Ratio Mathematica Volume 51, 2024

Fixed Point Results for (ψ, ϕ)-Contractive Mapping in $G_{\mathcal{F}}$-Metric Space

Kapil Jain*
Jatinderdeep Kaur†
Satvinder Singh Bhatia‡

Abstract

The main goal of this work is to present $G_{\mathcal{F}}$-metric space, a new generalization of G-metric space. A comparison between the classes of G-metric spaces, GP-metric spaces, G_b-metric spaces, generalized G_b-metric spaces, and G^*-metric spaces and the class of $G_{\mathcal{F}}$-metric spaces is also presented. We examine a few fundamental aspects of this newly defined abstract space. Proving the Banach contraction principle and the fixed point result for (ψ, ϕ)-contractive mapping in the context of $G_{\mathcal{F}}$-metric spaces is the paper's secondary goal.
Keywords: Fixed point, G-metric space, G_b-metric space, $\mathcal{F}$-metric space, Contractive mapping.
2020 AMS subject classifications: 47H10; 54H25.[1]

*Department of Mathematics, RKSD College, Kaithal, Haryana, India; kjain_phdp16@thapar.edu.
†School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, Punjab, India; jkaur@thapar.edu.
‡School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, Punjab, India; ssbhatia@thapar.edu.
[1]Received on November 2, 2023. Accepted on January 15, 2024. Published on January 31, 2024. DOI:10.23755/rm.v51i0.1455. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Gahler [1] proposed the idea of 2-metric space, which is an extension of the famous concept of metric space (X,d). Various writers have demonstrated that there is no relation between the two functions. For example, Ha *et al.* [2] demonstrate that the 2-metric does not necessarily need to be continuous. Dhage [3] introduced the concept of D-metric space, a new class of generalized metric space, in 1992. Most of the assertions about the basic topological structure of D-metric space were later proved inappropriate by Mustafa and Sims [4], Naidu *et al.* [5, 6]. Therefore, Mustafa and Sims [7] created a more suitable concept, known as G-metric space.

Definition 1.1. *[7] Let $\mathcal{A}$ be a non-empty set and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function satisfying:*
$(G1)$ $G(\zeta, \eta, \vartheta) = 0$ *if* $\zeta = \eta = \vartheta$;
$(G2)$ $0 < G(\zeta, \zeta, \eta)$, *for all* $\zeta, \eta \in \mathcal{A}$ *with* $\zeta \neq \eta$;
$(G3)$ $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, *for all* $\zeta, \eta, \vartheta \in \mathcal{A}$ *with* $\vartheta \neq \eta$;
$(G4)$ $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \cdots$ (*symmetric in its variables*);
$(G5)$ $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta)$, *for all* $\zeta, \eta, \vartheta, a \in \mathcal{A}$.
The pair $(\mathcal{A}, G)$ is a G-metric space, and the function G is referred to as a generalized metric or a G-metric on $\mathcal{A}$.

Example 1.1. *Assume that the set of real numbers is $\mathcal{A}$, define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ as*

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, \text{ for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

G is therefore a G-metric on $\mathcal{A}$.

[8, 9, 10, 11, 12, 13, 15, 14, 16] has more results and more information in G-metric spaces. As a generalisation of partial metric space [17] and G-metric space, Zand and Nezhad [18] presented GP-metric space in 2011.

Definition 1.2. *[18] Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function such that the following conditions hold:*
(G_p1) $\zeta = \eta = \vartheta$ *if* $G(\zeta, \eta, \vartheta) = G(\zeta, \zeta, \zeta) = G(\eta, \eta, \eta) = G(\vartheta, \vartheta, \vartheta)$;
(G_p2) $G(\zeta, \zeta, \zeta) \leq G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, *for all* $\zeta, \eta, \vartheta \in \mathcal{A}$;
(G_p3) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \cdots$ (*symmetric in its variables*);
(G_p4) $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta) - G(a, a, a)$, *for all* $\zeta, \eta, \vartheta, a \in \mathcal{A}$.
Then, the function G is called a GP-metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is a GP-metric space.

Later, in 2013, Parvaneh *et al.* [19] discovered that (G_p2) makes GP-metric spaces symmetric. Because those G-metric spaces are nonsymmetric, GP-metric spaces do not generalize them (see Example 1, [7]). Parvaneh *et al.* [19] redefined GP-metric space in light of this by modifying the inequality (G_p2) to read as follows:

$$(G_p2') \quad G(\zeta,\zeta,\zeta) \leq G(\zeta,\zeta,\eta) \leq G(\zeta,\eta,\vartheta), \text{ for all } \zeta,\eta,\vartheta \in \mathcal{A} \text{ with } \eta \neq \vartheta.$$

Example 1.2. *[18] Let $\mathcal{A} = [0,\infty)$ and define a map $G : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to [0,\infty)$ by setting*

$$G(\zeta,\eta,\vartheta) = max\{\zeta,\eta,\vartheta\}, \text{ for all } \zeta,\eta,\vartheta \in \mathcal{A}.$$

Consequently, $(\mathcal{A}, G)$ is a GP-metric space but not a G-metric space since $G(1,1,1) = 1 \neq 0$, i.e., $(G1)$ does not hold.

Further details about GP-metric spaces are provided in papers [20, 21, 22, 23, 24, 25, 26, 27, 28]. By merging the ideas of G-metric spaces and b-metric spaces[30], Aghajani *et al.* introduced the notion of G_b-metric spaces in [29] as follows:

Definition 1.3. *[29] Let $s \geq 1$ be a real number and let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to [0,\infty)$ be a function such that:*
(G_b1) $G(\zeta,\eta,\vartheta) = 0$ if $\zeta = \eta = \vartheta$;
(G_b2) $0 < G(\zeta,\zeta,\eta)$, for all $\zeta,\eta \in \mathcal{A}$ with $\zeta \neq \eta$;
(G_b3) $G(\zeta,\zeta,\eta) \leq G(\zeta,\eta,\vartheta)$, for all $\zeta,\eta,\vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
(G_b4) $G(\zeta,\eta,\vartheta) = G(\zeta,\vartheta,\eta) = G(\eta,\vartheta,\zeta) = \cdots$ (symmetric in its variables);
(G_b5) $G(\zeta,\eta,\vartheta) \leq s[G(\zeta,a,a) + G(a,\eta,\vartheta)]$, for all $\zeta,\eta,\vartheta,a \in \mathcal{A}$.
Then, on $\mathcal{A}$, the function G is referred to as a G_b-metric or a generalized b-metric, and the pair $(\mathcal{A}, G)$ is a G_b-metric space or a generalized b-metric space. A G-metric space is a G_b-metric space with $s = 1$, but the opposite is not true in general.

Example 1.3. *[29] Let $\mathcal{A} = \mathbb{R}$ represent the set of real numbers. Define $G : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to [0,\infty)$ as follows:*

$$G(\zeta,\eta,\vartheta) = \frac{1}{9}(|\zeta-\eta| + |\eta-\vartheta| + |\vartheta-\zeta|)^2, \text{ for all } \zeta,\eta,\vartheta \in \mathcal{A}.$$

Hence, on $\mathcal{A}$, G is a G_b-metric but not a G-metric.

