in L$, we will show that the value $z \Swarrow^{n_1} y = \sup\{x \in L \mid x \&_{n_1n_2} y \preceq z\}$ is a maximum. Note that, $\perp \in \{x \in L \mid x \&_{n_1n_2} y \preceq z\}$, since $\perp \&_{n_1n_2} y = \perp \preceq z$ holds. Therefore, the set $\{x \in L \mid x \&_{n_1n_2} y \preceq z\}$ is not empty. We will consider the next partition of the set $\{x \in L \mid x \&_{n_1n_2} y \preceq z\} = X_1 \cup X_2$, where:

$$
\begin{aligned}
X_1 &= \{x \in L \mid x \npreceq n_1(y),\ x \&_{n_1n_2} y \preceq z\} \\
X_2 &= \{x \in L \mid x \preceq n_1(y),\ x \&_{n_1n_2} y \preceq z\}
\end{aligned}
$$

The set X_2 is not empty, since $\perp \preceq n_1(y)$ and $\perp \&_{n_1n_2} y = \perp \preceq z$ holds. In addition, by Proposition 21, $x \&_{n_1n_2} y = \perp$, for all $x \preceq n_1(y)$. Hence, we have $\sup(X_2) = \sup\{x \in L \mid x \preceq n_1(y),\ x \&_{n_1n_2} y \preceq z\} = \sup\{x \in L \mid x \preceq n_1(y)\} = n_1(y)$ and clearly $n_1(y)$ belongs to X_2.

Suppose that $X_1 \neq \varnothing$, then there exists $x_0 \in X_1$ such that $x_0 \npreceq n_1(y)$ and $x_0 \&_{n_1n_2} y \preceq z$. In particular, by definition of $\&_{n_1n_2}$, we have that $x_0 \& y = x_0 \&_{n_1n_2} y \preceq z$. Hence, we can ensure that $x_0 \in \{x \in L \mid x \& y \preceq z\}$. Since $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(L, \preceq)$, we have that $x_0 \preceq z \swarrow y = \max\{x \in L \mid x \& y \preceq z\}$. In additon, as $x_0 \npreceq n_1(y)$, we obtain that $z \swarrow y \npreceq n_1(y)$. Moreover, by the definition of $\&_{n_1n_2}$ and the adjoint property, we obtain that $(z \swarrow y) \&_{n_1n_2} y = (z \swarrow y) \& y \preceq z$. Hence, $z \swarrow y \in X_1$. Consequently, we obtain the following chain of inequalities:

$$
\begin{aligned}
\sup(X_1) &= \sup\{x \in L \mid x \npreceq n_1(y),\ x \&_{n_1n_2} y \preceq z\} \\
&= \sup\{x \in L \mid x \npreceq n_1(y),\ x \& y \preceq z\} \\
&\preceq \sup\{x \in L \mid x \& y \preceq z\} \\
&= z \swarrow y
\end{aligned}
$$

Therefore, since $z \swarrow y \in X_1$, by the definition of supremum, we obtain that $\sup(X_1) = z \swarrow y$ and it is a maximum.

Notice that, we have proven that if $X_1 \neq \varnothing$ then we obtain that $z \swarrow y \npreceq n_1(y)$. As $\swarrow$ is compatible with n_1, then $n_1(y) \preceq z \swarrow y$. Therefore, we conclude that $z \Swarrow^{n_1} y = \sup\{x \in L \mid x \&_{n_1n_2} y \preceq z\} = \sup(X_1 \cup X_2) = \sup\{z \swarrow y, n_1(y)\} = z \swarrow y$, and, since $z \swarrow y \in X_1$, the supremum in the definition of $z \Swarrow^{n_1} y$ is a maximum and, specifically, $z \Swarrow^{n_1} y = \max\{z \swarrow y, n_1(y)\}$.

Now, suppose that $X_1 = \varnothing$. Hence, $z \Swarrow^{n_1} y = \sup\{x \in L \mid x \&_{n_1 n_2} y \leq z\} = \sup(X_2) = n_1(y)$, which belongs to X_2 and so, is the maximum of the set. On the other hand, in this case, given $x' \in \{x \in L \mid x \& y \leq z\}$, if $x' \not\leq n_1(y)$, then we obtain that $x' \&_{n_1 n_2} y = x' \& y \leq z$, that is $x' \in X_1$, which leads us to a contradiction. Consequently, $x' \leq n_1(y)$ and so, $z \swarrow y = \max\{x \in L \mid x \& y \leq z\} \leq n_1(y)$ and we just obtain $z \Swarrow^{n_1} y = \max\{z \swarrow y, n_1(y)\}$.

In the following, we will prove that $\&_{n_1 n_2}$ is an order-preserving operator in the first argument. That is, given $x_1, x_2 \in L$ such that $x_1 \leq x_2$, we will obtain that $x_1 \&_{n_1 n_2} y \leq x_2 \&_{n_1 n_2} y$, for all $y \in L$, distinguishing the following three cases:

- Case $x_1 \not\leq n_1(y)$ and $x_2 \not\leq n_1(y)$: By definition of the operator $\&_{n_1 n_2}$ and taking into account that $\&$ is an order preserving operator, we obtain that $x_1 \&_{n_1 n_2} y = x_1 \& y \leq x_2 \& y = x_2 \&_{n_1 n_2} y$, for all $y \in L$.

- Case $x_1 \leq n_1(y)$ and $x_2 \leq n_1(y)$: Straigthfordwardly, by definition of the operator $\&_{n_1 n_2}$, we can ensure that $x_1 \&_{n_1 n_2} y = x_2 \&_{n_1 n_2} y = \bot$. Hence, $x_1 \&_{n_1 n_2} y \leq x_2 \&_{n_1 n_2} y$, for all $y \in L$.

- Case $x_1 \leq n_1(y)$ and $x_2 \not\leq n_1(y)$: Applying the definition of the operator $\&_{n_1 n_2}$, we have that $x_1 \&_{n_1 n_2} y = \bot \leq x_2 \& y = x_2 \&_{n_1 n_2} y$, for all $y \in L$.

As a consequence, we conclude that $\&_{n_1 n_2}$ is an order-preserving operator in the first argument.

