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Bubbles and Crashes: A Tale of Quantiles

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Abstract

Periodically collapsing bubbles, if they exist, induce asymmetric dynamics in asset prices. In this paper, I show that unit root quantile autoregressive models can approximate such dynamics by allowing the largest autoregressive root to take values below unity at low quantiles, which correspond to price crashes, and above unity at upper quantiles, that correspond to bubble expansions. On this basis, I employ two unit root tests based on quantile regressions to detect bubbles. Monte Carlo simulations suggest that the two tests have good size and power properties, and can outperform recursive least-squares-based tests that allow for time variation in persistence. The merits of the two tests are further illustrated in three empirical applications that examine Bitcoin, U.S. equity and U.S. housing markets. In the empirical applications, special attention is given to the issue of controlling for economic fundamentals. The estimation results indicate the presence of asymmetric dynamics that closely match those of the simulated bubble processes.

Keywords: rational bubbles; unit root quantile autoregressions; cryptocurrencies; U.S. house prices; S&P 500
JEL Classification: C12, C22, G10, R30

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1 Introduction

Over the last decades, and especially after the financial crisis of 2007–09, a large literature has emerged that deals with the development and application of econometric tests of speculative bubbles. Motivated by the theoretical predictions of rational expectation models, this literature has mainly concentrated on testing for an explosive root in asset prices. In this vein, early studies on bubble detection employed conventional integration tests based on ordinary least squares, such as the standard Augmented Dickey Fuller (ADF). A major shortcoming of such tests is that they frequently fail to detect explosive dynamics when bubbles periodically collapse (Evans, 1991; van Norden, 1996). One of the reasons for this failure is that market crashes generate extreme realizations in the left tail of the conditional asset price distribution. Because the least-squares estimator is highly sensitive to extreme values, conventional tests often indicate stationarity even though the series under examination is inherently explosive.

In recent years, new econometric methodologies have been proposed that attempt to deal with this shortcoming by allowing for time variation in persistence (Astill et al., 2023, 2018; Homm and Breitung, 2012). Two tests that have gained substantial popularity are the supremum ADF (SADF) of Phillips et al. (2011) and the generalized SADF (GSADF) derived by Phillips et al. (2015a,b). To deal with the effect of market crashes on the test's performance, the SADF and GSADF use a recursive least-squares algorithm that estimates ADF regressions on subsamples of data and retrieves the maximum ADF statistic. The SADF employs a forward expanding window, while its extension, the GSADF, tests for exuberance using all possible subsamples of a time series given a minimum window size. Between the two procedures, the GSADF is particularly attractive because it minimizes the impact of previous boom-bust episodes on estimation and thereby is consistent with multiple changes in regime. Since the seminal work of Phillips et al. (2011) and Phillips et al. (2015a,b), numerous studies have contributed to the literature on recursive right-tailed unit root testing (e.g., Harvey et al., 2015, 2020, 2016; Phillips and Shi, 2018, 2019).

The present paper proposes an alternative approach for bubble detection based on unit root quantile autoregressive models (Galvao, 2009; Koenker and Xiao, 2004, 2006; Koenker et al., 2017). Compared to ordinary least squares, quantile methods provide a more robust and efficient approach in the presence of outliers and/or non-Gaussian error distributions. The main advantage of unit root quantile autoregressions, however, is that they offer a mechanism for estimating the full range of conditional quantile functions rather than relying on a single measure of conditional central tendency. Thus, instead of allowing for time variation in the degree of persistence like the SADF and GSADF tests, unit root tests based on quantile regressions allow for heterogeneous dynamics across quantiles. As shown in Sections 4 and 5, allowing for asymmetric dynamics is crucial for bubble detection. Intuitively, in the presence of periodically collapsing bubbles, autoregressive coefficient estimates increase with the quantile, taking values below unity at low quartiles, that correspond to market crashes, and above unity at upper quantiles, which correspond to bubble eruptions. Thus, researchers can test for speculative bubbles by examining the unit root property at high quantiles.

In this paper, I employ two unit root tests based on quantile autoregressions for bubble detection. The first is the coefficient-based U_n test and the second is a right-sided version of the Kolmogorov-Smirnov type QKS_α test proposed by Koenker and Xiao (2004). Monte Carlo simulation experiments indicate that both tests have good size properties under different error distributions and lag length specifications. Moreover, the two tests outperform the SADF and GSADF in detecting periodically collapsing bubbles by, in many cases, a substantial margin. The superior power properties of U_n and QKS_α can be attributed to the fact that, being full-sample tests, they use information from all bubble episodes collectively. Whereas, the SADF and GSADF test statistics are based on a single subsample of data which, in the presence of multiple relatively short-lived bubble episodes, may not provide a sufficiently strong signal to reject the null hypothesis (see, Phillips et al., 2011, Section 6.2).[1]

In addition to the Monte Carlo experiments, I employ unit root quantile autoregressions to examine the presence of speculative dynamics in three distinct asset markets. The first empirical application deals with a leading cryptoasset known for its remarkably wild price fluctuations, Bitcoin. The second examines the S&P 500 index, which is a key benchmark for U.S. equity performance; and the final investigates U.S. housing, which constitutes a critical component of U.S. household wealth and the U.S. economy as a whole. Depending on data availability, I account for economic fundamental influences when testing for bubbles by adopting either a direct approach based on observed measures (i.e., price-dividend and price-rent ratios) or/and a recently proposed indirect approach based on futures prices (Pavlidis et al., 2017, 2018). The empirical applications provide novel insights about the persistence properties of the time series. In summary, the results for the unit root quantile autoregressive models suggest that there is substantial heterogeneity in persistence across quantiles for all three assets, with the observed pattern of autoregressive coefficient estimates closely resembling that for periodically collapsing bubbles. Not surprisingly, among the three assets, Bitcoin is found to be the most speculative.

The rest of the paper is structured as follows. Section 2 describes the theoretical asset pricing framework. Section 3 provides an outline of unit root tests based on quantile autoregressions. Sections 4 and 5 demonstrate the applicability of quantile autoregression models for bubble detection and investigate the finite-sample properties of unit root tests through Monte Carlo simulations. Section 6 outlines the indirect approach adopted to account for market fundamentals. Section 7 presents the three empirical applications, and the final section concludes.

[1] The fact that U_n and QKS_α are full-sample tests also implies that they do not come with an accompanying date-stamping procedure. That is, the proposed quantile approach provides rich information about the persistence of a series across quantiles but not over time. Note, however, that for rational periodically collapsing bubbles, such as those proposed by Evans (1991), date-stamping is not meaningful because these processes alternate between two explosive regimes, with the switch from the high growth to the low growth regime involving a market crash, see Section 2. Hence, with the exception of the periods of collapse, these bubble processes grow exponentially throughout the sample.

2 Speculative Dynamics in Asset Prices

I consider rational expectation models in which the spot price of an asset, P_t, consists of an economic fundamentals component, x_t, and a speculative bubble component, B_t:

$$P_t = x_t + B_t. \tag{1}$$

There are many asset pricing models that take this general form. Among others, these include the dividend-discount model for stocks (Campbell and Shiller, 1988), Pindyck's (1993) model of rational commodity pricing, the present-value relation for housing prices (Glaeser and Nathanson, 2015; Meese and Wallace, 1994), and the monetary model of exchange rate determination (Devereux and Smith, 2021). In these theoretical formulations, the fundamental component of asset prices is a function of dividends, convenience yields, housing rents, and relative money supplies and relative income, respectively, and is typically assumed to follow a unit root process. On the contrary, in empirical applications, the intrinsic value of an asset as well as its time series properties are unknown. This gives rise to the well-known joint hypothesis problem, which states that econometric tests for bubbles actually examine a composite hypothesis of no bubbles and the correct model for fundamentals. For simplicity, I assume for now that x_t follows a random walk process:

$$x_t = x_{t-1} + \epsilon_t, \qquad \epsilon_t \sim iid(0, \sigma_\epsilon^2), \tag{2}$$

and return to the important issue of controlling for unobserved economic fundamentals in empirical applications in Section 6.