Numerous researchers demonstrated different findings in G_b-metric spaces; refer to [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. In [29], the authors proposed the idea of G_b-metric space. The term G_b-metric space was also used by Jain and Kaur in [43], although it referred to a different abstract space. Jain *et al.* [44] renamed this abstract space as 'generalized G_b-metric space', and its definition is as follows:

Definition 1.4. *[44] Let $\mathcal{A}$ be a non-empty set and $s \geq 1$ be a real number. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function satisfying:*
(gG_b1) $G(\zeta, \eta, \vartheta) = 0$ *if* $\zeta = \eta = \vartheta$;
(gG_b2) $0 < G(\zeta, \zeta, \eta)$, *for all* $\zeta, \eta \in \mathcal{A}$ *with* $\zeta \neq \eta$;
(gG_b3) $G(\zeta, \zeta, \eta) \leq s\, G(\zeta, \eta, \vartheta)$, *for all* $\zeta, \eta, \vartheta \in \mathcal{A}$ *with* $\vartheta \neq \eta$;
(gG_b4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \cdots$ (*symmetric in its variables*);
(gG_b5) $G(\zeta, \eta, \vartheta) \leq s[G(\zeta, a, a) + G(a, \eta, \vartheta)]$, *for all* $\zeta, \eta, \vartheta, a \in \mathcal{A}$.
The pair $(\mathcal{A}, G)$ is a generalized G_b-metric space, and the function G is referred to as a generalized G_b-metric on $\mathcal{A}$. The following example shows that while it is evident that every G_b-metric space is a generalized G_b-metric space, the converse is not true:

Example 1.4. *[44] For every $\zeta, \eta, \vartheta \in \mathbb{R}$, define a mapping $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ as follows:*

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2.$$

In that case, $(\mathbb{R}, G)$ is not a G_b-metric space, but it is a generalized G_b-metric space with $s = 2$. To calculate $G(\zeta, \eta, \vartheta) = |1 - 3|^2 + |3 - 2|^2 + |2 - 1|^2 = 6$ and $G(\zeta, \eta, \eta) = 2|1 - 3|^2 = 8$, let $\zeta = 1$, $\eta = 3$, and $\vartheta = 2$. Consequently, $G(\zeta, \eta, \eta) \nleq G(\zeta, \eta, \vartheta)$, that is, (G_b3), is not true.

After that, Jain *et al.* [44] introduced G^*-metric space to generalize GP-metric space and generalized G_b-metric space.

Definition 1.5. *[44] Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty]$ be a mapping, where $\mathcal{A}$ is a non-empty set. If there is an $\alpha > 0$ such that for every $\zeta, \eta, \vartheta \in \mathcal{A}$, the following conditions hold, then we say that G is a G^*-metric on $\mathcal{A}$:*
$(Gg1)$ $G(\zeta, \eta, \vartheta) = 0$ *implies* $\zeta = \eta = \vartheta$;
$(Gg2)$ $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \cdots$ (*symmetric in its variables*);
$(Gg3)$ *if* $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$, *then*

$$G(\zeta, \eta, \vartheta) \leq \alpha \left(\limsup_{n\to\infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right),$$

$$\text{where } C_{\mathcal{A}}(G, \zeta) = \left\{ \{\zeta_n\} \subset \mathcal{A} \mid \lim_{n,m\to\infty} G(\zeta_n, \zeta_m, \zeta) = G(\zeta, \zeta, \zeta) < \infty \right\}.$$

The pair $(\mathcal{A}, G)$ in this instance is referred to as a G^*-metric space with constant α.

Example 1.5. *[44] Assume that $\mathcal{A} = \mathcal{B} \cup \{0\}$, where $\mathcal{B} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty]$ be a mapping defined so that G satisfies $(Gg2)$ and*

$$G(\zeta, \eta, \vartheta) = \begin{cases} \zeta + \eta + \vartheta, & \text{if atleast one of } \zeta, \eta, \vartheta \text{ is } 0; \text{ or} \\ & \text{if } \zeta = \frac{1}{n}, \eta = \frac{1}{n+m}, \vartheta = \frac{1}{n+l}, \text{ where } n, m, l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{A}, G)$ is a G^*-metric space with constant α. However, $G(0.5, 0.5, 0.5) = 5 \neq 0$, $(\mathcal{A}, G)$ is not a generalized G_b-metric space. Also, $(\mathcal{A}, G)$ is not a GP-metric space as for $\zeta = \frac{1}{10}$ and $\eta = \frac{1}{5}$, $G(\zeta, \zeta, \zeta) = 5 \nleq \frac{2}{5} = G(\zeta, \zeta, \eta)$, that is, (G_p2') is not true.

Meanwhile, in 2018, Jleli and Samet [45] established an exciting generalization of metric space as follows.
Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:
$(\mathcal{F}1)$ f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.
$(\mathcal{F}2)$ For every sequence $\{t_n\}$ in $(0, \infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} f(t_n) = -\infty.$$

Definition 1.6. *[45] Let $\mathcal{A}$ be a non-empty set and let $D : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*
$(D1)$ *$(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, $D(\zeta, \eta) = 0$ if and only if $\zeta = \eta$.*
$(D2)$ *$D(\zeta, \eta) = D(\eta, \zeta)$, for all $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$.*
$(D3)$ *For every $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 2$, and for every $\{u_1, u_2, \cdots, u_n\} \subset \mathcal{A}$ with $(u_1, u_n) = (\zeta, \eta)$, we have*

$$D(\zeta, \eta) > 0 \text{ implies } f(D(\zeta, \eta)) \leq f\left(\sum_{i=1}^{n-1} D(u_i, u_{i+1})\right) + \alpha.$$

Then, the function D is said to be an $\mathcal{F}$-metric on $\mathcal{A}$, and the pair $(\mathcal{A}, D)$ is said to be an $\mathcal{F}$-metric space.

We refer to [46, 48, 49, 47, 50] for more details on $\mathcal{F}$-metric spaces. Now, motivated to the work done in [45], we define a new generalization of G-metric space as in the following section.

2 $G_{\mathcal{F}}$-Metric Space

Definition 2.1. *Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a mapping. Let there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*
$(G_{\mathcal{F}}1)$ *$G(\zeta, \eta, \vartheta) = 0$ if and only if $\zeta = \eta = \vartheta$.*
$(G_{\mathcal{F}}2)$ *$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$, $\zeta \neq \eta$.*
$(G_{\mathcal{F}}3)$ *$G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \cdots$ (symmetric in its variables).*
$(G_{\mathcal{F}}4)$ *For every $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 3$, and every $\{a_1, a_2, \cdots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$, $G(\zeta, \eta, \vartheta) > 0$ implies*

$$f(G(\zeta, \eta, \vartheta)) \leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha.$$

Then, the function G is called a $G_{\mathcal{F}}$-metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is said to be a $G_{\mathcal{F}}$-metric space.

2.1 Examples

Example 2.1. *Every G-metric space is a $G_{\mathcal{F}}$-metric space. Let $(\mathcal{A}, G)$ be a G-metric space. Then, G is a $G_{\mathcal{F}}$ metric on $\mathcal{A}$, as $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ can be obtained from $(G1)$, $(G2)$, $(G3)$ and $(G4)$. Also, with $\alpha = 0$ and $f(t) = \frac{-1}{t^2}$, $(G_{\mathcal{F}}2)$ and $(G_{\mathcal{F}}4)$ are satisfied using $(G3)$ and $(G5)$.*

Now, we construct an example of a $G_{\mathcal{F}}$-metric space which is a G_b-metric space as well, but not a G-metric space.