On the other hand, notice that the inequality $x \leq n_1(y)$ is equivalent to $y \leq n_2(x)$, by conditions (2) and (3) of Definition 10. Therefore, the conjunctor $\&_{n_1 n_2}$ can be also expressed by means of n_2 as follows:

$$x \&_{n_1 n_2} y = \begin{cases} x \& y & \text{if } y \not\leq n_2(x) \\ \bot & \text{if } y \leq n_2(x) \end{cases}$$

By using the last expression of the conjunctor $\&_{n_1 n_2}$ and following an analogous reasoning to the previous one, we obtain that the operator $\Nwarrow_{n_2} \colon L \times L \to L$ defined as $z \Nwarrow_{n_2} x = \sup\{y \in L \mid x \&_{n_1 n_2} y \leq z\}$, for all $x, z \in L$, is a maximum. Specifically, we obtain that $z \Nwarrow_{n_2} x = \max\{y \in L \mid x \&_{n_1 n_2} y \leq z\} = \max\{z \nwarrow x,\, n_2(x)\}$, for all $x, z \in L$. In addition, we have that $\&_{n_1 n_2}$ is an order-preserving operator in the second argument. Therefore, we conclude that $(\&_{n_1 n_2}, \Swarrow^{n_1}, \Nwarrow_{n_2})$ is an adjoint triple.

Finally, it remains to prove that $n_1 = n_{S_{n_1}}$ and $n_2 = n_{N_{n_2}}$. The equality $n_1(y) = n_{S_{n_1}}(y)$ is deduced from the following chain of equalities $n_{S_{n_1}}(y) = \bot \Swarrow^{n_1} y = \max\{\bot \swarrow y,\, n_1(y)\} = n_1(y)$, since by hypothesis $\bot \swarrow y \leq n_1(y)$ holds for all $y \in L$. In a similar way, we can prove that the equality $n_2(x) = n_{N_{n_2}}(x)$ is satisfied, for all $x \in L$. $\square$

In order to clarify the previous results, we will include the following example.

Example 24. Consider the pair of weak negations (n_1, n_2) and the complete lattice $(L, \preceq)$ depicted in Figure 1 of Example 12. Now, we will consider the operators $\&$, $\swarrow$ and $\nwarrow$ displayed in Table 7. It is easy to check that $(\&, \swarrow, \nwarrow)$ is an adjoint triple by using either Proposition 2 or Proposition 3.

Table 7: Adjoint triple $(\&, \swarrow, \nwarrow)$ of Example 24.

$\&$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	c	c	c
b	$\bot$	c	c	c	c
c	$\bot$	c	c	c	c
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

$\swarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\top$	a	$\bot$	$\bot$	$\bot$
b	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	c	c	c	c
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\top$	a	$\bot$	$\bot$	$\bot$
b	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Taking into account Definition 22 and Table 7, we can conclude that the implication $\swarrow$ is compatible with n_1 and the implication $\nwarrow$ is compatible with n_2. Besides, it is easy to see that the inequalities $\bot \swarrow y \preceq n_1(y)$ and $\bot \nwarrow x \preceq n_2(x)$ are verified, for all $x, y \in L$. Therefore, since the hypothesis of Proposition 23 are satisfied, we obtain that the mappings $\&_{n_1n_2}, \Swarrow^{n_1}, \Nwarrow_{n_2} \colon L \times L \to L$ defined as:

$$x \ \&_{n_1n_2} \ y = \begin{cases} x \,\&\, y & \text{if} \quad x \npreceq n_1(y) \\ \bot & \text{if} \quad x \preceq n_1(y) \end{cases}$$

$$z \ \Swarrow^{n_1} \ y = \max\{z \swarrow y,\ n_1(y)\} \qquad z \ \Nwarrow_{n_2} \ x = \max\{z \nwarrow x,\ n_2(x)\}$$

for all $x, y, z \in L$, form an adjoint triple. Making simple computations, we obtain that the adjoint triple $(\&_{n_1n_2}, \Swarrow^{n_1}, \Nwarrow_{n_2})$ is the one given in Table 8. Moreover, the equalities $n_1(y) = n_{S_{n_1}}(y)$ and $n_2(x) = n_{N_{n_2}}(x)$ are satisfied straightforwardly, for all $x, y \in L$. $\square$

We are interested in generalizing the compatibility property shown in Definition 22 in order to address a wider range of situations from which adjoint

Table 8: Adjoint triple ($\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2}$) of Example 24.

$\&_{n_1n_2}$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	c
b	$\bot$	c	c	c	c
c	$\bot$	c	c	c	c
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

$\swarrow^{n_1}$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	a	a	a	$\bot$
a	$\top$	a	a	a	$\bot$
b	$\top$	a	a	a	$\bot$
c	$\top$	c	c	c	c
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow_{n_2}$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	c	$\bot$	$\bot$	$\bot$
a	$\top$	c	$\bot$	$\bot$	$\bot$
b	$\top$	c	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

triples generating a given pair of weak negations can be defined. From now on, given (n_1, n_2) a pair of weak negations defined on a complete lattice $(L, \preceq)$ and $g\colon L \times L \to L$ an arbitrary mapping, we will say that g *is compatible with* n_1 if either $n_1(y) \preceq g(z, y)$ or $g(z, y) \preceq n_1(y)$ holds, for all $y, z \in L$. The notion of an arbitrary mapping $h\colon L \times L \to L$ being *compatible with* n_2 is defined analogously.

The following technical result, related to the compatibility property previously mentioned, will be useful in order to obtain the second mechanism for defining adjoint triples generating such pair of weak negations.

Proposition 25. *Given a pair of weak negations (n_1, n_2), two mappings $f, g\colon L\times L \to L$, such that f preserves the supremum of non-empty sets in the first argument, that is*

$$f\left(\bigvee_{x_i \in X} x_i, y\right) = \bigvee_{x_i \in X} f(x_i, y), \textit{ for all non-empty set } X \subseteq L \textit{ and } y \in L$$

and g is defined for all $y, z \in L$ as $g(z, y) = \sup\{x \in L \mid f(x, y) \preceq z\}$, satisfying that g is compatible with n_1. For each $y \in L$, subset $X \subseteq L$, and the partition of the set $X = X_1 \cup X_2$, where $X_1 = \{x \in X \mid x \npreceq n_1(y)\} \neq \varnothing$ and $X_2 = \{x \in X \mid x \preceq n_1(y)\} \neq \varnothing$, we obtain that $f(x_2, y) \preceq f(x_1, y)$, for all $x_1 \in X_1$ and $x_2 \in X_2$.

