Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

The structure of quasi-transitive graphs avoiding a minor with applications to the domino problem

(Extended abstract)

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Abstract

An infinite graph is quasi-transitive if its vertex set has finitely many orbits under the action of its automorphism group. In this paper we obtain a structure theorem for locally finite quasi-transitive graphs avoiding a minor, which is reminiscent of the Robertson-Seymour Graph Minor Structure Theorem. We prove that every locally finite quasi-transitive graph G avoiding a minor has a tree-decomposition whose torsos are finite or planar; moreover the tree-decomposition is canonical, i.e. invariant under the action of the automorphism group of G. As applications of this result, we prove the following.

- Every locally finite quasi-transitive graph attains its Hadwiger number, that is, if such a graph contains arbitrarily large clique minors, then it contains an infinite clique minor. This extends a result of Thomassen (1992) who proved it in the 4-connected case and suggested that this assumption could be omitted.

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- Locally finite quasi-transitive graphs avoiding a minor are accessible (in the sense of Thomassen and Woess), which extends known results on planar graphs to any proper minor-closed family.
- Minor-excluded finitely generated groups are accessible (in the group-theoretic sense) and finitely presented, which extends classical results on planar groups.
- The domino problem is decidable in a minor-excluded finitely generated group if and only if the group is virtually free, which proves the minor-excluded case of a conjecture of Ballier and Stein (2018).

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DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-059`

1 Introduction

1.1 A structure theorem

A central result in modern graph theory is the Graph Minor Structure Theorem of Robertson and Seymour [16], later extended to infinite graphs by Diestel and Thomas [8]. This theorem states that any graph G avoiding a fixed minor has a *tree-decomposition*, such that each piece of the decomposition, called a *torso*, is close to being embeddable on a surface of bounded genus. A natural question is the following: if the graph G has non trivial symmetries, can we make these symmetries apparent in the tree-decomposition? In other worlds, do graph avoiding a fixed minor have a tree-decomposition as above, but with the additional constraint that the decomposition is *canonical*, i.e., invariant under the automorphism group of G? In this paper we answer this question positively for infinite, locally finite graphs G that are *quasi-transitive*, i.e., the vertex set of G has finitely orbits under the action of the automorphism group of G. This additional restriction has the advantage of making the structure theorem much cleaner: instead of being almost embeddable on a surface of bounded genus, each piece of the tree-decomposition is now simply finite or planar.

Theorem 1.1. *Every locally finite quasi-transitive graph avoiding the countable clique as a minor has a canonical tree-decomposition whose torsos are finite or planar.*

We also give the following more precise version of this result, which might be useful for applications.

Theorem 1.2. *Every locally finite quasi-transitive graph G avoiding the countable clique as a minor has a canonical tree-decomposition with adhesion at most 3 in which each torso is a minor of G, and is planar or has bounded treewidth.*

Interestingly, the proof does not use the original structure theorem of Robertson and Seymour [16] and its extension to infinite graphs by Diestel and Thomas [8]. Instead, we rely mainly on a series of results and tools introduced by Grohe [13] to study decompositions of finite 3-connected graphs into quasi-4-connected components, together with a result of Thomassen [18] on locally finite quasi-4-connected graphs. The main technical contribution of our work consists in extending the results of Grohe to infinite, locally finite graphs and in addition, making sure that the decompositions we obtain are canonical (in a certain weak sense). Our proof crucially relies on a recent result of Carmesin, Hamann, and Miraftab [7], which shows that there exists a canonical tree-decomposition that distinguishes all tangles of a given order (in our case, of order 4).

We now discuss some applications of Theorem 1.1.

1.2 Hadwiger number

As a consequence of Theorem 1.1, we obtain a result on the Hadwiger number of locally finite quasi-transitive graphs. The *Hadwiger number* of a graph G is the supremum of the sizes of all complete minors in G. We say that a graph *attains its Hadwiger number* if the supremum above is attained, that is if it is either finite, or G contains an infinite clique minor. Thomassen [18] proved that every locally finite quasi-transitive 4-connected graph attains its Hadwiger number, and suggested that the 4-connectedness assumption might be unnecessary. We prove that this is indeed the case.