Example 2.2. *Let $\mathcal{A} = \{a, b, c\}$ and define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ as follows: $G(a, a, a) = G(b, b, b) = G(c, c, c) = 0$, $G(a, a, b) = G(a, b, b) = 1$, $G(a, a, c) = G(a, c, c) = 1.2$, $G(b, b, c) = G(b, c, c) = 1.3$, $G(a, b, c) = 3.3$, and assume that $(G_{\mathcal{F}}3)$ holds. Then G is a $G_{\mathcal{F}}$-metric on $\mathcal{A}$ with $f(t) = \ln(t)$, $t > 0$, and $\alpha = \ln(1.5)$. Also, G is a G_b-metric on $\mathcal{A}$ with $s = 1.5$, but G is not a G-metric on $\mathcal{A}$ as $G(a, b, c) = 3.3 \nleq 2.3 = G(a, b, b) + G(b, b, c)$.*

See another example of a $G_{\mathcal{F}}$-metric space which is a generalized G_b-metric space as well, but not a G_b-metric space.

Example 2.3. *Let $\mathcal{A} = \{1, 2, 3, \cdots, l-2\} \cup \{l - \frac{1}{n} \mid n \in \mathbb{N}\}$, where l be a fixed natural number such that $l \geq 5$ and $\mathcal{B} = \{1, 2, 3\}$. Consider a mapping $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ defined by*

$$G(\zeta, \eta, \vartheta) = \begin{cases} |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2, & \text{if } (\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}, \\ |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, & \text{otherwise.} \end{cases}$$

Then, $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ are satisfied for mapping G.
Let $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \cdots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$.
Let $P = \{i = 1, 2, 3, \cdots, n-2 \mid a_i, a_{i+1} \in \mathcal{B}\}$ and $Q = \{1, 2, 3, \cdots, n-2\} - P$.
Case 1: If $(\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}$ and $a_{n-1} \in \mathcal{B}$, we have

$$\begin{aligned} & G(\zeta, \eta, \vartheta) \\ = \ & |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ \leq \ & 2|\zeta - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ \leq \ & 2\left(\sum_{i \in P} |a_i - a_{i+1}| + \sum_{i \in Q} |a_i - a_{i+1}|\right) + 2|a_{n-1} - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ \leq \ & 2\left(\sum_{i \in P} |a_i - a_{i+1}|^2 + \sum_{i \in Q} |a_i - a_{i+1}|\right) + 2|a_{n-1} - \eta|^2 + |\eta - \vartheta|^2 \\ & + 2|\vartheta - a_{n-1}|^2 + 2|a_{n-1} - \zeta|^2 \end{aligned}$$

$$\begin{aligned}
&\leq\ 2\left(2\sum_{i\in P}|a_i-a_{i+1}|^2+2\sum_{i\in Q}|a_i-a_{i+1}|\right)+2|a_{n-1}-\eta|^2\\
&\quad+2|\eta-\vartheta|^2+2|\vartheta-a_{n-1}|^2+2|a_{n-1}-\zeta|^2\\
&\leq\ 10\Bigg(\left(2\sum_{i\in P}|a_i-a_{i+1}|^2+2\sum_{i\in Q}|a_i-a_{i+1}|\right)+|a_{n-1}-\eta|^2\\
&\quad+|\eta-\vartheta|^2+|\vartheta-a_{n-1}|^2\Bigg)\\
&=\ 10\left(\sum_{i=1}^{n-2}G(a_i,a_{i+1},a_{i+1})+G(a_{n-1},\eta,\vartheta)\right).
\end{aligned}$$

Case 2: If $(\zeta,\eta,\vartheta)\in\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ and $a_{n-1}\notin\mathcal{B}$, we have

$$\begin{aligned}
&G(\zeta,\eta,\vartheta)\\
&=\ |\zeta-\eta|^2+|\eta-\vartheta|^2+|\vartheta-\zeta|^2\\
&\leq\ 2|\zeta-\eta|+2|\eta-\vartheta|+2|\vartheta-\zeta|\\
&\leq\ 2\left(\sum_{i\in P}|a_i-a_{i+1}|+\sum_{i\in Q}|a_i-a_{i+1}|\right)+2|a_{n-1}-\eta|+2|\eta-\vartheta|\\
&\quad+2|\vartheta-a_{n-1}|+2|a_{n-1}-\zeta|\\
&\leq\ 2\left(2\sum_{i\in P}|a_i-a_{i+1}|^2+2\sum_{i\in Q}|a_i-a_{i+1}|\right)+2|a_{n-1}-\eta|+2|\eta-\vartheta|\\
&\quad+2|\vartheta-a_{n-1}|+2|a_{n-1}-\zeta|\\
&\leq\ 2l\Bigg(\left(2\sum_{i\in P}|a_i-a_{i+1}|^2+2\sum_{i\in Q}|a_i-a_{i+1}|\right)+|a_{n-1}-\eta|+|\eta-\vartheta|\\
&\quad+|\vartheta-a_{n-1}|\Bigg)\\
&=\ 2l\left(\sum_{i=1}^{n-2}G(a_i,a_{i+1},a_{i+1})+G(a_{n-1},\eta,\vartheta)\right).
\end{aligned}$$

Case 3: If $(\zeta,\eta,\vartheta)\notin\mathcal{B}\times\mathcal{B}\times\mathcal{B}$, then using Case 1 and Case 2, we have

$$\begin{aligned}
G(\zeta,\eta,\vartheta)\ &=\ |\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|\\
&\leq\ l\left(\sum_{i=1}^{n-2}G(a_i,a_{i+1},a_{i+1})+G(a_{n-1},\eta,\vartheta)\right).
\end{aligned}$$

Thus, combining all cases, we have

$G(\zeta,\eta,\vartheta) > 0$ *implies* $G(\zeta,\eta,\vartheta) \leq 2l\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right)$.

Therefore, by taking $f(t) = \ln(t)$*,* $t > 0$*, and* $\alpha = \ln(2l)$*,* $(G_{\mathcal{F}}4)$ *and* $(G_{\mathcal{F}}2)$ *are obtained. Hence,* G *is a* $G_{\mathcal{F}}$*-metric on* $\mathcal{A}$*. Also,* G *is a generalized* G_b*-metric on* $\mathcal{A}$ *with* $s = 2l$*. But* G *is not a* G_b*-metric on* $\mathcal{A}$ *as* $G(1,1,3) = 8 \nleq 6 = G(1,2,3)$*, i.e.,* (G_b3) *does not hold.*

In the following example, we see the existence of a $G_{\mathcal{F}}$-metric and G^*-metric space, which is not a generalized G_b-metric space.