Proof. Given $y \in L$, a subset $X \subseteq L$, the partition of the set $X = X_1 \cup X_2$, such that $X_1 = \{x \in X \mid x \npreceq n_1(y)\} \neq \varnothing$ and $X_2 = \{x \in X \mid x \preceq n_1(y)\} \neq \varnothing$, and $x_1 \in X_1$, we consider the element:

$$g(f(x_1, y), y) = \sup\{x \in L \mid f(x, y) \preceq f(x_1, y)\}$$

From now on, we will denote the set $\{x \in L \mid f(x, y) \preceq f(x_1, y)\}$ as X_f. Applying that f preserves the supremum of non-empty sets in the first argument and $X_f \neq \varnothing$, since clearly $x_1 \in X_f$, we obtain that:

$$f(x_1, y) \preceq f\left(\bigvee_{x \in X_f} x, y\right) = \bigvee_{x \in X_f} f(x, y) \preceq f(x_1, y)$$

Therefore, $f(x_1,y) = f\left(\bigvee_{x\in X_f} x,y\right) = f(g(f(x_1,y),y),y)$, because $\bigvee_{x\in X_f} x = g(f(x_1,y),y)$. As a consequence, $g(f(x_1,y))$ is in X_f and it is the maximum. Hence, $x_1 \preceq g(f(x_1,y))$ and we obtain that $g(f(x_1,y),y) \npreceq n_1(y)$ and by the compatibility of g, the inequality $n_1(y) \preceq g(f(x_1,y),y)$ holds.

Finally, given $x_2 \in X_2$, as $x_2 \preceq n_1(y)$ and f is order-preserving in the first argument, we obtain the result:

$$f(x_2,y) \preceq f(n_1(y),y) \preceq f(g(f(x_1,y),y),y) = f(x_1,y)$$

□

An analogous result to Proposition 25 is obtained assuming that f preserves the supremum of non-empty sets in the second argument and that the mapping $h\colon L\times L \to L$ defined as $h(z,x) = \sup\{y \in L \mid f(x,y) \preceq z\}$, for all $x,z \in L$, is compatible with n_2.

These results are interesting to weaken the restrictions in the hypothesis of Proposition 23 and allow more general operators than adjoint triples. The following proposition gives a weaker sufficient condition to define adjoint triples generating a given pair of weak negations.

Proposition 26. *Let (n_1,n_2) be a pair of weak negations, $f,g,h\colon L\times L \to L$ three mappings such that f preserves the supremum of non-empty sets in both arguments, g is defined as $g(z,y) = \sup\{x \in L \mid f(x,y) \preceq z\}$, is compatible with n_1 satisfying that $g(\bot,y) \preceq n_1(y)$ and h is defined as $h(z,x) = \sup\{y \in L \mid f(x,y) \preceq z\}$, is compatible with n_2 satisfying that $h(\bot,x) \preceq n_2(x)$, for all $x,y,z \in L$. The triple $(\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2})$ composed of the following operators:*

$$x \;\&_{n_1n_2}\; y = \begin{cases} f(x,y) & \text{if} \quad x \npreceq n_1(y) \\ \bot & \text{if} \quad x \preceq n_1(y) \end{cases}$$

$$z \swarrow^{n_1} y = \max\{g(z,y),\, n_1(y)\} \qquad z \nwarrow_{n_2} x = \max\{h(z,x),\, n_2(x)\}$$

is an adjoint triple with respect to $(L,\preceq)$ verifying that $n_1 = n_{S_{n_1}}$ and $n_2 = n_{N_{n_2}}$.

Proof. In order to prove that $(\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2})$ is an adjoint triple, by using Proposition 2, we only need to see that & preserves the supremum in both arguments.

Given $X \subseteq L$ and $y \in L$, we will consider the following partition of the set $X = X_1 \cup X_2$, where $X_1 = \{x \in X \mid x \npreceq n_1(y)\}$ and $X_2 = \{x \in X \mid x \preceq n_1(y)\}$. Notice that, the set X_2 is a non-empty set since $\bot \preceq n_1(y)$ and therefore $\bot \in X_2$.

If $X_1 = \varnothing$, then $X = X_1 \cup X_2 = X_2$ and so, $\bigvee_{x \in X} x \leq n_1(y)$, and we obtain that

$$\bigvee_{x \in X} (x \,\&_{n_1 n_2}\, y) = \bot = \left(\bigvee_{x \in X} x \right) \&_{n_1 n_2}\, y$$

Otherwise, if $X_1 \neq \varnothing$, taking into account that f preserves the supremum of non-empty sets in the first argument and the definition of the operator $\&_{n_1 n_2}$, we obtain the following chain of equalities:

$$\begin{aligned}
\bigvee_{x \in X} (x \,\&_{n_1 n_2}\, y) &= \left(\bigvee_{x_i \in X_1} (x_i \,\&_{n_1 n_2}\, y) \right) \vee \left(\bigvee_{x_j \in X_2} (x_j \,\&_{n_1 n_2}\, y) \right) \\
&= \bigvee_{x_i \in X_1} (x_i \,\&_{n_1 n_2}\, y) \\
&= \bigvee_{x_i \in X_1} f(x_i, y) \\
&\overset{(1)}{=} \bigvee_{x_i \in X_1} f(x_i, y) \vee \bigvee_{x_j \in X_2} f(x_j, y) \\
&= \bigvee_{x \in X_1 \cup X_2} f(x, y) \\
&\overset{(2)}{=} f\left(\bigvee_{x \in X} x, y \right) \\
&\overset{(3)}{=} \left(\bigvee_{x \in X} x \right) \&_{n_1 n_2}\, y
\end{aligned}$$

where (1) is obtained by Proposition 25, (2) is given because f preserves the supremum of non-empty sets in the first argument, and (3) holds because $X_1 \neq \varnothing$ and so, $\bigvee_{x \in X} x \not\leq n_1(y)$.

Analogously, we prove the equality $x \,\&_{n_1 n_2} \left(\bigvee_{y \in Y} y \right) = \bigvee_{y \in Y} (x \,\&_{n_1 n_2}\, y)$, for any $Y \subseteq L$ and $x \in L$. Hence, we conclude that $(\&_{n_1 n_2}, \swarrow^{n_1}, \nwarrow_{n_2})$ is an adjoint triple with respect to $(L, \leq)$. Consequently, by Proposition 2, we have that:

$$z \swarrow^{n_1} y = \max\{x \in L \mid x \,\&_{n_1 n_2}\, y \leq z\}, \text{ for all } y \in L \text{ and } z \in L.$$

$$z \nwarrow_{n_2} x = \max\{y \in L \mid x \,\&_{n_1 n_2}\, y \leq z\}, \text{ for all } x \in L \text{ and } z \in L.$$

Now, given $y, z \in L$, we will prove that $z \swarrow^{n_1} y = \max\{g(z, y), n_1(y)\}$, distinguishing two cases:

- Case $z \swarrow^{n_1} y \preceq n_1(y)$: applying the definition of $\&_{n_1n_2}$, we have that $n_1(y) \&_{n_1n_2} y = \perp \preceq z$. Therefore, we can ensure that $n_1(y) \in \{x \in L \mid x \&_{n_1n_2} y \preceq z\}$. As $z \swarrow^{n_1} y = \max\{x \in L \mid x \&_{n_1n_2} y \preceq z\}$ and, by hypothesys, $z \swarrow^{n_1} y \preceq n_1(y)$ holds, we can conclude that $z \swarrow^{n_1} y = n_1(y)$.