With regard to the second component of the spot price, rational bubbles satisfy the condition (see, Diba and Grossman, 1988)

$$E_t(B_{t+1}) = (1 + r)B_t, \tag{3}$$

where r is a positive constant derived from the structural model describing the economy and $E_t(\cdot)$ is the expectation operator. Aside from the linear AR(1), several nonlinear bubble processes have been proposed that meet (3). These nonlinear processes appear more realistic in that they allow the bubble to exhibit rich dynamics, growing exponentially in some periods and crashing in others (Blanchard and Watson, 1982; Evans, 1991; Homm and Breitung, 2012). I adopt the widely used periodically collapsing bubble process proposed by Evans (1991),

$$B_{t+1} = \begin{cases} (1+r)B_t\eta_{t+1}, & \text{if } \mathrm{B}_t \leq b, \\ \left[\lambda + \frac{1}{\pi}(1+r)\zeta_{t+1}\left(B_t - \frac{1}{(1+r)}\lambda\right)\right]\eta_{t+1}, & \text{if } B_t > b, \end{cases} \tag{4}$$

where λ and b are positive constants that satisfy $\lambda < (1 + r)b$; $\eta_{t+1} = \exp(\kappa_{t+1} - \sigma_\kappa^2/2)$ with $\kappa_{t+1} \sim \mathcal{N}(0, \sigma_\kappa^2)$ is a lognormal variable scaled to have a mean of unity; and ζ_{t+1} is a Bernoulli process that takes the value of one with probability π and the value of zero with probability $1 - \pi$. By taking expectations of both sides, it is easy to verify that the expected gross growth rate of the bubble is $1+r$ and thus the process satisfies the condition for a rational bubble. The bubble process has also the appealing property that, conditional on a positive initial value B_0, B_t remains positive for all future time periods $t > 0$. Consider, for example, a bubble that starts below the threshold b. The bubble initially grows exponentially at a constant expected rate of $1 + r$. Eventually the bubble exceeds b and its expected growth rate, conditional on the bubble not collapsing, increases to $(1 + r)/\pi$. When the bubble collapses, it drops to a positive expected value of λ, and the cycle begins again.

A direct implication of condition (3) and Equation (1) is that, if bubbles are present in asset markets, then the spot price will display explosive dynamics. Starting with the seminal paper of Diba and Grossman (1988), a large literature has developed that employs right-tailed unit root tests to examine the presence of speculative dynamics in asset markets on the basis of this implication. This literature has mainly focused on least-squares unit root tests, leaving tests based on quantile autoregressions unexplored.

3 Unit Root Quantile Autoregressions

Consider the Augmented Dickey Fuller (ADF) regression equation,

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \sum_{j=1}^{q} \alpha_{j+1} \Delta y_{t-j} + u_t. \tag{5}$$

Following Koenker and Xiao (2004, 2006), I adopt the corresponding unit root quantile autoregression (QAR) model given by[2]

$$Q_{y_t}(\tau|\Delta y_{t-1}, \ldots, \Delta y_{t-q}) = \alpha_0(\tau) + \alpha_1(\tau) y_{t-1} + \sum_{j=1}^{q} \alpha_{j+1}(\tau) \Delta y_{t-j} + Q_u(\tau). \tag{6}$$

In the above QAR model, the τth conditional quantile of y_t, $Q_{y_t}(\tau|\Delta y_{t-1}, \ldots, \Delta y_{t-q})$, is a linear function of the lagged value of the series, q lagged first differences, and the τth quantile of u_t, $Q_u(\tau)$. Notice that regression parameter values are allowed to vary across quantiles. As an implication, the null hypothesis of a unit root,

$$H_0 : \alpha_1(\tau) = 1, \tag{7}$$

may be rejected in favour of the one-sided alternative of explosive dynamics in y_t,

$$H_1 : \alpha_1(\tau) > 1, \tag{8}$$

at some but not all quantiles.

Letting $\mathcal{F}_t$ denote the σ-field generated by $\{u_s, s \leq t\}$, and defining $x_t = (1, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-q})^\intercal$

[2] For a detailed exposition of the general unit root quantile autoregression with deterministic terms and covariates see (Galvao, 2009).

and $\alpha(\tau) = (\alpha_0(\tau) + Q_u(\tau), \alpha_1(\tau), \dots, \alpha_{q+1}(\tau))^\intercal$, Equation (6) can be written more compactly as

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = x_t^\intercal \alpha(\tau). \tag{9}$$

For a selected value of τ, parameter estimation requires solving the minimization problem

$$\min_{\alpha \in \mathcal{R}} \sum_{t=1}^{n} \rho_\tau(y_t - x_t^\intercal \alpha), \tag{10}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ is the check loss function of Koenker and Bassett Jr (1978). The solution $\hat{\alpha}(\tau)$ for a given τ is called the τth autoregression quantile; while, viewed as a function of τ, $\hat{\alpha}(\tau)$ is referred to as the QAR process.

Koenker and Xiao (2004) propose several unit root tests based on (6). The two tests employed in this paper are the coefficient-based $U_n(\tau)$ (analogous to the coefficient-based ADF) and a right-sided version of their Kolmogorov-Smirnov type test, QKS_α. The latter examines the unit root property over a range of quantiles, $\tau \in \mathcal{T}$, by exploiting the QAR process. The two test statistics are given by

$$U_n(\tau) = n(\hat{\alpha_1}(\tau) - 1) \qquad \text{and} \qquad \text{QKS}_\alpha = \sup_{\tau \in \mathcal{T}} \; U_n(\tau). \tag{11}$$

In practice, the $U_n(\tau)$ statistic can be computed over a grid of values, and the QKS_α statistic can be obtained by taking the maximum value.

Under the null, both test statistics have nonstandard limiting distributions that depend on nuisance parameters. Accurate finite-sample critical values can be computed, however, using the following bootstrap unit root procedure:

1. Let $\omega_t = \Delta y_t$. Impose the null of a unit root and estimate the restricted regression

$$\omega_t = \sum_{j=1}^{q} \xi_j \omega_{t-j} + v_t,$$

   by ordinary least squares to obtain the coefficient estimates $\hat{\xi}_1, \dots, \hat{\xi}_q$ and the residuals $\hat{v}$.

2. Generate bootstrap residuals, v_t^b, by sampling with replacement draws from the centered residuals $\hat{v}_t - 1/(n-q) \sum_{j=q+1}^{n} \hat{v}_j$.

3. Use the bootstrap residuals and the estimated coefficients to recursively generate bootstrap samples for first differences,

$$\omega_t^b = \sum_{j=1}^{q} \hat{\xi}_j \omega_{t-j}^b + v_t^b,$$

   and for levels,

$$y_t^b = y_{t-1}^b + w_t^b,$$

   with $\omega_j^b = \Delta y_j$ for $j = 1, \dots, q$ and $y_1^b = y_1$.

4. Estimate the quantile autoregression (6) and compute the coefficient-based and Kolmogorov-Smirnov statistics, $U_n^b(\tau)$ and QKS_α^b.

5. Repeat steps (2) to (4) N times to approximate the limiting distributions under the null, and compute the bootstrap p-value as the percentage of times the simulated statistic is as or more extreme than the original.

4 Periodically Collapsing Bubbles and Quantile Autoregressions

To illustrate the applicability of unit root quantile autoregressions for bubble detection, I first examine a single realization from the theoretical asset pricing model described by Equations (1), (2), and (4). The sample size n is set equal to 200 observations and the values of the structural parameters describing the economy are $\pi = 0.5$, $r = 0.015$, $\lambda = 0.5$, $b = 1$, $\sigma_k = 0.05$, $\sigma_\epsilon = 0.7$, $B_0 = 0.5$, and $x_0 = 30$. Following previous studies, I scale up the bubble process by a factor of 20.

Figure 1 displays the simulated price and fundamental series. What stands out is the large bubble eruption occurring before the middle of the sample period, with the asset price increasing from less than 40 to almost 80 units in a short period of time. At the peak of the bubble, the speculative component constitutes over half of the asset price. This remarkable market boom ends at $t = 93$ with a dramatic market collapse that erases all capital gains. A second bubble eruption takes place in the last part of the sample, but in this case the associated price rally is not substantial. This is due to the fact that, on the one hand, the bubble eruption is smaller in magnitude and, on the other, it coincides with a reduction in the value of fundamentals. Nonetheless, there is a sizeable market crash at t =169 when the bubble pops and the asset price implodes by 26 percent.

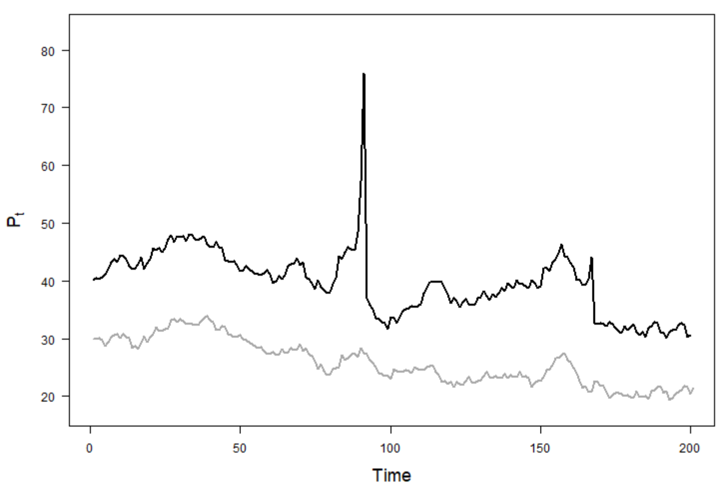

Figure 1: Simulated price (black line) and economic fundamental (grey line) series.

The market booms and crashes are also apparent in Figure 2a. This figure presents the scatter plot of the simulated price series against its lagged value. Superimposed on the plot are the fitted quantile regression lines corresponding to $\tau = \{0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95\}$ and the fitted least-squares line. To facilitate the analysis, the accompanying Figure 2b shows the QAR slope coefficient estimates against the same set of quantiles.