Example 2.4. *Let* $\mathcal{A} = \mathbb{R}$ *and* $\mathfrak{B} = [0,1] \times [0,1] \times [0,1] - \{(\zeta,\zeta,\zeta) \mid \zeta \in [0,1]\}$ *and* $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0,\infty)$ *be a mapping defined by*

$$G(\zeta,\eta,\vartheta) = \begin{cases} 0, & \text{if } \zeta = \eta = \vartheta, \\ \frac{1}{3}(|\zeta-\eta| + |\eta-\vartheta| + |\vartheta-\zeta|), & \text{if } (\zeta,\eta,\vartheta) \in \mathfrak{B}, \\ 2^{(|\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|)}, & \text{otherwise}. \end{cases}$$

Then, G *satisfies* $(G_{\mathcal{F}}1)$ *and* $(G_{\mathcal{F}}3)$ *directly from the definition of* G*.*
Let $f(t) = \frac{-1}{\sqrt{t}}$*,* $t > 0$*, and* $\alpha = 1$*. Then, for any* $(\zeta,\eta,\vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ *such that* $G(\zeta,\eta,\vartheta) > 0$ *and for* $n \geq 3$*,* $\{a_1, a_2, \cdots, a_{n-1}\} \subset \mathcal{A}$ *with* $a_1 = \zeta$*, we consider the following cases:*
<u>Case 1</u>: If $(\zeta,\eta,\vartheta) \in \mathfrak{B}$*, then*

$$\begin{aligned} G(\zeta,\eta,\vartheta) &= \frac{1}{3}(|\zeta-\eta| + |\eta-\vartheta| + |\vartheta-\zeta|) \\ &\leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \end{aligned}$$

which implies that

$$\begin{aligned} f\left(G(\zeta,\eta,\vartheta)\right) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha. \end{aligned}$$

Case 2 : If $(\zeta,\eta,\vartheta)\notin\mathfrak{B}$, then

$$\begin{aligned} & f\left(G(\zeta,\eta,\vartheta)\right)-f\left(\sum_{i=1}^{n-2}G(a_i,a_{i+1},a_{i+1})+G(a_{n-1},\eta,\vartheta)\right)-\alpha \\ = & \frac{-1}{\sqrt{2^{(|\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|)}}}+\frac{1}{\sqrt{\sum_{i=1}^{n-2}G(a_i,a_{i+1},a_{i+1})+G(a_{n-1},\eta,\vartheta)}}-1 \\ \leq & \frac{-1}{\sqrt{2^{(|\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|)}}}+1-1\leq 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}}4)$ holds.
Also, for $\zeta,\eta,\vartheta\in\mathcal{A}$ with $\zeta\neq\eta$, $\eta\neq\vartheta$, we consider the following two cases:
Case 1: If $(\zeta,\zeta,\eta)\in\mathfrak{B}$, then

$$G(\zeta,\zeta,\eta)=\frac{1}{3}(2|\zeta-\eta|)\leq G(\zeta,\eta,\vartheta),$$

which implies that

$$f\left(G(\zeta,\zeta,\eta)\right)\leq f\left(G(\zeta,\eta,\vartheta)\right)\leq f\left(G(\zeta,\eta,\vartheta)\right)+\alpha.$$

Case 2: If $(\zeta,\zeta,\eta)\notin\mathfrak{B}$, then

$$f\left(G(\zeta,\zeta,\eta)\right)=\frac{-1}{\sqrt{2^{(2|\zeta-\eta|)}}}\leq\frac{-1}{\sqrt{2^{(|\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|)}}}\leq f\left(G(\zeta,\eta,\vartheta)\right)+\alpha.$$

Thus, $(G_{\mathcal{F}}2)$ holds; hence, G is a $G_{\mathcal{F}}$-metric on $\mathcal{A}$. But G is not a generalized G_b-metric on $\mathcal{A}$ as for any $s\geq 1$, and for $n\in\mathbb{N}$,

$$\begin{aligned} 2^{4n} &= G(2n+1,1,1) \\ &\leq s\left(G(2n+1,n+1,n+1)+G(n+1,1,1)\right) \\ &= s(2^{2n}+2^{2n}), \end{aligned}$$

which gives $2^{2n-1}\leq s$, therefore, by taking $n\to\infty$, we have a contradiction.

The next example assures the existence of a $G_{\mathcal{F}}$-metric space which is not a G^*-metric space.

Example 2.5. *Let $\mathcal{B}=\left\{\frac{1}{n}\mid n\in\mathbb{N}\right\}$ and $\mathcal{A}=\mathcal{B}\cup\mathbb{N}\cup\{0\}$. Let $G:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to[0,\infty)$ be a mapping defined by*

$$G(\zeta,\eta,\vartheta)=\begin{cases} 0, & \text{if } \zeta=\eta=\vartheta; \\ |\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|, & \text{if } \zeta,\eta,\vartheta\in\mathcal{B};\ \text{or} \\ & \text{if atleast one of } \zeta,\eta,\vartheta \text{ is } 0; \\ 1, & \text{otherwise}. \end{cases}$$

First, we prove that G is not a G^*-metric. Suppose that G is a G^*-metric with constant $\beta > 0$.
Let $\zeta = 0$ and $\zeta_n = \frac{1}{n}$, then $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$. Therefore, for $\eta, \vartheta \in \mathbb{N}$, using $(Gg3)$, we get

$$G(\zeta, \eta, \vartheta) \leq \ \beta \left(\limsup_{n\to\infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right),$$

that is,

$$|\eta| + |\eta - \vartheta| + |\vartheta| \leq \beta(1 + 0) = \beta.$$

Taking limit $\eta, \vartheta \to \infty$, we have a contradiction. Thus, $(\mathcal{A}, G)$ is not a G^*-metric space.
We now prove that $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, where $f(t) = \frac{-1}{t}$ and $\alpha = 1$. $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ hold obviously. Now, for $(G_{\mathcal{F}}2)$, let $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\zeta \neq \eta$, $\eta \neq \vartheta$. Then consider the following two cases:
Case 1: If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$. Therefore,

$$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Case 2 : If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then $G(\zeta, \eta, \vartheta) \geq 1$. Thus,

$$\begin{aligned} & f(G(\zeta, \zeta, \eta)) - f(G(\zeta, \eta, \vartheta)) - \alpha \\ = \ & \frac{-1}{G(\zeta, \zeta, \eta)} + \frac{1}{G(\zeta, \eta, \vartheta)} - 1 \\ \leq \ & \frac{-1}{G(\zeta, \zeta, \eta)} + 1 - 1 < 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}}2)$ holds.
Now, for $(G_{\mathcal{F}}4,)$ let $\zeta, \eta, \vartheta \in \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \cdots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$. Consider the following two cases:
Case 1: If If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then

$$G(\zeta, \eta, \vartheta) \leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta),$$

therefore,

$$\begin{aligned} f\left(G(\zeta, \eta, \vartheta)\right) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right) + \alpha. \end{aligned}$$

<u>Case 2</u> *: If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then* $\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \geq 1$. *Thus,*

$$
\begin{aligned}
& f(G(\zeta, \eta, \vartheta)) - f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) - \alpha \\
= \; & \frac{-1}{G(\zeta, \eta, \vartheta)} + \frac{1}{\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)} - 1 \\
\leq \; & \frac{-1}{G(\zeta, \eta, \vartheta)} + 1 - 1 < 0.
\end{aligned}
$$

Thus, $(G_{\mathcal{F}}4)$ holds. Hence, $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$-metric space.