  In order to assert that $z \swarrow^{n_1} y = \max\{g(z, y),\ n_1(y)\}$, we also need to prove that $g(z, y) \preceq n_1(y)$. We will proceed by reductio ad absurdum. Hence, we will assume that $g(z, y) \npreceq n_1(y)$ and we will arrive to a contradiction. If $\{x \in L \mid f(x, y) \preceq z\} = \varnothing$, then

  $$g(z, y) = \sup\{x \in L \mid f(x, y) \preceq z\} = \perp$$

  which implies that $g(z, y) = \perp \preceq n_1(y)$ and leads us to a contradiction. Otherwise, the set $\{x \in L \mid f(x, y) \preceq z\} \neq \varnothing$, since f preserves the supremum of non-empty sets and by the definition of $\&_{n_1n_2}$, we obtain that:

  $$\begin{array}{rcl} g(z, y) \ \&_{n_1n_2}\ y & = & f(g(z, y), y) \\ & = & f(\sup\{x \in L \mid f(x, y) \preceq z\}, y) \\ & = & \sup\{f(x, y) \in L \mid f(x, y) \preceq z\} \\ & \preceq & z \end{array}$$

  Therefore, $g(z, y) \preceq \max\{x \in L \mid x \&_{n_1n_2} y \preceq z\} = z \swarrow^{n_1} y$, which also leads to a contradiction since $z \swarrow^{n_1} y = n_1(y)$, as it was proved above. Thus, in this case, we have $z \swarrow^{n_1} y = \max\{g(z, y),\ n_1(y)\}$.

- Case $z \swarrow^{n_1} y \npreceq n_1(y)$: by the definition of $\&_{n_1n_2}$ and considering that $z \swarrow^{n_1} y \in \{x \in L \mid x \&_{n_1n_2} y \preceq z\}$, we have that $f(z \swarrow^{n_1} y, y) = (z \swarrow^{n_1} y) \&_{n_1n_2} y \preceq z$. Therefore, $z \swarrow^{n_1} y \in \{x \in L \mid f(x, y) \preceq z\}$ and consequently $z \swarrow^{n_1} y \preceq g(z, y)$, since $g\colon L \times L \to L$ is defined as $g(z, y) = \sup\{x \in L \mid f(x, y) \preceq z\}$. On the other hand, given $x_0 \in \{x \in L \mid f(x, y) \preceq z\}$, we have that:

  $$x_0 \ \&_{n_1n_2}\ y = \begin{cases} f(x_0, y) & \text{if} \quad x_0 \npreceq n_1(y) \\ \perp & \text{if} \quad x_0 \preceq n_1(y) \end{cases}$$

  Consequently, $x_0 \&_{n_1n_2} y \preceq z$ and so, $x_0 \in \{x \in L \mid x \&_{n_1n_2} y \preceq z\}$. Since $z \swarrow^{n_1} y = \max\{x \in L \mid x \&_{n_1n_2} y \preceq z\}$, we have that $x_0 \preceq z \swarrow^{n_1} y$. Therefore, $x_0 \preceq z \swarrow^{n_1} y$ holds, for each $x_0 \in \{x \in L \mid f(x, y) \preceq z\}$. Specifically, $g(z, y) = \sup\{x \in L \mid f(x, y) \preceq z\} \preceq z \swarrow^{n_1} y$ and, as a consequence, we obtain that $z \swarrow^{n_1} y = g(z, y)$.

  To ensure that $z \swarrow^{n_1} y = \max\{g(z, y),\ n_1(y)\}$, we also need to prove that $n_1(y) \preceq g(z, y)$. Notice that, g is compatible with n_1 by hyphotesis, then either the inequality $n_1(y) \preceq g(z, y)$ or the inequality $g(z, y) \preceq n_1(y)$ holds, for all $y, z \in L$. Suppose that $g(z, y) \preceq n_1(y)$ is verified, taking into account that $z \swarrow^{n_1} y = g(z, y)$, we can conclude that $z \swarrow^{n_1} \preceq n_1(y)$. This last inequality leads us to a contradiction since we are assuming that $z \swarrow^{n_1} y \npreceq n_1(y)$. Hence, in this case, we also have that $z \swarrow^{n_1} y = \max\{g(z, y),\ n_1(y)\}$.

Following an analogous reasoning to the previous one, we obtain that $z \nwarrow_{n_2} x = \max\{y \in L \mid x \,\&_{n_1 n_2}\, y \preceq z\} = \max\{h(z, x),\ n_2(x)\}$, for all $x, z \in L$.

Finally, it remains to prove that $n_1 = n_{S_{n_1}}$ and $n_2 = n_{N_{n_2}}$. The equality $n_1(y) = n_{S_{n_1}}(y)$ is deduced considering that $n_{S_{n_1}}(y) = \bot \swarrow^{n_1} y = \max\{g(\bot, y)\,, n_1(y)\} = n_1(y)$, for all $y \in L$, since the inequality $g(\bot, y) \preceq n_1(y)$ is verified, for all $y \in L$, by hypothesis. We can prove that the equality $n_2(x) = n_{N_{n_2}}(x)$ is satisfied, for all $x \in L$, in an analogous way. □

Now, we will illustrate the procedure given in Proposition 26 in order to build adjoint triples generating a given pair of weak negations.

Example 27. Consider again the pair of weak negations (n_1, n_2) and the complete lattice $(L, \preceq)$ depicted in Figure 1 of Example 12. Consider also the mappings f, g and h defined in Table 9. According to Tables 1 and 9, we can ensure that g is compatible with n_1, h is compatible with n_2 and the inequalities $g(\bot, y) \preceq n_1(y)$ and $h(\bot, x) \preceq n_2(x)$ hold, for all $x, y \in L$. It is also easy to see that the mapping f preserves the supremum of non-empty sets in both arguments. Therefore, applying Proposition 26, we have that $(\&_{n_1 n_2}, \swarrow^{n_1}, \nwarrow_{n_2})$ is an adjoint triple, which definition is given by Table 8 of Example 24, and verifies that $n_1 = n_{S_{n_1}}$ and $n_2 = n_{N_{n_2}}$.