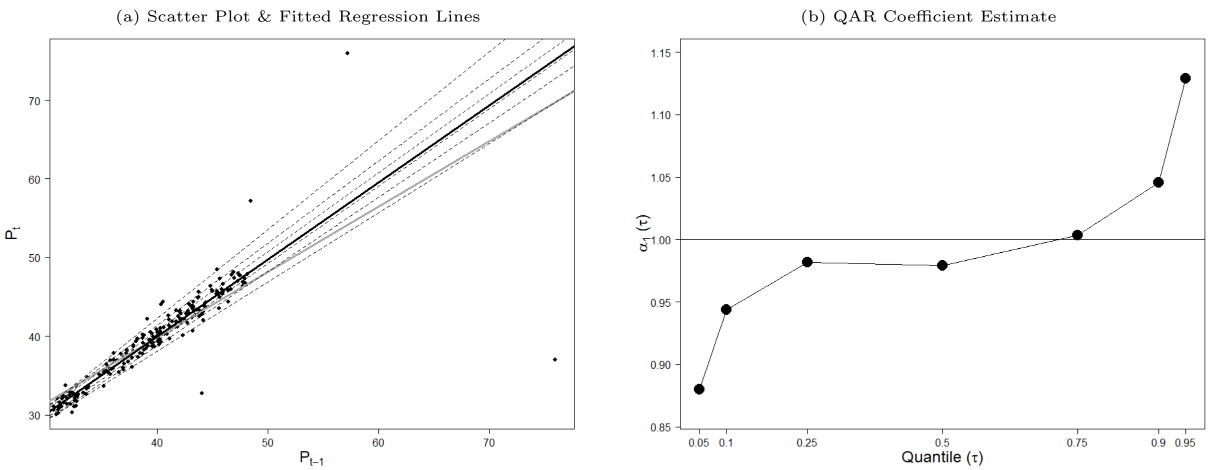

Figure 2: Scatter plot of simulated P_t against P_{t-1} with fitted quantile and least-squares regression lines (left). Median regression (solid black line), quantile regressions at $\tau = \{0.05, 0.1, 0.25, 0.75, 0.9, 0.95\}$ (dashed lines), and least-squares regression (solid grey line). QAR coefficient estimates against τ (right).

Starting with the least-squares results, it is evident that, between the extreme observations generated by market booms and market crashes, the latter have the largest influence on the regression line. Specifically, the abrupt price declines drive the least-squares estimate far below unity ($\hat{\alpha}_1 = 0.83$), falsely indicating a stationary process. This finding highlights the sensitivity of standard unit root tests to outliers and is in accord with previous studies that show that such tests have virtually no power to detect periodically collapsing bubbles (Evans, 1991; Franses and Haldrup, 1994; Harvey et al., 2001; Lucas, 1995; Phillips et al., 2011).

The quantile regression results, on the other hand, convey much richer information regarding the properties of the simulated asset price series. As can be seen from Figures 2a and 2b, the results suggest the presence of asymmetric dynamics, with the estimates for the autoregressive root increasing in a nonlinear fashion as we move from low to high quantiles. Specifically, the curve depicting the QAR process $\hat{\alpha}_1(\tau)$ is relative flat in the central region, while it displays a steep incline at the boundaries. At low quantiles, $\tau = \{0.05, 0.1\}$, the autoregressive coefficient estimates take small values, substantially below unity due to the market crashes. At $\tau = \{0.25, 0.5, 0.75\}$, the estimates are close to unity. While, at the upper quantiles $\tau = \{0.9, 0.95\}$ that correspond to bubble eruptions, they exceed unity, indicating that the time series exhibits explosive behaviour.

To generalize the above results, I now conduct a set of Monte Carlo experiments with 10,000 replications. For these experiments, the sample size n is set equal to 100, 200, 400, and 1,000 observations, and the probability π of the bubble not collapsing is set equal to 25, 50, 75 and 90 percent. The values for the remaining structural parameters are the same as in the previous exercise. Figure 3 and Table 1 report mean α_1 estimates for quantile and least-squares autoregressions of order one, respectively.

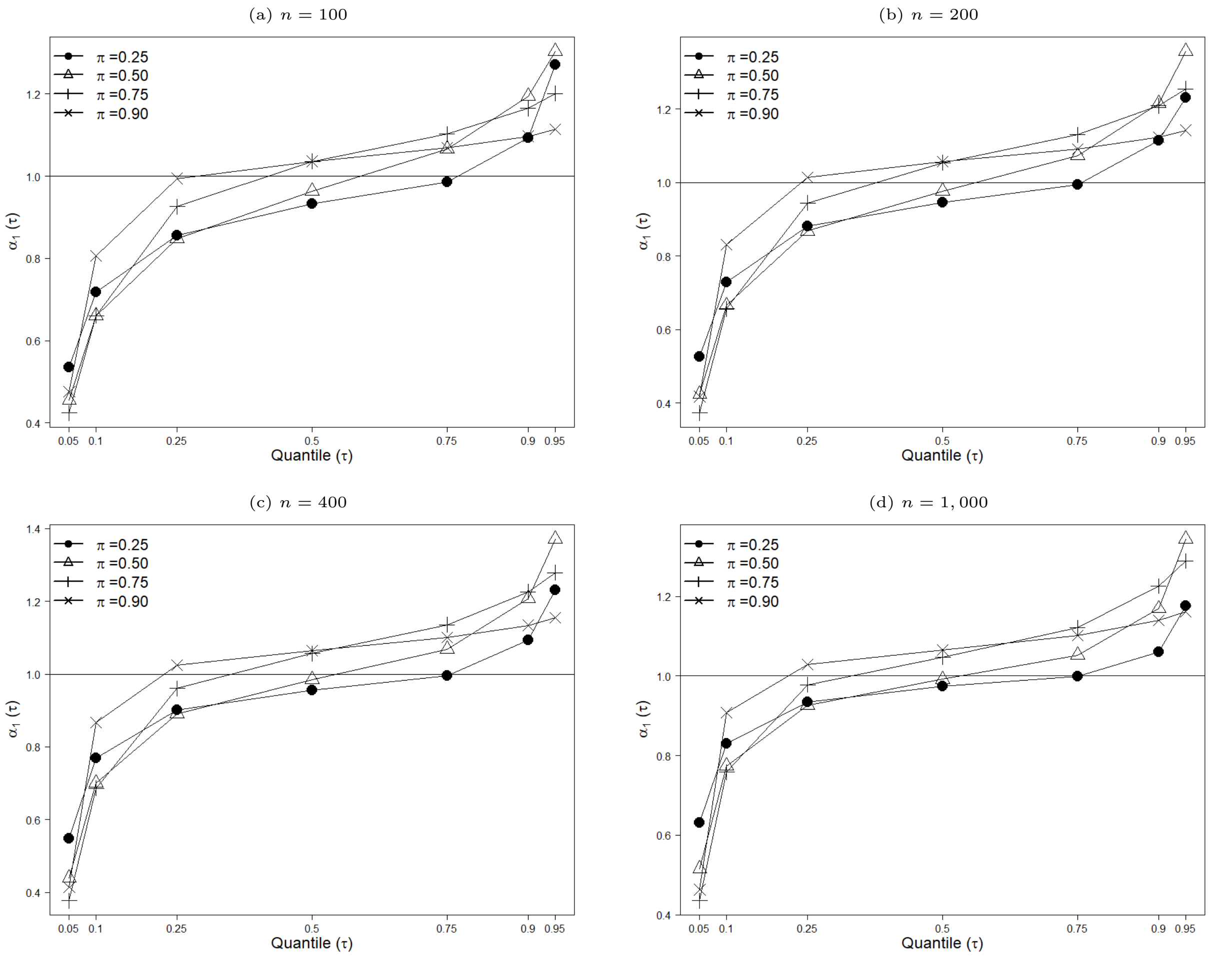

Figure 3: Mean quantile estimates of the QAR coefficient for different sample sizes n and probabilities of the bubble erupting π.

Several interesting conclusions emerge by visual inspection of Figure 3. The main conclusion is that, irrespective of the sample size and of the probability of collapse, the autoregressive coefficient estimates display a similar overall pattern: They monotonically increase with τ, always starting from values below unity at low quantiles and exceeding unity at higher quantiles. A closer look at the figure reveals that the range of mean α_1 estimates is wide, from less than 0.4 at the fifth quantile to almost 1.4 at the 95th quantile. Furthermore, it reveals that the quantile at which mean estimates exceed unity heavily depends on the probability of the bubble not collapsing. Not surprisingly, higher values of π are associated with lower quantiles. The curves corresponding to lower πs, however, exhibit a steeper incline at upper quantiles and thus take large values near the boundary ($\tau = 0.95$). This last observation can be explained by the fact that the number of sample periods characterized by a bubble erupting is positively related to π, while the intensity of the bubble, as measured by $(1 + r)/\pi$, negatively. That is, lower values of π result in fewer but more intense bubble eruptions, raising the autoregressive coefficient estimate more rapidly at upper quantiles. The main implication of the above observations is that researchers can test for speculative bubbles by examining the null hypothesis of a unit root against the alternative of explosive dynamics at sufficiently high quantiles.