Remark 2.1. *Abstract spaces in Example 1.2, Example 1.3, Example 1.4, and Example 1.5 are not $G_{\mathcal{F}}$-metric spaces. A relation among the classes of abstract spaces discussed so far is described in the following diagram (Figure 1):*

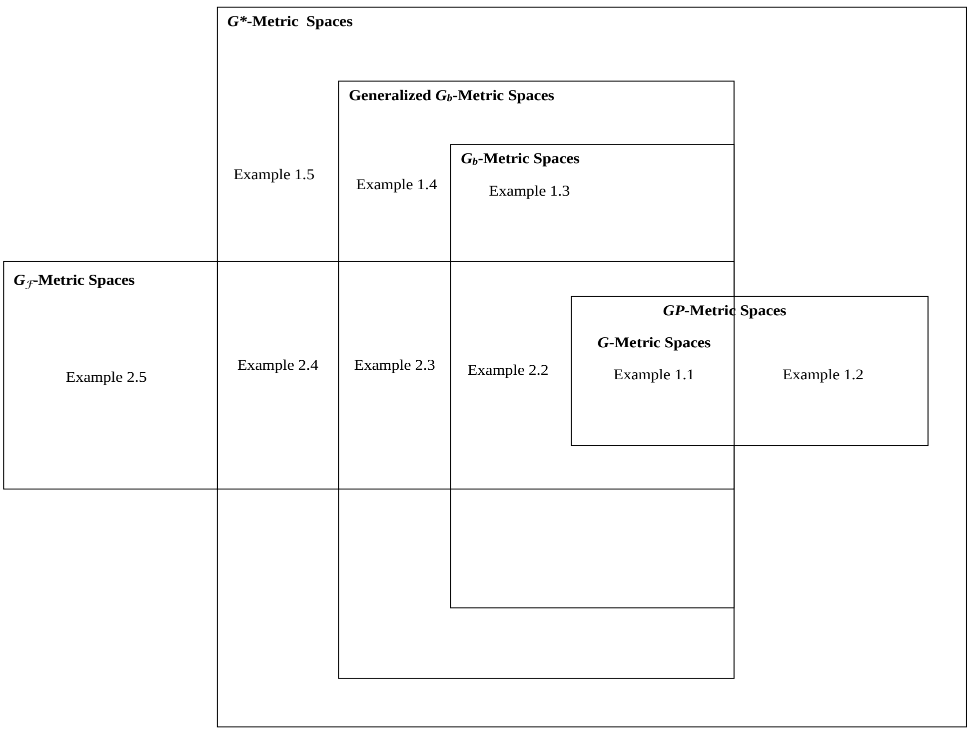

Figure 1: A comparison among various classes of abstract space.

2.2 Some Basic Concepts

This section defines a few fundamental ideas and examines their characteristics within the context of $G_{\mathcal{F}}$-metric space.

Definition 2.2. *Let A be a subset of $G_{\mathcal{F}}$-metric space $(\mathcal{A}, G)$. Then A is said to be a $G_{\mathcal{F}}$-open set if for every $\zeta \in A$, there exists $r > 0$ such that $B(\zeta, r) \subseteq A$, where*

$$B(\zeta, r) = \{\eta \in \mathcal{A} \mid G(\zeta, \eta, \eta) < r\}.$$

Let $\tau_{G_{\mathcal{F}}}$ be the collection of all such open sets, then $\tau_{G_{\mathcal{F}}}$ is a topology on $\mathcal{A}$.

Definition 2.3. *Let $\{\zeta_n\}$ be a sequence of points in $G_{\mathcal{F}}$-metric space $(\mathcal{A}, G)$. Then sequence $\{\zeta_n\}$ is said to be $G_{\mathcal{F}}$-convergent to $\zeta_0 \in \mathcal{A}$ if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(\zeta_n, \zeta_m, \zeta_0) < \epsilon$, for all $n, m \geq n_0$.*

Proposition 2.1. *Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, then*

$$f(G(\zeta, \eta, \eta)) \leq f(2G(\zeta, \zeta, \eta)) + \alpha, \quad for\ all\ \ \zeta, \eta \in \mathcal{A}\ \ with\ \ \zeta \neq \eta.$$

Proof. For $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$, using $(G_{\mathcal{F}}1)$, $(G_{\mathcal{F}}4)$ and $(G_{\mathcal{F}}3)$, we have,

$$f(G(\zeta, \eta, \eta)) \leq f(G(\eta, \zeta, \zeta) + G(\zeta, \zeta, \eta)) + \alpha = f(2G(\zeta, \zeta, \eta)) + \alpha.$$

□

Proposition 2.2. *Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then, the following statements are equivalent.*
(I) $G(\zeta_n, \zeta_n, \zeta_0) \to 0\ \ as\ \ n \to \infty$.
(II) $G(\zeta_n, \zeta_0, \zeta_0) \to 0\ \ as\ \ n \to \infty$.
(III) $G(\zeta_n, \zeta_m, \zeta_0) \to 0\ \ as\ \ n, m \to \infty$.

Proof. First, we prove (I) implies (II); for this, let $G(\zeta_n, \zeta_n, \zeta_0) \to 0$ as $n \to \infty$. If for infinitely many n, $\zeta_n \neq \zeta_0$, then for those n, using Proposition 2.1, we have

$$f(G(\zeta_n, \zeta_0, \zeta_0)) \leq f(2G(\zeta_n, \zeta_n, \zeta_0)) + \alpha.$$

Taking $n \to \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_0, \zeta_0) \to 0$. and for rest of n, $G(\zeta_n, \zeta_0, \zeta_0) = 0$, thus in overall $G(\zeta_n, \zeta_0, \zeta_0) \to 0$.
Now we prove (II) implies (III); for this, let $G(\zeta_n, \zeta_0, \zeta_0) \to 0$ as $n \to \infty$.

If for infinitely many m and n, $G(\zeta_n,\zeta_m,\zeta_0)\neq 0$, then for those (m,n), using $(G_{\mathcal{F}}4)$, we have

$$f\left(G(\zeta_n,\zeta_m,\zeta_0)\right)\leq f\left(G(\zeta_n,\zeta_0,\zeta_0)+G(\zeta_0,\zeta_0,\zeta_m)\right)+\alpha.$$

Taking $n,m\to\infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n,\zeta_m,\zeta_0)\to 0$, and for rest of n,m, $G(\zeta_n,\zeta_m,\zeta_0)=0$, thus in overall $G(\zeta_n,\zeta_m,\zeta_0)\to 0$ as $n,m\to\infty$.
Also, (III) implies (I) obviously. □

Proposition 2.3. *Let $(\mathcal{A},G)$ be a $G_{\mathcal{F}}$-metric space with $(f,\alpha)\in\mathcal{F}\times[0,\infty)$. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ which is $G_{\mathcal{F}}$-convergent, converges to a unique element in $\mathcal{A}$.*

Proof. Let, if possible that sequence $\{\zeta_n\}$ converges to ζ and η in $\mathcal{A}$ with $\zeta\neq\eta$, then, by $(G_{\mathcal{F}}4)$

$$f\left(G(\zeta,\eta,\eta)\right)\leq f\left(G(\zeta,\zeta_n,\zeta_n)+G(\zeta_n,\eta,\eta)\right)+\alpha,$$

taking $n\to\infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction. □

Definition 2.4. *Let $(\mathcal{A},G)$ be a $G_{\mathcal{F}}$-metric space. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ is said to be $G_{\mathcal{F}}$-Cauchy sequence if, for each $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $G(\zeta_n,\zeta_m,\zeta_l)<\epsilon$, for all $n,m,l\geq n_0$.*

Proposition 2.4. *Let $(\mathcal{A},G)$ be a $G_{\mathcal{F}}$-metric space with $(f,\alpha)\in\mathcal{F}\times[0,\infty)$. Then, the following statements are equivalent.*
(I) *Sequence $\{\zeta_n\}$ in $\mathcal{A}$ is $G_{\mathcal{F}}$-Cauchy sequence.*
(II) $G(\zeta_n,\zeta_m,\zeta_l)\to 0$ *as* $n,m,l\to\infty$.
(III) $G(\zeta_n,\zeta_m,\zeta_m)\to 0$ *as* $n,m\to\infty$.