It is important to highlight that Proposition 26 requires a weaker constraint than Proposition 23. Notice that, when $X = \varnothing$ and $y = \top$, we have that:

$$f\left(\bigvee_{x_i \in X} x_i, y\right) = f\,(\bot, \top) = a \neq \bot = \bigvee_{x_i \in X} f(x_i, \top) = \bigvee_{x_i \in X} f(x_i, y)$$

Therefore, f does not preserve the supremum in the first argument, for all $X \subseteq L$ and $y \in L$, and it cannot be the conjunctor of an adjoint triple. As a consequence, we can ensure that the procedure given in Proposition 26 provides adjoint triples build from more general operators. □

Table 9: Mappings f, g and h of Example 27.

f	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	a
a	$\bot$	a	c	c	c
b	$\bot$	c	c	c	c
c	$\bot$	c	c	c	c
$\top$	b	$\top$	$\top$	$\top$	$\top$

g	$\bot$	a	b	c	$\top$
$\bot$	c	$\bot$	$\bot$	$\bot$	$\bot$
a	c	a	$\bot$	$\bot$	$\bot$
b	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	c	c	c	c
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

h	$\bot$	a	b	c	$\top$
$\bot$	c	$\bot$	$\bot$	$\bot$	$\bot$
a	$\top$	a	$\bot$	$\bot$	$\bot$
b	c	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

The possibility of considering general mappings that do not need to preserve the supremum of the empty set is very interesting, since the number of operators to be considered increases considerably. Notice that, preserving the supremum of the empty set is an important property in many frameworks [5, 26, 39]. Having the possibility of removing such a property is a remarkable achievement.

The next result proves that the set of all adjoint triples generating a given pair of weak negations has the structure of a complete join-semilattice. From now on, given a pair of weak negations (n_1, n_2) defined on a complete lattice $(L, \preceq)$, we denote the set of all adjoint triples generating (n_1, n_2) as $\mathcal{T}_{n_1 n_2}$.

Theorem 28. *Let (n_1, n_2) be a pair of weak negations defined on a complete lattice $(L, \preceq)$. We have that the pair $(\mathcal{T}_{n_1 n_2}, \sqsubseteq_{\mathcal{T}_{n_1 n_2}})$ forms a complete join-semilattice, where $\sqsubseteq_{\mathcal{T}_{n_1 n_2}}$ is the ordering relation defined as:*

$$(\&^j_{n_1 n_2}, \swarrow^{n_{1j}}, \nwarrow_{n_{2j}}) \sqsubseteq_{\mathcal{T}_{n_1 n_2}} (\&^k_{n_1 n_2}, \swarrow^{n_{1k}}, \nwarrow_{n_{2k}}) \qquad \text{iff} \qquad x \,\&^j_{n_1 n_2}\, y \preceq x \,\&^k_{n_1 n_2}\, y$$

for all $x, y \in L$ and $(\&^j_{n_1 n_2}, \swarrow^{n_{1j}}, \nwarrow_{n_{2j}}), (\&^k_{n_1 n_2}, \swarrow^{n_{1k}}, \nwarrow_{n_{2k}}) \in \mathcal{T}_{n_1 n_2}$. Moreover, the greatest element is the adjoint triple $(\&^g_{n_1 n_2}, \swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ defined, for all $x, y, z \in L$, as follows:

$$x \,\&^g_{n_1 n_2}\, y = \begin{cases} \top & \text{if} \quad x \not\preceq n_1(y) \\ \bot & \text{if} \quad x \preceq n_1(y) \end{cases}$$

$$z \swarrow^{n_{1g}} y = \begin{cases} n_1(y) & \text{if} \quad z \neq \top \\ \top & \text{if} \quad z = \top \end{cases} \qquad z \nwarrow_{n_{2g}} x = \begin{cases} n_2(x) & \text{if} \quad z \neq \top \\ \top & \text{if} \quad z = \top \end{cases}$$

Proof. First of all, given the family $\{(\&^i_{n_1 n_2}, \swarrow^{n_{1i}}, \nwarrow_{n_{2i}})\}_{i \in I} \subseteq \mathcal{T}_{n_1 n_2}$, where I nonempty index set, we will prove that the mappings $\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup} \colon L \times L \to L$

defined, for all $x, y, z \in L$, as:

$$\begin{aligned}
x \,\&_{\sup}\, y &= \bigvee_{i\in I}\{x \,\&^{i}_{n_1n_2}\, y\}\\
z \swarrow^{\sup} y &= \bigwedge_{i\in I}\{z \swarrow^{n_{1i}} y\}\\
z \nwarrow_{\sup} x &= \bigwedge_{i\in I}\{z \nwarrow_{n_{2i}} x\}
\end{aligned}$$

form an adjoint triple with respect to $(L, \preceq)$ by the adjoint property. We assume that the inequality $x \,\&_{\sup}\, y \preceq z$ is verified, where $x, y, z \in L$, that is, $\bigvee_{i\in I}\{x \,\&^{i}_{n_1n_2}\, y\} \preceq z$ holds. Applying the supremum property, we have that the inequality $x \,\&^{i}_{n_1n_2}\, y \preceq z$ is satisfied, for all $i \in I$. Taking into account that $(\&^{i}_{n_1n_2}, \swarrow^{n_{1i}}, \nwarrow_{n_{2i}})$ is an adjoint triple, we obtain that $x \,\&^{i}_{n_1n_2}\, y \preceq z$ is equivalent to $x \preceq z \swarrow^{n_{1i}} y$, for all $i \in I$. By the infimum property, we have that $x \preceq \bigwedge_{i\in I}\{z \swarrow^{n_{1i}} y\} = z \swarrow^{\sup} y$ holds.

As the previous deductions are equivalences, if we suppose that $x \preceq z \swarrow^{\sup} y = \bigwedge_{i\in I}\{z \swarrow^{n_{1i}} y\}$, the we obtain that $\bigvee_{i\in I}\{x \,\&^{i}_{n_1n_2}\, y\} \preceq z$, that is $x \,\&_{\sup}\, y \preceq z$.

The another equivalence $x \,\&_{\sup}\, y \preceq z$ if and only if $y \preceq z \nwarrow_{\sup} x$ can be proved in a similar way. Hence, $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ is an adjoint triple with respect to $(L, \preceq)$.

Notice that, considering the point-wise ordering between the conjunctors, we obtain that $\&_{\sup}$ is the supremum of $\{\&^{i}_{n_1n_2}\}_{i\in I}$. Therefore, given a non-empty index set I and the family $\{(\&^{i}_{n_1n_2}, \swarrow^{n_{1i}}, \nwarrow_{n_{2i}})\}_{i\in I} \subseteq \mathcal{T}_{n_1n_2}$, we have that $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ is the supremum of the family $\{(\&^{i}_{n_1n_2}, \swarrow^{n_{1i}}, \nwarrow_{n_{2i}})\}_{i\in I}$ in $\mathcal{T}_{n_1n_2}$.