Table 1: Mean least-squares α_1 estimates in the presence of periodically collapsing bubbles

	$n = 100$	$n = 200$	$n = 400$	$n = 1,000$
$\pi = 0.25$	0.745	0.702	0.672	0.652
$\pi = 0.50$	0.753	0.722	0.705	0.694
$\pi = 0.75$	0.814	0.802	0.797	0.794
$\pi = 0.90$	0.885	0.883	0.887	0.890

Notes: n and π denote the sample size and the probability of the bubble erupting, respectively.

It is also interesting to look at the results for median (least absolute deviations) and least-squares regressions. For median regressions, average $\alpha_1(0.5)$ estimates are positively related to the probability of the bubble erupting (see Figure 3). When this probability is high (i.e., $\pi = \{0.75, 0.9\}$), they take values above unity. In contrast, the average least-squares estimates reported in Table 1 are always below unity, with a minimum value of 0.652 (for $\pi = 0.25$, $n = 1,000$), and a maximum of 0.89 (for $\pi = 0.9$, $n = 1,000$). Hence, unit root tests based on least absolute deviations are expected to outperform their least-squares counterparts in detecting bubbles, but perform worse compared to tests focusing on upper quantiles. I examine the performance of unit root tests next.

5 Monte Carlo Results: Empirical Size and Power

This section deals with the empirical size and power properties of $U_n(\tau)$ and QKS_α. Since bubbles drive the autoregressive coefficient above unity at high quantiles, I set $\tau = \{0.8, 0.85, 0.9, 0.95\}$ for the $U_n(\tau)$ test and, I consider the range $\mathcal{T} = [0.8, 0.95]$ for QKS_α and a step size of 0.01. In addition to $U_n(\tau)$ and QKS_α, I also present results for the standard ADF, the SADF of Phillips et al. (2011) and the GSADF test of Phillips et al. (2015a). These tests are described in the Appendix. For all experiments, the number of Monte Carlo simulations is set equal to 1,000, the number of bootstrap repetitions N is 2,000, and the sample sizes n are $100, 200, 300$ and 400. Finally, the nominal significance level is set equal to five percent.

5.1 Empirical Size

I conduct two sets of size experiments to examine the properties of the tests under different error distributions and lag lengths. The data generating process is a driftless random walk,

$$y_t = y_{t-1} + u_t, \tag{12}$$

where the error term, u_t, is an i.i.d. random variable. Following Koenker and Xiao (2004), in the first set of experiments, I consider the standard normal and two heavy-tailed distributions for u_t: the Student-t with three degrees of freedom, t_3, and the Student-t with two degrees of freedom, t_2. In the latter case, errors have infinite variance.

Table 2: Size of unit root tests with $\mathcal{N}(0,1)$, t_3 and t_2 errors

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$\mathcal{N}(0,1)$									
n =100	0.053	0.056	0.063	0.052	0.057	0.055	0.050	0.042	0.049
n =200	0.059	0.045	0.050	0.050	0.045	0.046	0.046	0.044	0.035
n =300	0.057	0.058	0.064	0.060	0.051	0.051	0.057	0.050	0.043
n =400	0.050	0.050	0.053	0.052	0.055	0.048	0.056	0.055	0.045
t_3									
n =100	0.055	0.047	0.048	0.057	0.050	0.044	0.056	0.064	0.085
n =200	0.056	0.058	0.050	0.046	0.037	0.043	0.040	0.060	0.089
n =300	0.061	0.064	0.060	0.063	0.051	0.043	0.055	0.052	0.094
n =400	0.064	0.057	0.068	0.061	0.051	0.044	0.052	0.053	0.116
t_2									
n =100	0.055	0.043	0.042	0.038	0.043	0.036	0.047	0.083	0.119
n =200	0.053	0.042	0.044	0.050	0.051	0.040	0.053	0.093	0.144
n =300	0.041	0.047	0.049	0.053	0.065	0.052	0.044	0.082	0.146
n =400	0.048	0.058	0.053	0.052	0.055	0.054	0.043	0.078	0.156

Notes: n denotes the sample size.

According to the results in Table 2, there are no substantial deviations of the empirical size from the nominal significance level for $U_n(\tau)$ and QKS_α. Thus, quantile unit root tests perform well for all sample sizes and distributions, even for the case of infinite-variance errors. These findings complement the Monte Carlo results of Koenker and Xiao who do not focus on high quantiles, but instead look at unit root tests based on median quantile autoregressions and examine a two-tailed Kolmogorov-Smirnov type test with $\mathcal{T} = [0.1, 0.9]$. Regarding the least-squares tests, the standard ADF does not exhibit any significant size distortions and the SADF appears to be only slightly oversized for t_2 errors. The more flexible GSADF test, however, is slightly oversized for t_3 and moderately oversized for t_2 errors, with the size distortions increasing with the sample size. The maximum rejection rate of the GSADF is 15.6 percent for $n = 400$ and t_2 errors. Overall, aside from extremely heavy-tailed data, recursive least-squares tests perform reasonably well.

For the second set of size experiments, I focus on standard normal errors and consider lag lengths $q = 1, 3$ and 6. Table 3 reports the simulation results. The $U_n(\tau)$, QKS_α and ADF tests exhibit, again, good size properties with rejection rates close to the five percent level. The size properties of SADF and GSADF under different lag settings have already been examined in Phillips et al. (2015a). In line with the study of Phillips et al., the results in Table 3 indicate that the two tests exhibit size distortions that increase with the lag length and decrease with the sample size. The distortions are particularly severe for the GSADF test, reaching values as high as 67 percent. As argued by Phillips et al., the greater distortions for the GSADF test can be attributed to the

Table 3: Size of unit root tests for different lag lengths

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$n = 100$									
$q = 1$	0.029	0.046	0.043	0.050	0.055	0.053	0.063	0.054	0.095
$q = 3$	0.025	0.044	0.051	0.061	0.053	0.061	0.047	0.099	0.252
$q = 6$	0.023	0.033	0.046	0.048	0.054	0.054	0.066	0.187	0.670
$n = 200$									
$q = 1$	0.033	0.040	0.048	0.049	0.055	0.047	0.047	0.054	0.081
$q = 3$	0.033	0.032	0.037	0.037	0.050	0.042	0.051	0.084	0.190
$q = 6$	0.031	0.044	0.048	0.055	0.078	0.069	0.051	0.140	0.479
$n = 300$									
$q = 1$	0.033	0.043	0.046	0.052	0.076	0.070	0.052	0.061	0.093
$q = 3$	0.033	0.043	0.054	0.059	0.060	0.062	0.068	0.087	0.208
$q = 6$	0.030	0.045	0.043	0.046	0.047	0.061	0.044	0.095	0.383
$n = 400$									
$q = 1$	0.041	0.041	0.041	0.046	0.056	0.052	0.051	0.057	0.086
$q = 3$	0.036	0.041	0.045	0.040	0.051	0.057	0.048	0.060	0.172
$q = 6$	0.033	0.043	0.050	0.039	0.044	0.039	0.053	0.098	0.381

Notes: n and q denote the sample size and the lag length, respectively.

smaller sample sizes in the flexible-window procedure for larger values of q. Due to this property, the authors recommend selecting a very small lag length for empirical use of the SADF and GSADF test procedures.

5.2 Empirical Power

Having examined the empirical size properties of the tests, I now evaluate their ability to detect periodically collapsing bubbles. The design of the power experiments is the same as that of Section 2 with the exception that the structural parameter r takes values in $\{0.01, 0.015, 0.02\}$. Thus, the experiments allow us to investigate the role of the following factors in the performance of the tests: the sample size n, the probability of the bubble erupting π, the structural parameter r, and, for $U_n(\tau)$, the quantile τ. Rejection rates are reported in Table 4.

Overall, the results for the unit root tests based on quantile autoregressions are very encouraging. Among these tests, the QKS_α and $U_n(0.95)$ are found to perform best, with the former displaying the same or slightly higher power than the latter. The two tests outperform the SADF and GSADF for all simulation settings by, in many cases, a substantial margin. The mean relative difference between the QKS and SADF (GSADF) is 39.7 (17.5) percent, and the mean absolute difference is 21 (9.5) percentage points. Interestingly, the largest differences in power between least-squares and quantile regression based tests occur when the probability of the bubble erupting is low and thus there are many bubble eruptions but of short duration. For the QKS_α-SADF pair, the maximum relative difference is 95.4 percent ($r = 0.015, \pi = 0.25$, $n = 300$) and, for the QKS_α-GSADF pair, it is 77.7 percent ($r = 0.015, \pi = 0.25$, $n = 100$). Thus, in the presence of many, short-lived bubble episodes, it is much more informative to examine the persistence of asset prices at high quantiles over the entire sample period, instead of evaluating the persistence in the mean over subsamples. For large values of n and π, power differences remain substantial for the QKS_α-SADF pair but they become negligible for the QKS_α-GSADF since both tests almost always detect bubbles.