Proof. (I) and (II) are equivalent using Definition 2.4.
Now (II) implies (III) obviously, so we prove (III) implies (II). If for infinitely many n,m and l, $G(\zeta_n,\zeta_m,\zeta_l)\neq 0$, then for those (n,m,l), using $(G_{\mathcal{F}}4)$, we have

$$f\left(G(\zeta_n,\zeta_m,\zeta_l)\right)\leq f\left(G(\zeta_n,\zeta_m,\zeta_m)+G(\zeta_m,\zeta_m,\zeta_l)\right)+\alpha.$$

Taking $n,m,l\to\infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n,\zeta_m,\zeta_l)\to 0$, and for rest of n,m,l, $G(\zeta_n,\zeta_m,\zeta_l)=0$, thus in overall $G(\zeta_n,\zeta_m,\zeta_l)\to 0$ as $n,m,l\to\infty$. □

Proposition 2.5. *Every $G_{\mathcal{F}}$-convergent sequence is a $G_{\mathcal{F}}$-Cauchy sequence.*

Remark 2.2. *In Example 2.3, sequence* $\{l - \frac{1}{n}\}$ *is* $G_{\mathcal{F}}$*-Cauchy sequence but not a* $G_{\mathcal{F}}$*-convergent sequence.*

Definition 2.5. *A* $G_{\mathcal{F}}$*-metric space is said to be* $G_{\mathcal{F}}$*-complete if every* $G_{\mathcal{F}}$*-Cauchy sequence is a* $G_{\mathcal{F}}$*-convergent sequence.*

Remark 2.3. *In Example 2.4,* $(\mathcal{A}, G)$ *is a* $G_{\mathcal{F}}$*-complete metric space as every* $G_{\mathcal{F}}$*-Cauchy sequence is a* $G_{\mathcal{F}}$*-convergent sequence.*

Definition 2.6. *Let* $(\mathcal{A}, G)$ *be a* $G_{\mathcal{F}}$*-metric space and* $\mathcal{B} \subseteq \mathcal{A}$*. Then closure of* $\mathcal{B}$ *is denoted by* $\overline{\mathcal{B}}$ *and defined by*

$$\overline{\mathcal{B}} = \{\zeta \in \mathcal{A} \mid \zeta \in \mathcal{B} \text{ or } B(\zeta, r) \cap \mathcal{B} \text{ is infinite set for every } r > 0\}.$$

The following propositions are needed in the main results of this paper.

Proposition 2.6. *Let* $(\mathcal{A}, G)$ *be a* $G_{\mathcal{F}}$*-metric space with* $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ *and* f *is a continuous function. If* $\{\zeta_n\}$ *is* $G_{\mathcal{F}}$*-convergent to* ζ *with* $\zeta \neq b$ *or* $\zeta \neq c$*, where* b *and* c *are real constants, then*

$$\begin{aligned} f\left(G(\zeta, b, c)\right) - \alpha \leq \liminf_{n\to\infty} f\left(G(\zeta_n, b, c)\right) &\leq \limsup_{n\to\infty} f\left(G(\zeta_n, b, c)\right) \\ &\leq f\left(G(\zeta, b, c)\right) + \alpha. \end{aligned}$$

Proof. Without loss of generality, consider $\zeta \neq b$, so except first finitely many n, $\zeta_n \neq b$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned} f\left(G(\zeta, b, c)\right) - \alpha &\leq \liminf_{n\to\infty} f\left(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, b, c)\right) \\ &= \liminf_{n\to\infty} f\left(G(\zeta_n, b, c)\right) \\ &\leq \limsup_{n\to\infty} f\left(G(\zeta_n, b, c)\right) \\ &\leq \limsup_{n\to\infty} f\left(G(\zeta_n, \zeta, \zeta) + G(\zeta, b, c)\right) + \alpha \\ &= \limsup_{n\to\infty} f\left(G(\zeta, b, c)\right) + \alpha \\ &= f\left(G(\zeta, b, c)\right) + \alpha. \end{aligned}$$

□

Proposition 2.7. *Let* $(\mathcal{A}, G)$ *be a* $G_{\mathcal{F}}$*-metric space with* $(f, \alpha) \in \mathcal{F} \times [0, \infty)$*, and* f *is a continuous function. If* $\{\zeta_n\}$ *and* $\{\eta_n\}$ *are* $G_{\mathcal{F}}$*-convergent to* ζ *and* η*, respectively, and* c *is a real constant such that* $c \neq \zeta$ *or* $c \neq \eta$*, then*

$$\begin{aligned} f\left(G(\zeta, \eta, c)\right) - 2\alpha \leq \liminf_{n\to\infty} f\left(G(\zeta_n, \eta_n, c)\right) &\leq \limsup_{n\to\infty} f\left(G(\zeta_n, \eta_n, c)\right) \\ &\leq f\left(G(\zeta, \eta, c)\right) + 2\alpha. \end{aligned}$$

Proof. Without loss of generality, assume that $c \neq \zeta$; therefore, except first finitely many n, $c \neq \zeta_n$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned}
f\left(G(\zeta,\eta,c)\right) - 2\alpha &\leq \liminf_{n\to\infty} f\left(G(\zeta,\zeta_n,\zeta_n) + G(\zeta_n,\eta,c)\right) - \alpha \\
&= \liminf_{n\to\infty} f\left(G(\zeta_n,\eta,c)\right) - \alpha \\
&\leq \liminf_{n\to\infty} f\left(G(\eta,\eta_n,\eta_n) + G(\eta_n,\zeta_n,c)\right) \\
&= \liminf_{n\to\infty} f\left(G(\zeta_n,\eta_n,c)\right) \\
&\leq \limsup_{n\to\infty} f\left(G(\zeta_n,\eta_n,c)\right) \\
&\leq \limsup_{n\to\infty} f\left(G(\zeta_n,\zeta,\zeta) + G(\zeta,\eta_n,c)\right) + \alpha \\
&= \limsup_{n\to\infty} f\left(G(\zeta,\eta_n,c)\right) + \alpha \\
&\leq \limsup_{n\to\infty} f\left(G(\eta_n,\eta,\eta) + G(\eta,\zeta,c)\right) + 2\alpha \\
&= \limsup_{n\to\infty} f\left(G(\eta,\zeta,c)\right) + 2\alpha. \\
&= f\left(G(\zeta,\eta,c)\right) + 2\alpha.
\end{aligned}$$

□

3 Fixed Point Results in $G_{\mathcal{F}}$-Metric Space

This section deals with some fixed point results in the context of $G_{\mathcal{F}}$-metric space. Our first result is the following theorem.