Now, we need to prove that the triple $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ belongs to $\mathcal{T}_{n_1n_2}$. Since $(\&^{i}_{n_1n_2}, \swarrow^{n_{1i}}, \nwarrow_{n_{2i}}) \in \mathcal{T}_{n_1n_2}$, for every $i \in I$, we can ensure that the equalities $n_{S_{n_{1_i}}}(y) = \bot \swarrow^{n_{1i}} y = n_1(y)$ and $n_{N_{n_{2i}}}(x) = \bot \nwarrow_{n_{2i}} x = n_2(x)$ hold, for all $x, y \in L$. Hence, taking into account the definition of $\swarrow^{\sup}$ and $\nwarrow_{\sup}$, we obtain:

$$n_{S_{\sup}}(y) = \bot \swarrow^{\sup} y = \bigwedge_{i\in I}\{\bot \swarrow^{n_{1i}} y\} = n_1(y)$$

$$n_{N_{\sup}}(x) = \bot \nwarrow_{\sup} y = \bigwedge_{i\in I}\{\bot \nwarrow_{n_{2i}} x\} = n_2(x)$$

Therefore, we have proven that $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ is an adjoint triple in $\mathcal{T}_{n_1n_2}$. As a consequence, we can ensure that $(\mathcal{T}_{n_1n_2}, \sqsubseteq_{\mathcal{T}_{n_1n_2}})$ is a complete join-semilattice.

Now, we will see that $(\&^{g}_{n_1n_2}, \swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is an adjoint triple with respect to $(L, \preceq)$. By Proposition 3, we need to prove that $x \,\&^{g}_{n_1n_2}\, y = \min\{z \in L \mid x \preceq z \swarrow^{n_{1g}} y\} = \min\{z \in L \mid y \preceq z \nwarrow_{n_{2g}} x\}$, for all $x, y \in L$, being $\swarrow^{n_{1g}}$ and $\nwarrow_{n_{2g}}$

order-preserving operators in the first argument. We begin proving that, for each $x, y \in L$, the value $x \&^g_{n_1n_2} y = \inf\{z \in L \mid x \leq z \swarrow^{n_{1g}} y\}$ is a minimum. Given $x, y \in L$, the following two cases have to be considered:

- If $x \leq n_1(y)$, then every $z \in L$ satisfies $x \leq z \swarrow^{n_{1g}} y$, since $z \swarrow^{n_{1g}} y$ is either $\top$ or $n_1(y)$. Consequently, $x \&^g_{n_1n_2} y = \inf\{z \in L \mid x \leq z \swarrow^{n_{1g}} y\} = \inf\{z \in L\} = \bot$ and $\bot \in \{z \in L \mid x \leq z \swarrow^{n_{1g}} y\}$, hence $x \&^g_{n_1n_2} y$ is a minimum.
- When $x \nleq n_1(y)$, the element $z = \top$ is the unique value verifying that $x \leq z \swarrow^{n_{1g}} y$. Therefore, $x \&^g_{n_1n_2} y = \inf\{z \in L \mid x \leq z \swarrow^{n_{1g}} y\} = \inf\{\top\} = \top$. Hence, $x \&^g_{n_1n_2} y$ is a minimum.

Therefore, we have that $x \&^g_{n_1n_2} y = \min\{z \in L \mid x \leq z \swarrow^{n_{1g}} y\}$, for all $x, y \in L$.

Now, we will prove that $\swarrow^{n_{1g}}$ is an order-preserving operator in the first argument. Given $z_1, z_2 \in L$, suppose that $z_1 \leq z_2$ and distinguish the following cases:

- Case $z_1 \neq \top$ and $z_2 \neq \top$: by definition of the operator $\swarrow^{n_{1g}}$, we obtain that $z_1 \swarrow^{n_{1g}} y = z_2 \swarrow^{n_{1g}} y = n_1(y)$. Hence, $z_1 \swarrow^{n_{1g}} y \leq z_2 \swarrow^{n_{1g}} y$, for all $y \in L$.
- Case $z_1 = \top$ and $z_2 = \top$: by definition of the operator $\swarrow^{n_{1g}}$, we can ensure that $z_1 \swarrow^{n_{1g}} y = z_2 \swarrow^{n_{1g}} y = \top$. Thus, $z_1 \swarrow^{n_{1g}} y \leq z_2 \swarrow^{n_{1g}} y$, for all $y \in L$.
- Case $z_1 = \top$ and $z_2 \neq \top$: Applying the definition of the operator $\swarrow^{n_{1g}}$, we have that $z_1 \swarrow^{n_{1g}} y = n_1(y) \leq \top = z_2 \swarrow^{n_{1g}} y$, for all $y \in L$.

As a consequence, we conclude that $\swarrow^{n_{1g}}$ is an order-preserving operator in the first argument.

Notice that, the inequality $x \leq n_1(y)$ is equivalent to $y \leq n_2(x)$, by conditions (2) and (3) of Definition 10. Therefore, the conjunctor $\&^g_{n_1n_2}$ verifies that:

$$x \&^g_{n_1n_2} y = \begin{cases} \top & \text{if } x \nleq n_1(y) \\ \bot & \text{if } x \leq n_1(y) \end{cases} = \begin{cases} \top & \text{if } y \nleq n_2(x) \\ \bot & \text{if } y \leq n_2(x) \end{cases}$$

By using the last expression of $\&^g_{n_1n_2}$ and following an analogous reasoning to the previous one, we obtain that the operator defined as $x \&^g_{n_1n_2} y = \inf\{z \in L \mid y \leq z \nwarrow_{n_{2g}} x\}$, for all $x, y \in L$, is a minimum. In addition, we obtain that $\nwarrow_{n_{2g}}$ is an order-preserving operator in the first argument.

Moreover, we trivially obtain that $n_1 = n_{S_{n_{1g}}}$ and $n_2 = n_{N_{n_{2g}}}$ and therefore the adjoint triple $(\&^g_{n_1n_2}, \swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ belongs to $\mathcal{T}_{n_1n_2}$.