Table 4: Power of unit root tests in the presence of periodically collapsing bubbles

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$r = 0.01$, $\pi = 0.25$									
$n = 100$	0.077	0.260	0.300	0.325	0.339	0.375	0.024	0.226	0.211
$n = 200$	0.079	0.349	0.429	0.489	0.513	0.583	0.011	0.301	0.348
$n = 300$	0.060	0.423	0.499	0.588	0.634	0.701	0.001	0.361	0.474
$n = 400$	0.064	0.441	0.514	0.610	0.664	0.727	0.000	0.407	0.570
$r = 0.01$, $\pi = 0.50$									
$n = 100$	0.101	0.386	0.408	0.449	0.459	0.480	0.033	0.355	0.368
$n = 200$	0.113	0.540	0.609	0.665	0.692	0.729	0.014	0.490	0.593
$n = 300$	0.108	0.632	0.696	0.759	0.796	0.831	0.004	0.551	0.719
$n = 400$	0.120	0.675	0.736	0.791	0.833	0.866	0.003	0.622	0.813
$r = 0.01$, $\pi = 0.75$									
$n = 100$	0.321	0.535	0.555	0.580	0.592	0.607	0.044	0.465	0.496
$n = 200$	0.452	0.743	0.779	0.816	0.831	0.844	0.013	0.654	0.751
$n = 300$	0.535	0.829	0.880	0.906	0.923	0.930	0.017	0.733	0.867
$n = 400$	0.592	0.890	0.915	0.939	0.962	0.969	0.007	0.787	0.937
$r = 0.01$, $\pi = 0.90$									
$n = 100$	0.435	0.583	0.587	0.588	0.587	0.600	0.089	0.477	0.491
$n = 200$	0.651	0.820	0.829	0.852	0.842	0.863	0.038	0.690	0.777
$n = 300$	0.770	0.908	0.933	0.944	0.941	0.951	0.022	0.794	0.899
$n = 400$	0.848	0.961	0.966	0.969	0.978	0.978	0.016	0.838	0.948
$r = 0.015$, $\pi = 0.25$									
$n = 100$	0.091	0.350	0.392	0.432	0.479	0.535	0.007	0.335	0.351
$n = 200$	0.053	0.435	0.529	0.602	0.669	0.728	0.003	0.398	0.502
$n = 300$	0.057	0.475	0.569	0.673	0.762	0.813	0.000	0.447	0.617
$n = 400$	0.037	0.490	0.622	0.719	0.783	0.842	0.002	0.483	0.667
$r = 0.015$, $\pi = 0.50$									
$n = 100$	0.123	0.528	0.571	0.616	0.641	0.676	0.012	0.512	0.538
$n = 200$	0.101	0.693	0.761	0.805	0.858	0.871	0.005	0.637	0.760
$n = 300$	0.110	0.772	0.823	0.869	0.914	0.942	0.003	0.671	0.877
$n = 400$	0.090	0.796	0.869	0.916	0.951	0.963	0.002	0.677	0.922
$r = 0.015$, $\pi = 0.75$									
$n = 100$	0.441	0.702	0.735	0.767	0.795	0.800	0.029	0.643	0.704
$n = 200$	0.590	0.877	0.913	0.930	0.951	0.955	0.021	0.786	0.896

$n = 300$	0.709	0.954	0.963	0.981	0.986	0.989	0.011	0.813	0.964
$n = 400$	0.745	0.966	0.976	0.982	0.990	0.993	0.009	0.823	0.984
$r = 0.015$, $\pi = 0.90$									
$n = 100$	0.639	0.787	0.800	0.814	0.801	0.813	0.074	0.700	0.733
$n = 200$	0.828	0.940	0.953	0.963	0.959	0.969	0.034	0.841	0.927
$n = 300$	0.921	0.987	0.992	0.991	0.993	0.993	0.030	0.886	0.980
$n = 400$	0.956	0.989	0.991	0.994	0.994	0.994	0.019	0.894	0.988
$r = 0.02$, $\pi = 0.25$									
$n = 100$	0.077	0.390	0.459	0.527	0.564	0.630	0.005	0.405	0.435
$n = 200$	0.049	0.466	0.574	0.656	0.741	0.800	0.001	0.435	0.575
$n = 300$	0.039	0.497	0.624	0.730	0.796	0.858	0.000	0.439	0.643
$n = 400$	0.032	0.530	0.659	0.777	0.830	0.892	0.001	0.461	0.681
$r = 0.02$, $\pi = 0.50$									
$n = 100$	0.112	0.619	0.666	0.725	0.756	0.784	0.015	0.599	0.661
$n = 200$	0.091	0.768	0.835	0.864	0.910	0.925	0.006	0.662	0.844
$n = 300$	0.089	0.841	0.878	0.923	0.943	0.957	0.002	0.672	0.916
$n = 400$	0.080	0.865	0.914	0.952	0.979	0.988	0.000	0.667	0.934
$r = 0.02$, $\pi = 0.75$									
$n = 100$	0.544	0.817	0.837	0.872	0.889	0.900	0.025	0.753	0.831
$n = 200$	0.723	0.952	0.967	0.978	0.983	0.986	0.014	0.839	0.959
$n = 300$	0.811	0.984	0.993	0.992	0.993	0.996	0.008	0.857	0.983
$n = 400$	0.846	0.995	0.996	0.996	0.998	0.999	0.005	0.815	0.997
$r = 0.02$, $\pi = 0.90$									
$n = 100$	0.781	0.889	0.901	0.904	0.903	0.919	0.087	0.815	0.864
$n = 200$	0.930	0.978	0.985	0.988	0.989	0.993	0.032	0.892	0.975
$n = 300$	0.973	0.996	0.998	0.997	0.998	1.000	0.026	0.920	0.996
$n = 400$	0.986	1.000	1.000	1.000	1.000	1.000	0.016	0.917	0.999

Notes: n and π denote the sample size and the probability of the bubble erupting; r is a structural parameter that determines the growth rate of the bubble process.

In addition to the superior performance of QKS_α and $U_n(0.95)$ over their least-squares counterparts, several other conclusions emerge from Table 4. Focusing on the $U_n(\tau)$ tests with $\tau = \{0.80, 0.85, 0.9, 0.95\}$, rejection rates generally increase with the sample size n, the structural parameter r, the probability of the bubble erupting π, and the quantile τ. The positive relation between power and sample size is intuitively obvious. The same is true for the structural parameter r. This factor determines the value of the growth rate of the bubble process and, consequently, the degree of explosiveness of the asset price. When its value increases, the asset price becomes more explosive which raises power.

As far as the impact of the probability of the bubble erupting on the power of $U_n(\tau)$ is concerned, an increase in π has two effects that work in opposite directions. On the one hand, it lowers the growth rate of the ongoing bubble, which *ceteris paribus* leads to lower power of unit root tests. On the other, it increases the expected bubble duration which, in turn, raises power. This twofold impact reflects the complex dynamics at play in the model. The simulation results indicate that, at least for the simulation design used in this paper, the first effect outweighs the second so that higher values of π lead to higher rejections rates.

Regarding the last factor, the finding that the power of $U_n(\tau)$ generally increases with τ is in line with the analysis of the previous section which shows that higher quantiles are associated with larger autogressive root estimates. It should be noted, however, that the relationship between τ and the power of $U_n(\tau)$ depends on π and, for large values of π, it becomes flat. This observation can be attributed to the fact that, as the probability of the bubble regime increases, the estimated autoregressive roots exceed unity at lower quantiles (see Figure 3). Hence, the magnitude of differences in power across quantiles declines. It should also be noted that the distribution of data is more sparse for upper quantiles, which may lead to less precise estimates. As a consequence, increasing τ beyond a point may actually lead to a drop in power. From this perspective, the QKS_α test has the major advantage over $U_n(\tau)$ of allowing the researcher to remain agnostic about the optimal τ.

Turning to $U_n(0.5)$, the results of the previous section indicate that, for median regressions, average α_1 estimates are positively related to π, only exceeding unity when π is higher than 50 percent. Consistent with this result, the power of $U_n(0.5)$ is very low when π is at or below the 50 percent threshold, and abruptly increases as π changes from 50 to 75 percent. For large n, r and π, $U_n(0.5)$ exhibits good power properties and, in a few cases, outperforms the SADF test, but not the GSADF. In contrast, as expected, the standard ADF has virtually no power to detect bubbles in all cases. Before proceeding to the empirical applications, I turn to the issue of controlling for economic fundamentals.

6 Controlling for Economic Fundamentals

Testing for speculative bubbles is confounded by the fact that the fundamental value of assets is unobserved. Early studies have attempted to address this issue by utilizing observed variables suggested by theory, such as dividends and rents. The main drawback of such direct approaches is that they crucially depend on the highly unrealistic assumption that the true model for fundamentals is known. Model misspecification or omitted variables can lead to false inference in favour of bubbles, rendering direct approaches invalid (Gürkaynak, 2008).