Theorem 3.1. *Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$-complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T : \mathcal{A} \to \mathcal{A}$ be a mapping such that there exists a non-empty subset $\mathcal{B}$ of $\mathcal{A}$ with $T(\mathcal{B}) \subseteq \mathcal{B}$ and*

$$G(T\zeta, T\eta, T\vartheta) \leq \lambda G(\zeta, \eta, \vartheta), \quad for\ all \quad \zeta, \eta, \vartheta \in \overline{\mathcal{B}},$$

where $\lambda \in [0, 1)$. Then, T has a fixed point in $\mathcal{A}$. Moreover, if the fixed point belongs to $\mathcal{B}$, then T has a unique fixed point in $\mathcal{B}$.

Proof. Let $\zeta_0 \in \mathcal{B}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_n = T\zeta_{n-1}$, for all $n \in \mathbb{N}$. If $\zeta_n = \zeta_{n+1}$ for some $n \in \mathbb{N}$, then ζ_n is a fixed point of T. So let $\zeta_n \neq \zeta_{n+1}$, for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we have

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda G(\zeta_{n-1}, \zeta_n, \zeta_n).$$

Now, an easy induction gives that

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda^n G(\zeta_0, \zeta_1, \zeta_1).$$

Let $n, m \in \mathbb{N}$ with $m > n$; then, we have

$$\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1}) \leq \frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1).$$

If for infinitely many pairs (m, n) with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) \neq 0$, then for these m, n using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned} f\left(G(\zeta_n, \zeta_m, \zeta_m)\right) &\leq f\left(\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1})\right) + \alpha \\ &\leq f\left(\frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1)\right) + \alpha, \end{aligned}$$

taking $n, m \to \infty$ and using $(\mathcal{F}2)$, we get $G(\zeta_n, \zeta_m, \zeta_m) \to 0$.
Also, for rest of m, n with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) = 0$. Thus in overall,

$$G(\zeta_n, \zeta_m, \zeta_m) \to 0 \;\; as \;\; n, m \to \infty.$$

Thus, $\{\zeta_n\}$ is a $G_{\mathcal{F}}$-Cauchy sequence in $\mathcal{A}$, but $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$-complete metric space, therefore $\{\zeta_n\}$ is $G_{\mathcal{F}}$-convergent to some $\zeta' \in \mathcal{A}$. Suppose, if possible $G(T\zeta', \zeta', \zeta') > 0$, then using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned} f\left(G(T\zeta', \zeta', \zeta')\right) &\leq f\left(G(T\zeta', T\zeta_n, T\zeta_n) + G(T\zeta_n, \zeta', \zeta')\right) + \alpha \\ &\leq f\left(\lambda G(\zeta', \zeta_n, \zeta_n) + G(\zeta_{n+1}, \zeta', \zeta')\right) + \alpha. \end{aligned}$$

Taking $n \to \infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction.
Thus, $G(T\zeta', \zeta', \zeta') = 0$ and which implies that $T\zeta' = \zeta'$.
Now, if $\zeta' \in \mathcal{B}$ and $\eta \in \mathcal{B}$ be another fixed point of T, then

$$G(\zeta', \eta, \eta) = G(T\zeta', T\eta, T\eta) \leq \lambda G(\zeta', \eta, \eta)$$

which implies that $G(\zeta', \eta, \eta) = 0$ as $\lambda \in [0, 1)$. Therefore, $\zeta' = \eta$, that is, ζ' is a unique fixed point of T in $\mathcal{B}$. □

Example 3.1. *Consider $G_{\mathcal{F}}$-metric space $(\mathcal{A}, G)$ as in Example 2.4, which is a $G_{\mathcal{F}}$-complete metric space. Define a mapping $T : \mathcal{A} \to \mathcal{A}$ as*

$$T\zeta = \frac{\zeta(\zeta+1)}{4}, \text{ for all } \zeta \in \mathcal{A}.$$

Now, for $\mathcal{B}=[0,1]$, we see that $T(\mathcal{B})\subseteq\mathcal{B}$, and for $\zeta,\eta,\vartheta\in\overline{\mathcal{B}}$, we have

$$\begin{aligned}
&G(T\zeta,T\eta,T\vartheta)\\
&=\frac{1}{3}\Bigg(\left|\frac{\zeta(\zeta+1)}{4}-\frac{\eta(\eta+1)}{4}\right|+\left|\frac{\eta(\eta+1)}{4}-\frac{\vartheta(\vartheta+1)}{4}\right|\\
&\quad+\left|\frac{\vartheta(\vartheta+1)}{4}-\frac{\zeta(\zeta+1)}{4}\right|\Bigg)\\
&\leq\frac{1}{3}\times\frac{3}{4}(|\zeta-\eta|+|\eta-\vartheta|+|\vartheta-\zeta|)\\
&=\frac{3}{4}G(\zeta,\eta,\vartheta).
\end{aligned}$$

Thus, the hypothesis of Theorem 3.1 is satisfied. And we notice that T has two fixed points, 0 and 3. Also, 0 is the only fixed point of T in $\mathcal{B}$.

Example 3.2. *Consider $G_{\mathcal{F}}$-metric space $(\mathcal{A},G)$ as in Example 2.4, which is a $G_{\mathcal{F}}$-complete metric space. Define a mapping $T:\mathcal{A}\to\mathcal{A}$ as $T\zeta=\frac{\zeta}{2}$, for all $\zeta\in\mathcal{A}$.*
Then for $\mathcal{B}=[0,1]$, we can easily see that hypothesis of Theorem 3.1 is satisfied. Also, we notice that T has a unique fixed point in $\mathcal{A}$.

Corollary 3.1. *Let $(\mathcal{A},G)$ be a $G_{\mathcal{F}}$-complete metric space with $(f,\alpha)\in\mathcal{F}\times[0,\infty)$ and $T:\mathcal{A}\to\mathcal{A}$ be a mapping such that*

$$G(T\zeta,T\eta,T\vartheta)\leq\lambda G(\zeta,\eta,\vartheta),\quad for\ all\ \ \zeta,\eta,\vartheta\in\mathcal{A},$$

where $\lambda\in[0,1)$. Then, T has a unique fixed point in $\mathcal{A}$.

Proof. Take $\mathcal{B}=\mathcal{A}$ in Theorem 3.1. □

In the following result, we find the unique fixed point for (ψ,ϕ)-contractive mapping (see detail of (ψ,ϕ)-contractive mapping in [51, 52, 14, 53, 29, 32, 54]) in the setting of $G_{\mathcal{F}}$-complete metric space.