Theorem 1.3. *Every locally finite quasi-transitive graph attains its Hadwiger number.*

1.3 Accessibility in graphs

We now introduce the notion of accessibility in graphs considered by Thomassen and Woess [19]. To distinguish it from the related notion in groups (see below), we will call it vertex-accessibility in the remainder of the paper. A *ray* in an infinite graph G is an infinite one-way path in G. Two rays of G are equivalent if there are infinitely many disjoint paths between them in G (note that this is indeed an equivalence relation). An *end* of G is an equivalence class of rays in G. When there is a set X of vertices of G, two distinct components C_1, C_2 of $G - X$, and two distinct ends ω_1, ω_2 of G such that for each $i = 1, 2$, all rays of ω_i have infinitely many vertices in C_i, we say that X *separates* ω_1 and ω_2. A graph G is *vertex-accessible* if there is an integer k such that for any two ends ω_1, ω_2 in G, there is a set of at most k vertices that separates ω_1 and ω_2.

It was proved by Dunwoody [12] (see also [14, 15] for a more combinatorial approach) that locally finite quasi-transitive planar graphs are vertex-accessible. Here we extend the result to graphs excluding a countable clique K_∞ as a minor, and in particular to any proper minor-closed family.

Theorem 1.4. *Every locally finite quasi-transitive K_∞-minor-free graph is vertex-accessible.*

1.4 Accessibility in groups

The notion of vertex-accessibility introduced above is related to the notion of accessibility in groups. Given a finitely generated group Γ, and a finite set of generators S, the *Cayley* *graph* of Γ with respect to the set of generators S is the edge-labelled graph $\mathrm{Cay}(\Gamma, S)$ whose vertex set is the set of elements of Γ and where for every two $g, h \in \Gamma$ we put an edge (g, h) labelled with $a \in S$ when $h = a \cdot g$. It is known that the number of ends of a Cayley graph of a finitely generated group does not depend of the choice of generators, so we can talk about the number of ends of a finitely generated group. A classical theorem of Stallings [17] states that if a finitely generated group Γ has more than one end, it can be split as a non-trivial free product with finite amalgamation, or as an HNN-extension over a finite subgroup. If any group produced by the splitting still has more than one end we can keep splitting it using Stallings theorem. If the process eventually finishes (with Γ being obtained from finitely many 0-ended or 1-ended groups using free products with amalgamation and HNN-extensions), then Γ is said to be *accessible*. Thomassen and Woess [19] proved that a finitely generated group is accessible if and only if at least one of its Cayley graphs is vertex-accessible, if and only if all of its Cayley graphs are vertex-accessible.

A finitely generated group is *minor-excluded* if at least one of its Cayley graphs avoids a finite minor. Similarly a finitely generated group is K_∞*-minor-free* if one of its Cayley graphs avoids the countable clique as a minor, and *planar* if one of its Cayley graphs is planar. Note that planar groups are minor-excluded and Theorem 1.3 immediately implies that a finitely generated group is minor-excluded if and only if it is K_∞-minor-free.

Droms [9] proved that finitely generated planar groups are finitely presented, while Dunwoody [11] proved that finitely presented groups are accessible, which implies that finitely generated planar groups are accessible. Theorem 1.4 immediately implies the following, which extends this result to all minor-excluded finitely generated groups, and equivalently to all finitely generated K_∞-minor-free groups.

Corollary 1.5. *Every finitely generated K_∞-minor-free group is accessible.*

In fact, combining Theorem 1.1 with techniques introduced by Hamann [14, 15] in the planar case, we prove the following stronger result which also implies Corollary 1.5 using the result of Dunwoody [11] that all finitely presented groups are accessible.

Theorem 1.6. *Every finitely generated K_∞-minor-free group is finitely presented.*

1.5 The domino problem

We refer to [2] for a detailed introduction to the domino problem. A *coloring* of a graph G with colors from a set Σ is simply a map $V(G) \to \Sigma$. The domino problem for a finitely generated group Γ together with a finite generating set S is defined as follows. The input is a finite alphabet Σ and a finite set $\mathcal{F} = \{F_1, \ldots, F_p\}$ of forbidden *patterns*, which are colorings with colors from Σ of the closed neighborhood of the neutral element 1_Γ in the Cayley graph $\mathrm{Cay}(\Gamma, S)$, viewed as an edge-labelled subgraph of $\mathrm{Cay}(\Gamma, S)$. The problem then asks if there is a coloring of $\mathrm{Cay}(\Gamma, S)$ with colors from Σ, such that for each $v \in \Gamma$, the coloring of the closed neighborhood of v in $\mathrm{Cay}(\Gamma, S)$ (viewed as an edge-labelled subgraph of Cay(Γ, S)), is not isomorphic to any of the colorings $F_1, \ldots, F_p$, where we consider isomorphisms preserving the edge-labels.