To circumvent this obstacle, more recent studies have employed indirect approaches that exploit information about market fundamentals incorporated in derivative prices or survey data (Pavlidis et al., 2017, 2018). These studies show that, during the expansion phase of periodically collapsing bubbles, actual realizations of future spot prices and market expectations diverge. Under general conditions, this difference between actual prices and market expectations solely depends on the bubble process. As an implication, instead of using observed fundamental to proxy for intrinsic asset values, researchers can employ measures of market expectations.

To illustrate this most simply, consider the theoretical model of Section 2 and let F_t denote the market forecast for the price of the asset one period ahead

$$F_t = E_t(P_{t+1}) = E_t(x_{t+1}) + E_t(B_{t+1}). \tag{13}$$

It follows from (4) that, conditional on the bubble erupting, the market forecast for the speculative component of the asset price is biased

$$B_{t+1} - E_t(B_{t+1}) = (1+r)\left(\frac{1}{\pi}\eta_{t+1} - 1\right) B_t + \left(1 - \frac{1}{\pi}\right) \lambda\eta_{t+1}. \tag{14}$$

This bias arises because rational agents at time t correctly attach a nonzero probability to the bubble bursting at $t+1$. As a consequence, the expected growth rate of the bubble component $1+r$ is lower than the actual rate $(1+r)/\pi$. Being a function of B_t, the forecast error $B_{t+1} - E_t(B_{t+1})$ displays explosive dynamics. This property is propagated to the cumulative demeaned forecast errors for the asset price

$$P_t^f = \sum_{i=1}^{t} (\nu_i^f - \bar{\nu}^f), \tag{15}$$

where $\nu^f = P_{t+1} - E_t(P_{t+1})$ and $\bar{\nu}^f = 1/n \sum_{i=1}^{n} \nu^f$. Assuming that the forecast error for fundamentals is not explosive, researchers can test for bubbles by running unit root tests on P_t^f.[3] The main advantage of this approach is that, by exploiting the information incorporated in market expectations, it does not require the specification of market fundamentals and, thus, ameliorates the joint hypothesis problem. In practice, market expectations can be approximated by futures prices or survey data.

7 Empirical Applications

This section presents three empirical applications of the proposed bubble detection methods to Bitcoin, U.S. equity, and U.S. housing markets. All of these markets have a rich history of price run-ups and market crashes and, for this reason, they have attracted substantial attention by academics, policy makers and the media.

The Bitcoin Market. For the first application, I employ monthly data on Bitcoin spot and futures prices in U.S. dollars. The two series are downloaded from Bloomberg and span the period December 2017 to June 2023. The start date of the data set is chosen to coincide with the commencement of Bitcoin futures trading on the Chicago Mercantile Exchange.

Figure 4a shows the evolution of the spot price over time. Evidently, the series exhibits wild

[3]A wide range of processes for x_t satisfy the condition that the forecast error $x_{t+1} - E_t(x_{t+1})$ does not grow exponentially in-sample, including explosive linear AR models and nonlinear models, such as threshold and smooth transition autoregressive.

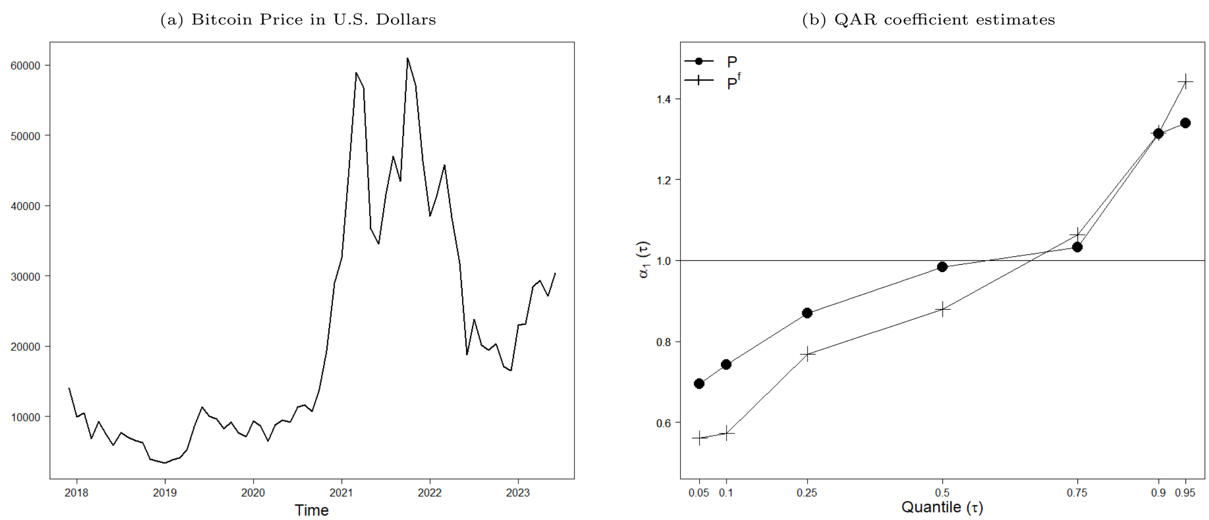

Figure 4: Bitcoin price in U.S. dollars (P_t) from December 2017 to June 2023 (left). Point estimates of the largest autoregressive root at different quantiles for P_t and P_t^f (right).

fluctuations within the sample. Starting at 10,706 dollars in September 2020, the price surged by 450 percent in a period of six months, reaching 58,960 dollars in March 2021. During the price surge, on January 11, the UK Financial Conduct Authority issued a warning about the risks of investments advertising high returns based on cryptoassets, highlighting that "If consumers invest in these types of product, they should be prepared to lose all their money".[4] A few days earlier, on January 8, Michael Hartnett, chief investment strategist at Bank of America Securities, referred to Bitcoin as the "mother of all bubbles". Consistent with these concerns, the Bitcoin price plummeted by 40 percent between March and June 2021 to 34,585 dollars. The cryptoasset experienced a notable rebound in the subsequent months, hitting an all-time high of 60,975 dollars in October 2021. This peak was, again, short-lived and by June 2022 Bitcoin had lost almost 70 percent of its value, reaching a low of 18,731 dollars.

Aside from its remarkably turbulent behaviour, testing for speculative dynamics in the Bitcoin market is interesting because, unlike other assets, cryptocurrencies lack traditional fundamental valuation metrics. Indeed, a widely held view among academics and policy makers is that Bitcoin is a purely speculative asset with an intrinsic value of zero (Benigno and Rosa, 2023). The following quote from Fabio Panetta, member of the Executive Board of the European Central Bank, is indicative of the prevailing view:[5] "Unbacked cryptos lack any intrinsic value, too. They are speculative assets. Investors buy them with the sole objective of selling them on at a higher price. In fact, they are a gamble disguised as an investment asset." On the contrary, those in investment and entrepreneurial circles claim that the price of Bitcoin reflects fundamental factors, such as the underlying blockchain technology (Pagnotta, 2022). In the absence of conventional

[4] https://www.fca.org.uk/news/news-stories/fca-warns-consumers-risks-investments-advertising-high-returns-based-cryptoassets

[5] https://www.ecb.europa.eu/press/blog/date/2023/html/ecb.blog230105~75d5aee900.es.html

valuation metrics, accounting for possible fundamental influences using futures prices can offer valuable insights.

As a preliminary exercise, I fit the unit root quantile autoregressive model (6) to Bitcoin P_t and P_t^f. In this and the remaining two empirical applications, I use the Akaike Information Criterion (AIC) with a maximum lag order of six to choose the lag length q in non-recursive unit root autoregressive models, and set q equal to zero for the SADF and GSADF tests following the recommendation of Phillips et al. (2015a). Figure 4b displays the QAR process. The α_1 estimates indicate the presence of asymmetric dynamics in both series that closely resemble those of the simulated bubbles of Section 4. Specifically, the values of $\hat{\alpha}_1$ start far below unity at low quantiles, they increase with τ and eventually exceed unity at $\tau \geq 0.75$, indicating that the two series exhibit explosive behaviour at upper quantiles. The range of values of $\hat{\alpha}_1$ is remarkably wide. For P_t^f, the lowest α_1 estimate is below 0.6 at the fifth quantile and the highest above 1.4 at the 95th. For P_t, the range is somewhat narrower, from close to 0.7 to slightly below 1.4.

Table 5: Unit root test statistics for Bitcoin

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
P	2	-1.084	10.695*	18.557*	19.998*	21.724*	22.337*	-1.378	6.589*	7.639*
P^f	0	-7.935	11.590*	20.182*	20.797*	29.186*	29.789*	-1.718	5.672*	6.321*

Notes: * denotes significance at the five percent level, q is the lag length selected by AIC.