Finally, we will demonstrate that $(\&^g_{n_1n_2}, \swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$ is the greatest element of $\mathcal{T}_{n_1n_2}$. Given $(\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2}) \in \mathcal{T}_{n_1n_2}$, we will prove that $x \,\&_{n_1n_2}\, y \preceq x \,\&^g_{n_1n_2}\, y$, for all $x, y \in L$. If $x \npreceq n_1(y)$, then the inequality $x \,\&_{n_1n_2}\, y \preceq \top = x \,\&^g_{n_1n_2}\, y$ holds. Otherwise, if $x \preceq n_1(y)$, then the equality $x \,\&_{n_1n_2}\, y = \bot$ is obtained for all $x, y \in L$, by Proposition 21. In addition, when $x \preceq n_1(y)$, we have that $x \,\&^g_{n_1n_2}\, y = \bot$. As a consequence, we obtain that $(\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2}) \sqsubseteq_{\mathcal{T}_{n_1n_2}} (\&^g_{n_1n_2}, \swarrow^{n_{1g}}, \nwarrow_{n_{2g}})$, for all $(\&_{n_1n_2}, \swarrow^{n_1}, \nwarrow_{n_2}) \in \mathcal{T}_{n_1n_2}$. $\square$

Therefore, the previous result has detailed the algebraic structure of the pair $(\mathcal{T}_{n_1n_2}, \sqsubseteq_{\mathcal{T}_{n_1n_2}})$, showing that the supremum of any subset of adjoint triples in a complete lattice always exists and this set also has a greatest element. However, this structure is not a complete lattice since the least adjoint triple does not exist, as the following example shows.

Example 29. Once again, we will consider the complete lattice $(L, \preceq)$ and the pair of weak negations (n_1, n_2) displayed in Figure 1 of Example 12. We can easily see that the triples $(\&^a_{n_1n_2}, \swarrow^{n_{1a}}, \nwarrow_{n_{2a}})$ and $(\&^b_{n_1n_2}, \swarrow^{n_{1b}}, \nwarrow_{n_{2b}})$ defined as:

$$x \,\&^a_{n_1n_2}\, y = \begin{cases} a & \text{if } x \npreceq n_1(y) \\ \bot & \text{if } x \preceq n_1(y) \end{cases} \qquad x \,\&^b_{n_1n_2}\, y = \begin{cases} b & \text{if } x \npreceq n_1(y) \\ \bot & \text{if } x \preceq n_1(y) \end{cases}$$

$$z \swarrow^{n_{1a}} y = \begin{cases} n_1(y) & \text{if } a \npreceq z \\ \top & \text{if } a \preceq z \end{cases} \qquad z \swarrow^{n_{1b}} y = \begin{cases} n_1(y) & \text{if } b \npreceq z \\ \top & \text{if } b \preceq z \end{cases}$$

$$z \nwarrow_{n_{2a}} x = \begin{cases} n_2(x) & \text{if } a \npreceq z \\ \top & \text{if } a \preceq z \end{cases} \qquad z \nwarrow_{n_{2b}} x = \begin{cases} n_2(x) & \text{if } b \npreceq z \\ \top & \text{if } b \preceq z \end{cases}$$

for all $x, y, z \in L$, are incomparable adjoint triples belonging to $\mathcal{T}_{n_1n_2}$. Indeed, the triples $(\&^a_{n_1n_2}, \swarrow^{n_{1a}}, \nwarrow_{n_{2a}})$ and $(\&^b_{n_1n_2}, \swarrow^{n_{1b}}, \nwarrow_{n_{2b}})$ are the minimal elements of $\mathcal{T}_{n_1n_2}$. In order to obtain an adjoint triple lesser than both previous ones, we need to consider a conjunctor operator being constantly bottom. However, the conjunctor operator being constantly bottom gives rise to an adjoint triple which does not belong to $\mathcal{T}_{n_1n_2}$ and therefore, $\mathcal{T}_{n_1n_2}$ has not a least element.

Obviously, we can find other adjoint triples generating the pair of weak negations (n_1, n_2), being greater than $(\&^a_{n_1n_2}, \swarrow^{n_{1a}}, \nwarrow_{n_{2a}})$ and $(\&^b_{n_1n_2}, \swarrow^{n_{1b}}, \nwarrow_{n_{2b}})$. By using the point-wise ordering among the adjoint conjunctors and Theorem 28, we have that these adjoint triples can be hierarchized in a proper non linear complete join-semilattice. $\square$

5. Conclusions and further work

This paper has proven that the Galois implications pairs associated with a (pair of) weak negations form a complete lattice, with a minimum and a maximum element. A similar study has been carried out with respect to adjoint triples generating these negations, obtaining that the set of adjoint triples forms a join-semilattice with a maximum element. In addition, the definitions of these operators, associated with a (pair of) weak negations, have been characterized. The characterization of Galois implications pairs has been introduced from a family of antitone Galois connections whereas the characterization of adjoint triples has been given through a family of operators where only the supremum of non-empty sets is required. These characterizations will be very useful in future applications, since they provide an extra flexibility level, allowing the use of a bigger range of operators. On the other hand, the hierarchies established in this paper allow the user to select the most suitable adjoint triple or pair, depending on the most useful negation operator to be considered in each practical problem.

In the future, the obtained results will be applied to particular frameworks, such as, in fuzzy relation equations, formal concept analysis, rough set theory, etc. Moreover, they will be used in real problems, such as, the ones associated with image processing.

References

[1] C. Alcalde, A. Burusco, J. Díaz, R. Fuentes-González, and J. Medina. Fuzzy property-oriented concept lattices in morphological image and signal processing. *Lecture Notes in Computer Science*, 7903:246–253, 2013.

[2] C. Alcalde, A. Burusco, J. C. Díaz-Moreno, and J. Medina. Fuzzy concept lattices and fuzzy relation equations in the retrieval processing of images and signals. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 25(Supplement-1):99–120, 2017.

[3] L. Antoni, S. Krajči, and O. Krídlo. Representation of fuzzy subsets by Galois connections. *Fuzzy Sets and Systems*, 326:52 – 68, 2017. Si: FSTA 2016.

[4] M. J. Asiain, H. Bustince, R. Mesiar, A. Kolesárová, and Z. Takáč. Negations with respect to admissible orders in the interval-valued fuzzy set theory. *IEEE Transactions on Fuzzy Systems*, 26(2):556–568, April 2018.

[5] E. Bartl. Minimal solutions of generalized fuzzy relational equations: Probabilistic algorithm based on greedy approach. *Fuzzy Sets and Systems*, 260(0):25 – 42, 2015.

[6] R. Bělohlávek and M. Trneckova. Factorization of matrices with grades via essential entries. *Fuzzy Sets and Systems*, 360:97 – 116, 2019.

[7] I. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, and B. De Baets. On the construction of adjunctions between a fuzzy preposet and an unstructured set. *Fuzzy Sets and Systems*, 320:81 – 92, 2017. Theme Logic and Algebra.

[8] I. P. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, and B. De Baets. Galois connections between a fuzzy preordered structure and a general fuzzy structure. *IEEE Transactions on Fuzzy Systems*, 26(3):1274–1287, 2018.