It turns out that the decidability of the domino problem for (Γ, S) is independent of the choice of the finite generating set S, hence we can talk of the decidability of the domino problem for a finitely generated group Γ. If we consider $\Gamma = (\mathbb{Z}^2, +)$, then the domino problem corresponds exactly to the well-known Wang tiling problem, which was shown to be undecidable by Berger in [6]. On the other hand, there is a simple greedy procedure to solve the domino problem in free groups, which admit trees as Cayley graphs. More generally, the domino problem is decidable in virtually free groups, which can equivalently be defined as finitely generated groups having a Cayley graph of bounded treewidth [2, 1]. A remarkable conjecture of Ballier and Stein [5] asserts that these groups are the only one having decidable domino problem.

Conjecture 1.7 (Domino problem conjecture [5])**.** *A finitely generated group has decidable domino problem if and only if it is virtually free.*

Recall that virtually free groups are precisely the groups having a Cayley graph of bounded treewidth, which is a property that is closed under taking minor. It is therefore natural to ask whether Conjecture 1.7 holds for minor-excluded groups (or equivalently, using Theorem 1.3 to K_∞-minor-free groups). Using Theorem 1.5, together with classical results on planar groups and recent results on fundamental groups of surfaces [3], we prove that this is indeed the case.

Theorem 1.8. *A finitely generated K_∞-minor-free group has decidable domino problem if and only if it is virtually free.*

2 Proof overview

2.1 Sketch of the proof of Theorems 1.1 and 1.2

Consider a locally finite quasi-transitive graph G. First note that if G is quasi-4-connected, the following result of Thomassen immediately implies that the trivial tree-decomposition has the desired properties, hence Theorem 1.2 can be seen as a generalization of it.

Theorem 2.1 ([18])**.** *Let G be a quasi-transitive, quasi-4-connected, locally finite graph which excludes the countable clique as a minor. Then G is planar or has finite treewidth.*

To deal with the more general case, the first step is to obtain a canonical tree-decomposition of G of adhesion at most 2 in which all torsos are minors of G that are 3-connected graphs, cycles, or complete graphs on at most 2 vertices. The existence of such a decomposition in the finite case is a well-known result of Tutte [20] and was proved in the locally finite case in [10]. For our proof we need to go one step further. A graph is said to be *quasi-*4*-connected* if it is 3-connected and for every set $S \subseteq V(G)$ of size 3 such that $G - S$ is not connected, $G - S$ has exactly two connected components and one of them consists of a single vertex.

A crucial step for us would be to prove a version of the following result of Grohe in which the tree-decomposition would be canonical, and which would hold for locally finite graphs.

Theorem 2.2 ([13])**.** *Every finite graph G has a tree-decomposition of adhesion at most* 3 *whose torsos are minors of G and are complete graphs on at most 4 vertices or quasi-*4*-connected graphs.*

However, as observed by Grohe, even in the finite case the decomposition he obtains is not canonical in general. Our main technical contribution is to extend Theorem 2.2 to locally finite graphs, while making sure that most of the construction (except the very end) is canonical. For this, we proceed in two steps. First, we use a result of [7] to find a canonical tree-decomposition of any 3-connected graph G that distinguishes all its tangles of order 4. Using this result, we show that we can assume that the graph under consideration admits a unique tangle $\mathcal{T}$ of order 4. We then follow the main arguments from [13] and show that G has a canonical tree-decomposition of adhesion 3 which is a star and whose torsos are all minors of G and finite, except for the torso H associated to the center of the star, which has the following property: there exists a matching $M \subseteq E(H)$ which is invariant under the automorphism group of G and such that the graph $H' := H/M$ obtained after the contraction of the edges of M is quasi-transitive, locally finite, and quasi-4-connected. In particular, Theorem 2.1 implies that H' is planar or has bounded treewidth. We then prove that even if H itself is not necessarily quasi-4-connected, it is still planar or has bounded treewidth, which is enough to conclude the proof of Theorem 1.2. The final step to prove Theorem 1.1 consists in refining the tree-decomposition to make sure that torsos of bounded treewidth are replaced by torsos of finite size (moreover, this refinement has to be done in a canonical way). □