To formally test for the presence of bubbles, I investigate the integration properties of the two series. The unit root test results presented in Table 5 suggest that the null hypothesis can be rejected in favour of the alternative of explosive dynamics at the five percent level by the all tests but two. Not surprisingly, the two tests that fail to reject the null are the standard ADF and $U_n(0.5)$, i.e., those with the lowest empirical power. The finding that Bitcoin prices display explosive dynamics is in accordance with the recent empirical literature on bubbles in cryptoassets (Harvey et al., 2020).[6] Our study extends this literature by showing that P_t^f also displays explosive dynamics. By doing so, it provides novel evidence, based on an agnostic approach about market fundamentals, in favor of speculative bubbles.

The U.S. Equity Market. As a second empirical application, I examine the behavior of the Standard & Poor's 500 (S&P 500) index. This index goes back to the 19th century and constitutes a leading U.S. economic indicator as well as a benchmark for mutual and exchange-traded fund performance. Due to its importance and long history, it is one of the most commonly examined series for speculative dynamics.

I utilize two data sets that cover the recent period, following the launch of the E-mini S&P 500 futures contracts, from January 1998 to June 2023. The first data set consists of the S&P 500 real

[6] Harvey et al. (2020) test for speculative bubbles in daily Bitcoin prices by running a novel sign-based recursive unit root test that is robust to deterministically time-varying volatility. Because time-varying volatility may be a feature of lower frequency data, as a robustness exercise, I run a wild-bootstrap version of the SADF and GSADF tests on P_t and P_t^f. The test statistics remain statistically significant at the five percent level.

price index and an observed stock market fundamentals series, namely real dividends. The second includes S&P 500 spot and E-mini futures prices.[7] The two data sets are downloaded from Robert Shiller's website and Bloomberg, respectively.

Figure 5a depicts the S&P 500 real price index. As can be seen from the figure, the period under examination is characterized by several market rallies and crashes. (For a historical perspective, see Shiller (2015).) The most notable include the dot-com bubble at the beginning of the sample period and the subsequent market crash; the severe market collapse associated with the 2007-09 financial crisis; the market crash at the onset of the Covid pandemic in February and March of 2020; and, finally, the price rally of 2020-21 and the abrupt price reversal of 2022.

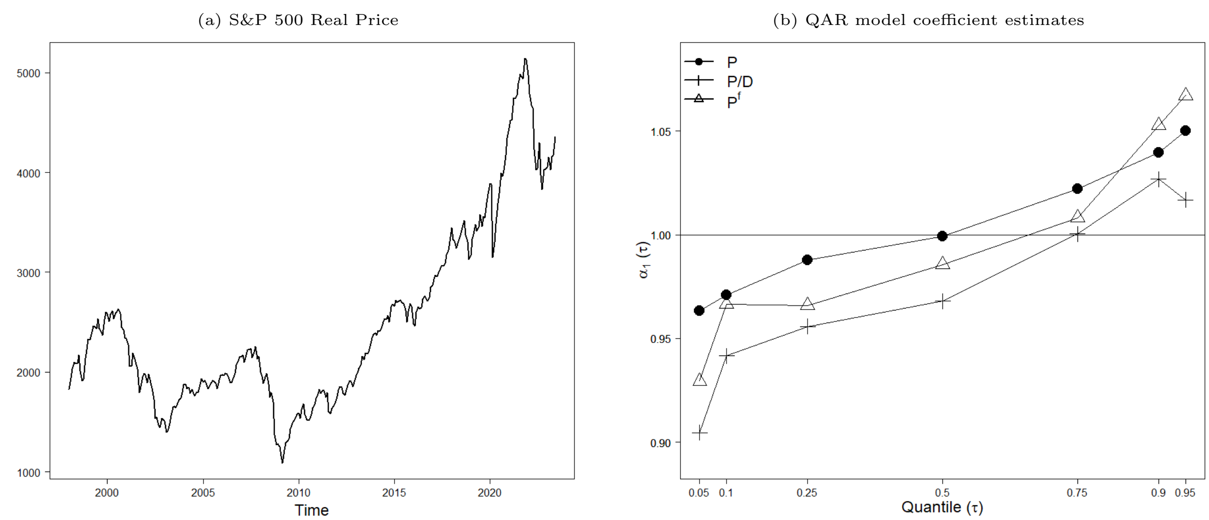

Figure 5: The S&P 500 real price index (P_t) from January 1998 to June 2023 (left). Point estimates of the largest autoregressive root at different quantiles for P_t, P_t/D_t and P_t^f (right).

Figure 5b displays the quantile autoregressive coefficient estimates for the S&P 500 real price index, P_t, the price-to-dividend ratio, P_t/D_t, and the price adjusted for fundamentals using information from the futures market, P_t^f. Like for Bitcoin, the results indicate the presence of asymmetric dynamics with $\hat{\alpha}_1$ increasing with the value of τ and exceeding unity at $\tau \geq 0.75$. The range of estimates is, however, substantially narrower compared to Bitcoin. The minimum estimate is close to 0.9 at $\tau = 0.05$ and the maximum is slightly above 1.05 at $\tau = 0.95$. This finding reflects the highly speculative nature of the market for cryptocurrencies. It is also interesting to note that, like for Bitcoin, the P_t^f series displays a higher degree of asymmetry (as measured by the difference in the degree of persistence near the extremes) compared to P_t. While, a comparison between P_t and P_t/D_t provides mixed results. Specifically, the α_1 estimates for P_t/D_t always lie below those for P_t so that absolute deviations from unity are larger for the former series at low but not at high quantiles.

[7] S&P 500 futures contracts expire four times a year in March, June, September, and December. Similarly to Chernenko et al. (2004) and Pavlidis et al. (2017), I obtain regularly spaced futures prices by linear interpolation of spot prices and near-term contracts.

Table 6: Unit root test statistics for S&P 500

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
P	1	-0.269	7.439*	11.570*	12.031*	15.205*	15.205*	-0.120	2.155*	2.397*
P/D	1	-9.742	0.715	8.386*	8.174*	5.121*	11.925*	-1.971	0.571	2.701*
P^f	2	-4.322	4.192*	6.126*	15.975*	20.362*	21.308*	-1.127	1.006	3.294*

Notes: * denotes significance at the five percent level, q is the lag length selected by AIC.

Table 6 presents the unit root test statistics for P_t, P_t/D_t, and P_t^f. The GSADF, QKS_α, and $U_n(\tau)$ tests with $\tau = \{0.85, 0.9, 0.95\}$ are consistent, indicating rejection of the null hypothesis for all series at the five percent level. On the contrary, the ADF and $U_n(0.5)$ statistics are not significant for any of the series, and the SADF statistic is only significant for P_t. Thus, in line with previous studies, the statistical evidence in this paper favours the presence of speculative dynamics in the S&P 500 (Homm and Breitung, 2012; Phillips et al., 2015a).

The U.S. Housing Market. The boom-bust episode in international housing markets from the late 1990s to the late 2000s generated a vast interest in the dynamics of house prices. A view shared by many academics and policy makers is that this episode was associated with house prices departing from their fundamental values, distorting investment decisions and leading to the 2008-09 global recession. Since then, many studies have provided evidence supporting this conjecture (Engsted et al., 2016; Pavlidis et al., 2016; Shi, 2017).

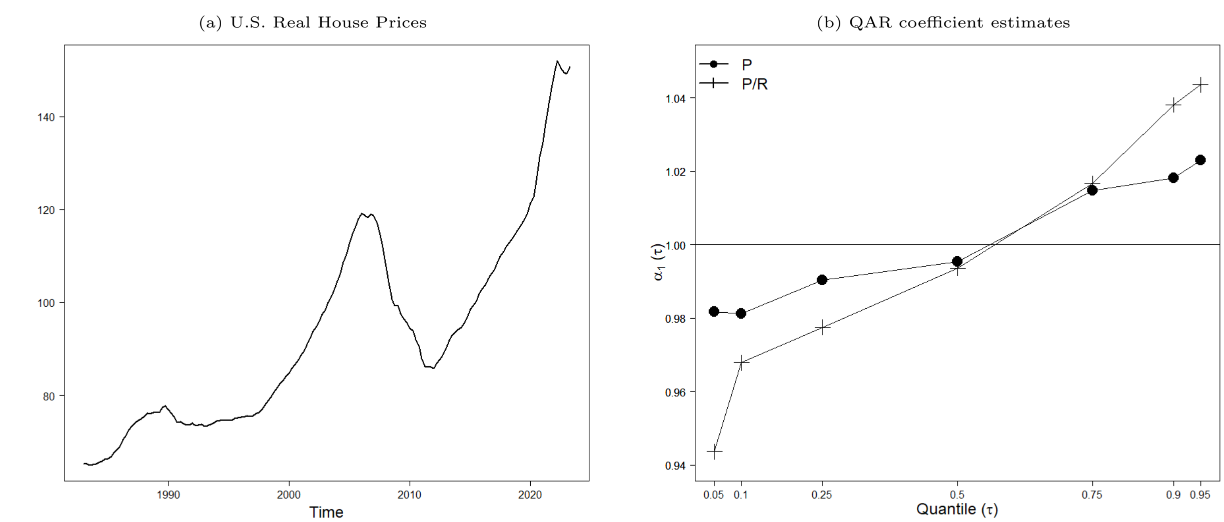

Figure 6: The U.S. real house price index (P_t) from 1983:Q1 to 2023:Q2 (left). Point estimates of the largest autoregressive root at different quantiles for P_t and P_t/R_t (right).