[9] I. Chajda. A representation of residuated lattices satisfying the double negation law. *Soft Computing*, 22(6):1773–1776, Mar 2018.

[10] P. Cintula, E. P. Klement, R. Mesiar, and M. Navara. Residuated logics based on strict triangular norms with an involutive negation. *Mathematical Logic Quarterly*, 52(3):269–282, 2006.

[11] P. Cintula, E. P. Klement, R. Mesiar, and M. Navara. Fuzzy logics with an additional involutive negation. *Fuzzy Sets and Systems*, 161(3):390–411, 2010. Fuzzy Logics and Related Structures.

[12] M. E. Cornejo, F. Esteva, J. Medina, and E. RamÃrez-Poussa. Relating adjoint negations with strong adjoint negations. In J. M. L. Kóczy, editor, *Proc. 7th European Symposium on Computational Intelligence and Mathematics*, pages 66–71, 2015.

[13] M. E. Cornejo, D. Lobo, and J. Medina. Syntax and semantics of multi-adjoint normal logic programming. *Fuzzy Sets and Systems*, 345:41 – 62, 2018. Theme : Logic.

[14] M. E. Cornejo, D. Lobo, and J. Medina. On the solvability of bipolar max-product fuzzy relation equations with the product negation. *Journal of Computational and Applied Mathematics*, 354:520 – 532, 2019.

[15] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. A comparative study of adjoint triples. *Fuzzy Sets and Systems*, 211:1–14, 2013.

[16] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Attribute reduction in multi-adjoint concept lattices. *Information Sciences*, 294:41 – 56, 2015.

[17] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Multi-adjoint algebras versus extended-order algebras. *Applied Mathematics & Information Sciences*, 9(2L):365–372, 2015.

[18] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Multi-adjoint algebras versus non-commutative residuated structures. *International Journal of Approximate Reasoning*, 66:119–138, 2015.

[19] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Adjoint negations, more than residuated negations. *Information Sciences*, 345:355 – 371, 2016.

[20] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Characterizing reducts in multi-adjoint concept lattices. *Information Sciences*, 422:364 – 376, 2018.

[21] C. Cornelis, J. Medina, and N. Verbiest. Multi-adjoint fuzzy rough sets: Definition, properties and attribute selection. *International Journal of Approximate Reasoning*, 55:412–426, 2014.

[22] D. Darais and D. V. Horn. Constructive Galois connections. *Journal of Functional Programming*, 29:e11, 2019.

[23] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[24] M. E. Della Stella and C. Guido. Associativity, commutativity and symmetry in residuated structures. *Order*, 30(2):363–401, 2013.

[25] K. Denecke, M. Erné, and S. L. Wismath, editors. *Galois Connections and Applications*. Kluwer Academic Publishers, Dordrecht. The Netherlands, 2004.

[26] J. C. Díaz-Moreno and J. Medina. Multi-adjoint relation equations: Definition, properties and solutions using concept lattices. *Information Sciences*, 253:100–109, 2013.

[27] J. C. Díaz-Moreno and J. Medina. Using concept lattice theory to obtain the set of solutions

of multi-adjoint relation equations. *Information Sciences*, 266(0):218–225, 2014.

[28] M. Erné, J. Koslowski, A. Melton, and G. Strecker. A primer on galois connections. In *York Academy of Science*, 1992.

[29] F. Esteva. Negaciones en retículos completos. *Stochastica*, I:49–66, 1975.

[30] F. Esteva and X. Domingo. Sobre funciones de negación en [0,1]. *Stochastica*, IV:141–166, 1980.

[31] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[32] F. Esteva, E. Trillas, and X. Domingo. Weak and strong negation functions in fuzzy set theory. In *Proc. XI Int. Symposium on Multivalued Logic*, pages 23–26, 1981.

[33] F. García-Pardo, I. Cabrera, P. Cordero, M. Ojeda-Aciego, and F. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences*, 286:173 – 187, 2014.

[34] G. Georgescu and A. Popescu. Non-commutative fuzzy structures and pairs of weak negations. *Fuzzy Sets and Systems*, 143:129–155, 2004.

[35] O. Krídlo and M. Ojeda-Aciego. *An Adjoint Pair for Intuitionistic L-Fuzzy Values*, pages 167–173. Springer International Publishing, Cham, 2019.

[36] N. Madrid and M. Ojeda-Aciego. Measuring inconsistency in fuzzy answer set semantics. *IEEE Transactions on Fuzzy Systems*, 19(4):605–622, Aug. 2011.

[37] N. Madrid, M. Ojeda-Aciego, J. Medina, and I. Perfilieva. L-fuzzy relational mathematical morphology based on adjoint triples. *Information Sciences*, 474:75 – 89, 2019.

[38] S. Massanet, J. Recasens, and J. Torrens. Fuzzy implication functions based on powers of continuous t-norms. *International Journal of Approximate Reasoning*, 83:265 – 279, 2017.

[39] J. Medina. Minimal solutions of generalized fuzzy relational equations: Clarifications and corrections towards a more flexible setting. *International Journal of Approximate Reasoning*, 84:33–38, 2017.

[40] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[41] J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Similarity-based unification: a multi-adjoint approach. *Fuzzy Sets and Systems*, 146:43–62, 2004.

[42] G. Moreno, J. Penabad, and C. Vázquez. Beyond multi-adjoint logic programming. *International Journal of Computer Mathematics*, 92(9):1956–1975, 2015.

[43] A. Pradera, G. Beliakov, H. Bustince, and B. D. Baets. A review of the relationships between implication, negation and aggregation functions from the point of view of material implication. *Information Sciences*, 329:357 – 380, 2016. Special issue on Discovery Science.

[44] A. Pradera, S. Massanet, D. Ruiz-Aguilera, and J. Torrens. The non-contradiction principle related to natural negations of fuzzy implication functions. *Fuzzy Sets and Systems*, 359:3 – 21, 2019. Theme: Many-valued Implications.

[45] S. Rasouli and Z. Zarin. On residuated lattices with left and right internal state. *Fuzzy Sets and Systems*, 373:37 – 61, 2019.

[46] W. San-Min. Logics for residuated pseudo-uninorms and their residua. *Fuzzy Sets and Systems*, 218(0):24–31, 2013. Theme: Logic and Algebra.

[47] M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, and H. Bustince. Strengthened ordered directionally monotone functions. links between the different notions of monotonicity. *Fuzzy Sets and Systems*, 357:151 – 172, 2019.

[48] Z. Shmuely. The structure of Galois connections. *Pacific Journal of Mathematics*, 54(2):209–225, 1974.

[49] E. Trillas. Sobre negaciones en la teoría de conjuntos difusos. *Stochastica*, III:47–60, 1979.