2.2 Sketch of the proof of Theorem 1.8

Let Γ be a finitely generated K_∞-minor-free group. Let S be a finite set of generators such that $G := \mathrm{Cay}(\Gamma, S)$ excludes the countable clique as a minor. If Γ has 0 or 2 ends, then the domino problem is known to be decidable in Γ. Assume now that Γ (and thus G) is one-ended. As G is vertex-transitive, a result of Thomassen [18] implies that G is planar. It is known that one-ended planar groups contain fundamental groups of surfaces as subgroups of finite index. For such groups the domino problem is known to be undecidable [3], and this directly implies that the domino problem is undecidable in Γ. Finally, assume that Γ has an infinite number of ends. Theorem 1.5 implies that Γ can be described as the fundamental group of a finite graph of group H, whose vertex-groups Γ_v $(v \in V(H))$ all have at most one end and are subgroups of Γ. If all the vertex-groups Γ_v have 0 ends (or equivalently, are finite), then Γ is virtually free and the domino problem is known to be decidable in Γ. Otherwise at least one vertex-group Γ_v is one-ended. A result of Babai [4] implies that since Γ_v is a subgroup of Γ, Γ_v is also K_∞-minor-free. The proof above then shows that in this case the domino problem in undecidable in Γ_v, and thus also in Γ. □

Acknowledgement We thank Marthe Bonamy, Vincent Delecroix, Reinhard Diestel, Matthias Hamann, Raphael Jacobs, Paul Knappe, Johannes Carmesin and Jan Kurkofka for the helpful discussions.

References

[1] Yago Antolín. On Cayley graphs of virtually free groups. *Groups Complexity Cryptology*, 3(2):301–327, 2011.

[2] Nathalie Aubrun, Sebastián Barbieri, and Emmanuel Jeandel. About the Domino Problem for Subshifts on Groups. In V. Berthé and M. Rigo, editors, *Sequences, Groups, and Number Theory*, Trends in Mathematics, pages 331–389. Birkhäuser, Cham, 2018.

[3] Nathalie Aubrun, Sebastián Barbieri, and Etienne Moutot. The domino problem is undecidable on surface groups. In Peter Rossmanith, Pinar Heggernes, and Joost-Pieter Katoen, editors, *44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26-30, 2019, Aachen, Germany*, volume 138 of *LIPIcs*, pages 46:1–46:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[4] László Babai. Some applications of graph contractions. *Journal of Graph Theory*, 1(2):125–130, 1977.

[5] Alexis Ballier and Maya Stein. The domino problem on groups of polynomial growth. *Groups, Geometry, and Dynamics*, 12(1):93–105, 2018.

[6] Robert L. Berger. The undecidability of the domino problem. *Memoirs of the American Mathematical Society*, 1966.

[7] Johannes Carmesin, Matthias Hamann, and Babak Miraftab. Canonical trees of tree-decompositions. *Journal of Combinatorial Theory, Series B*, 152:1–26, 2022.

[8] Reinhard Diestel and Robin Thomas. Excluding a countable clique. *Journal of Combinatorial Theory, Series B*, 76(1):41–67, 1999.

[9] Carl Droms. Infinite-ended groups with planar Cayley graphs. *Journal of Group Theory*, 9(4):487–496, 2006.

[10] Carl Droms, Brigitte Servatius, and Herman Servatius. The structure of locally finite two-connected graphs. *Electronic Journal of Combinatorics*, 2, 01 1998.

[11] Martin J. Dunwoody. The accessibility of finitely presented groups. *Inventiones Mathematicae*, 81:449–458, 1985.

[12] Martin J. Dunwoody. Planar graphs and covers. *arXiv preprint arXiv:0708.0920*, 2007.

[13] Martin Grohe. Quasi-4-connected components. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, volume 55 of *LIPIcs*, pages 8:1–8:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

[14] Matthias Hamann. Planar transitive graphs. *Electronic Journal of Combinatorics*, 25:4.8, 2015.

[15] Matthias Hamann. Accessibility in transitive graphs. *Combinatorica*, 38(4):847–859, 2018.

[16] Neil Robertson and Paul Seymour. Graph minors. XVI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003.

[17] John Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, 1972.

[18] Carsten Thomassen. The Hadwiger number of infinite vertex-transitive graphs. *Combinatorica*, 12:481–491, 1992.

[19] Carsten Thomassen and Wolfgang Woess. Vertex-transitive graphs and accessibility. *Journal of Combinatorial Theory, Series B*, 58(2):248–268, 1993.

[20] W.T. Tutte. *Graph Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984.