As a final application, I re-examine the integration properties of U.S. real house prices, P_t, and the U.S. house-price-to-rent ratio, P_t/R_t, using the quantile autoregressive model. The underlying series are downloaded from the OECD database and span the period from the 1st quarter of 1983 to the 2nd quarter of 2023. Regarding derivative prices, the Chicago Mercantile Exchange began trading housing futures contracts in May 2006 to allow investors to hedge against real estate risks. Unfortunately, since their launch, these securities have not attracted substantial attention by investors. Because the absence of sufficient liquidity can have a detrimental effect on the price discovery process, I omit futures prices from the analysis.

Figure 6a shows the evolution of the real house price index over time. Aside from the episode of the late 1990s/2000s, the U.S. housing market experienced a shorter and less intense boom-bust episode in the late 1980s/early 1990s; and displayed an impressive price surge after the Covid-19 recession. The latter rally in house prices raised issues of affordability and concerns about another U.S. housing bubble brewing, with the Federal Reserve Chairman Jerome Powell explicitly referring to a "housing bubble" and "housing prices going up at very unsustainable levels and overheating".[8]

Table 7: Unit root test statistics for U.S. house prices

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS$_\alpha$	ADF	SADF	GSADF
P	4	-0.732	2.265*	2.459*	2.841*	3.618*	3.726*	0.123*	9.903*	17.102*
P/R	4	-1.011	2.966*	4.161*	5.975*	6.835*	7.553*	-0.809	10.755*	19.662*

Notes: * denotes significance at the five percent level, q is the lag length selected by AIC.

Quantile autoregressive coefficient estimates and unit root test statistics are reported in Figure 6b and Table 7, respectively. Like for Bitcoin and S&P 500, there is a positive relationship between $\hat{\alpha}_1$ and τ. For house prices, the value of $\hat{\alpha}_1$ starts close to 0.98 at $\tau = 0.05$ and marginally exceeds 1.02 at $\tau = 0.95$. While, for the house-price-to-rent ratio, the coefficient curve is steeper, with a value of $\hat{\alpha}_1$ slightly above 0.94 at $\tau = 0.05$ and a value close to 1.04 at $\tau = 0.95$. Turning to the unit root test results, all statistics exceed their five percent critical value with the exception of $U_n(0.5)$, that fails to reject the null for both series, and the ADF, that only rejects the null for P_t. Thus, both recursive unit root tests and tests based on unit root quantile autoregressions detect explosive dynamics in the U.S. housing market.

8 Conclusion

In this paper, I introduced a novel approach for testing for periodically collapsing bubbles using unit root quantile autoregressions. The underlying idea is that, within a conditional quantile regression framework, bubble expansions drive the largest autoregressive root of asset price series above unity at upper quantiles. From the different quantile autoregression unit root tests proposed in the literature, I employed the coefficient-based U_n and a modified version of the Kolmogorov-Smirnov QKS$_\alpha$ of Koenker and Xiao (2004) to examine this hypothesis. Monte Carlo experiments demonstrated that the two tests have good finite-sample size and power properties, and can outperform the popular recursive least-squares SADF and GSADF.

An application to Bitcoin, the S&P 500 index, and the U.S. housing market, showed that the null of a unit root can be rejected at upper quantiles for all markets. Furthermore, it revealed

[8] Speech at the Brookings Institute Hutchins Center on Fiscal and Monetary Policy on 30th of November, 2022 `https://www.brookings.edu/events/federal-reserve-chair-jerome-powell-the-economic-outlook-and-the-labor-market/`

significant heterogeneity in persistence across quantiles, closely resembling the asymmetric dynamics implied by periodically collapsing bubbles. This finding is robust to accounting for economic fundamentals by using price-to-fundamental ratios and/or an indirect approach based on futures prices. Overall, the Monte Carlo experiments and the empirical applications suggest that unit root quantile autoregressions can provide new insights into the speculative dynamics characterizing asset markets and, thus, are a useful addition to the bubble detection toolkit of applied researchers.

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Appendix

The ADF, SADF, and GSADF Tests

This Appendix describes the ADF, SADF and GSADF tests, and provides technical details for the implementation of the latter two. At the heart of all tests lies the ADF regression equation:

$$y_t = \alpha_0^{r_1,r_2} + \alpha_1^{r_1,r_2} y_{t-1} + \sum\nolimits_{j=1}^{q} \alpha_{j+1}^{r_1,r_2} \Delta y_{t-j} + u_t, \qquad (16)$$

where $u_t \sim \mathcal{N}(0, \sigma^2_{r_1,r_2})$ is a Gaussian random variable, and $\alpha_0^{r_1,r_2}$, $\alpha_1^{r_1,r_2}$ and $\alpha_{j+1}^{r_1,r_2}$ with $j = 1, \ldots, k$ are regression coefficients. The above equation differs from Equation (5) in the manuscript in that it includes the superscripts r_1 and r_2. These superscripts denote fractions of the total sample size T that specify the starting and ending points of a subsample period. The null hypothesis of interest is that of a unit root, $H_0 : \alpha_1^{r_1,r_2} = 0$, against the alternative of explosive behavior in y_t, $H_1 : \alpha_1^{r_1,r_2} > 0$. The statistic corresponding to this null is given by,

$$\mathrm{ADF}_{r_1}^{r_2} = \widehat{\alpha}_1^{r_1,r_2} / \mathrm{s.e.}(\widehat{\alpha}_1^{r_1,r_2}). \tag{17}$$

ADF test. For the standard ADF test, the statistic in Equation (17) is obtained by estimating regression Equation (16) on the full sample of observations, i.e., by setting $r_1 = 0$ and $r_2 = 1$. Under the null of a unit root, the limit distribution of ADF_0^1 is given by,

$$\frac{\int_0^1 W dW}{\left(\int_0^1 W^2\right)^{1/2}},$$

where W is a Wiener process. Testing for exuberance entails comparing the ADF_0^1 statistic with the right-tailed critical value from its limit distribution.

Supremum ADF (SADF) test. Phillips et al. (2011) proposed a methodology that is consistent with a single boom-bust episode. Their methodology involves recursively estimating Equation (16) using a forward expanding sample. In this setting, the beginning of the subsample is held constant at $r_1 = 0$, while the end of the subsample, r_2, increases from r_0 (the minimum window size) to one (the entire sample period). Recursive estimation of Equation 16 yields a sequence of $ADF_0^{r_2}$ statistics. The supremum of this sequence, called the SADF statistic, is defined by,

$$\mathrm{SADF}(r_0) = \sup_{r_2 \in [r_0, 1]} \mathrm{ADF}_0^{r_2},$$

and has a limit distribution given by,

$$\sup_{r_2 \in [r_0,1]} \frac{\int_0^{r_2} W dW}{(\int_0^{r_2} W^2)^{1/2}}.$$

Similarly to the standard ADF test, rejection of the null of a unit root requires that the SADF statistic exceeds the right-tailed critical value from its limit distribution. However, contrary to the standard ADF test which examines the presence of explosive dynamics during the entire period, the alternative hypothesis of the SADF test is that of explosive dynamics in some part(s) of the sample.

Generalized SADF (GSADF) test. Phillips et al. (2015a,b) proposed an extension of the SADF, which has the same alternative hypothesis as the SADF, but which covers a larger number of subsamples. Specifically, given a minimum window size r_0, the methodology involves estimating the regression given in Equation (16) for all possible subsamples by allowing both the ending point, r_2, and the starting point, r_1, to change. This extra flexibility on the estimation window results in substantial power gains and makes the GSADF test better suited to detect the presence of multiple changes in regime.

Formally, the GSADF statistic is defined as,

$$\text{GSADF}(r_0) = \sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} \text{ADF}_{r_1}^{r_2},$$

and its limit distribution under the null is,

$$\sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} \left\{ \frac{\frac{1}{2} r_w [W(r_2)^2 - W(r_1)^2 - r_w] - \int_{r_1}^{r_2} W(r) dr [W(r_2) - W(r_1)]}{r_w^{1/2} \left\{ r_w \int_{r_1}^{r_2} W(r)^2 dr - \left[\int_{r_1}^{r_2} W(r) dr \right]^2 \right\}^{1/2}} \right\},$$

where $r_w = r_2 - r_1$ is the size of the expanding window. Again, rejection of the unit root hypothesis in favor of the alternative hypothesis of explosive behavior in some part(s) of the sample requires that the GSADF statistic exceeds the right-tailed critical value from its limit distribution.

The implementation of the unit root tests necessitates the selection of the minimum window size, r_0, and the distributions of the SADF and GSADF test statistics which are non-standard. Following Phillips et al. (2015a), for the minimum window size, we use the rule-of-thumb $r_0 = 0.01 + 1.8/\sqrt{T}$. To obtain finite-sample critical values, we simulate 2,000 random walk processes with $N(0,1)$ errors.