Theorem 3.2. *Let $(\mathcal{A},G)$ be a $G_{\mathcal{F}}$-complete metric space with $(f,\alpha)\in\mathcal{F}\times[0,\infty)$ such that f is a continuous function. Let $T:\mathcal{A}\to\mathcal{A}$ be a mapping such that*

$$\psi\left(f(M(\zeta,\eta,\vartheta))+4\alpha\right)\leq\psi\left(f(G(T\zeta,T\eta,T\vartheta))\right)-\phi\left(M(\zeta,\eta,\vartheta)\right),\quad(1)$$

for all $(\zeta,\eta,\vartheta)\in\mathcal{A}\times\mathcal{A}\times\mathcal{A}-\{(\zeta,\eta,\vartheta)\in\mathcal{A}\times\mathcal{A}\times\mathcal{A}\mid T\zeta=T\eta=T\vartheta\}$, where $\psi:\mathbb{R}\to\mathbb{R}$ is a continuous non-decreasing function and $\phi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\phi^{-1}(0)=\{0\}$ and

$$M(\zeta,\eta,\vartheta)=\max\{G(\zeta,\eta,\vartheta),\ G(\zeta,T\zeta,T\eta),\ G(\eta,T\eta,T\vartheta),\ G(\vartheta,T\vartheta,T\zeta)\}.$$

Then, T has a unique fixed point.

Proof. Let $\zeta_0 \in \mathcal{A}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_{n+1} = T\zeta_n,\ n = 0, 1, 2, \cdots$. If $\zeta_n = \zeta_{n+1}$, then ζ_n is a fixed point of T. Now, assume that $\zeta_n \neq \zeta_{n+1}$, for all n. Let $\theta_n = f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})),\ \ n = 1, 2, 3 \cdots$.
Now,

$$\begin{aligned}
&\psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))) \leq \psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha) \\
&\leq \psi(f(G(T\zeta_n, T\zeta_{n+1}, T\zeta_{n+2}))) - \phi\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) \\
&\leq \psi\left(\theta_{n+1}\right) - \phi\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right),
\end{aligned}$$

where

$$\begin{aligned}
M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) &= max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}),\ G(\zeta_n, T\zeta_n, T\zeta_{n+1}), \\
&\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}),\ G(\zeta_{n+2}, T\zeta_{n+2}, T\zeta_n)\} \\
&= max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}),\ G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), \\
&\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}),\ G(\zeta_{n+2}, \zeta_{n+3}, \zeta_{n+1})\}
\end{aligned}$$

and, therefore,

$$\begin{aligned}
f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) &= \max\{f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})),\ f(G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}))\} \\
&= \max\{\theta_n, \theta_{n+1}\}.
\end{aligned}$$

If $\theta_n < \theta_{n+1}$ for some n, then we have $\psi(\theta_{n+1}) \leq \psi(\theta_{n+1}) - \phi\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right)$, which gives that $M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, a contradiction, therefore $\theta_{n+1} \leq \theta_n$ for all n. Thus, $\{\theta_n\}$ is a non-increasing sequence. Suppose that $\{\theta_n\}$ is bounded below; then there exists a real θ such that $\lim_{n\to\infty} \theta_n = \theta$. Now,

$$\psi(\theta_n) \leq \psi(\theta_{n+1}) - \phi\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right)$$

taking limit supremum on both sides, we have

$$\begin{aligned}
\limsup_{n\to\infty} \psi(\theta_n) &\leq \limsup_{n\to\infty} \psi(\theta_{n+1}) - \liminf_{n\to\infty} \phi\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) \\
\text{i.e.,} \qquad \psi(\theta) &\leq \psi(\theta) - \phi\left(\liminf_{n\to\infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right)
\end{aligned}$$

which gives that $\liminf_{n\to\infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, therefore, by $(\mathcal{F}2)$
$\liminf_{n\to\infty} f\left(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) = -\infty$, i.e., $\lim_{n\to\infty} \theta_n = -\infty$. Hence by $(\mathcal{F}2)$,

$$\lim_{n\to\infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0. \tag{2}$$

Since $\zeta_n \neq \zeta_{n+1}$ for every n, therefore, by $(G_{\mathcal{F}}2)$,

$$f\left(G(\zeta_n, \zeta_n, \zeta_{n+1})\right) \leq f\left(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) + \alpha.$$

So, by using (2) and $(\mathcal{F}2)$, we have

$$\lim_{n\to\infty} G(\zeta_n,\zeta_n,\zeta_{n+1}) = 0. \tag{3}$$

Also, using Proposition 2.1, we have

$$\lim_{n\to\infty} G(\zeta_n,\zeta_{n+1},\zeta_{n+1}) = 0. \tag{4}$$

Next, we prove that $\{\zeta_n\}$ is a $G_{\mathcal{F}}$-Cauchy sequence. Suppose not, then there exists $\epsilon > 0$ such that we can find subsequences $\{\zeta_{m_k}\}$ and $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$G(\zeta_{n_k},\zeta_{m_k},\zeta_{m_k}) \geq \epsilon \tag{5}$$

this means that

$$G(\zeta_{n_k},\zeta_{m_k-1},\zeta_{m_k-1}) < \epsilon. \tag{6}$$

Now further, consider only those k for which left-hand side of (6) is greater than 0, and clearly, such k exists infinitely many.
Now,

$$\begin{aligned}
&\psi\left(f(M(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1})) + 4\alpha\right)\\
&= \psi\left(f(G(T\zeta_{n_k},T\zeta_{m_k-2},T\zeta_{m_k-1}))\right) - \phi\left(M(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1})\right)\\
&\leq \psi\left(f(G(\zeta_{n_k+1},\zeta_{m_k-1},\zeta_{m_k}))\right) - \phi\left(M(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1})\right),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
&M(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1})\\
&= max\{G(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1}),\ G(\zeta_{n_k},T\zeta_{n_k},T\zeta_{m_k-2}),\\
&\quad G(\zeta_{m_k-2},T\zeta_{m_k-2},T\zeta_{m_k-1}),\ G(\zeta_{m_k-1},T\zeta_{m_k-1},T\zeta_{n_k})\}\\
&= max\{G(\zeta_{n_k},\zeta_{m_k-2},\zeta_{m_k-1}),\ G(\zeta_{n_k},\zeta_{n_k+1},\zeta_{m_k-1}),\\
&\quad G(\zeta_{m_k-2},\zeta_{m_k-1},\zeta_{m_k}),\ G(\zeta_{m_k-1},\zeta_{m_k},\zeta_{n_k+1})\}.
\end{aligned} \tag{8}$$

Also, using $(G_{\mathcal{F}}4)$, (3), and (6), we have

$$\begin{aligned}
&\limsup_{k\to\infty} f\left(G(\zeta_{m_k-1},\zeta_{m_k},\zeta_{n_k+1})\right)\\
&\leq \limsup_{k\to\infty} f(G(\zeta_{n_k+1},\zeta_{n_k},\zeta_{n_k}) + G(\zeta_{n_k},\zeta_{m_k-1},\zeta_{m_k-1})\\
&\quad + G(\zeta_{m_k-1},\zeta_{m_k-1},\zeta_{m_k})) + \alpha\\
&\leq \limsup_{k\to\infty} f\left(G(\zeta_{n_k},\zeta_{m_k-1},\zeta_{m_k-1})\right) + \alpha\\
&\leq f(\epsilon) + \alpha.
\end{aligned} \tag{9}$$

Now, using (5), $(G_{\mathcal{F}}4)$, and $(G_{\mathcal{F}}2)$, we have

$$\begin{aligned} f(\epsilon) &\leq \limsup_{k\to\infty} f\left(G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k})\right) \\ &\leq \limsup_{k\to\infty} f\left(G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{n_k+1}) + G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})\right) + \alpha \\ &= \limsup_{k\to\infty} f\left(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})\right) + \alpha \\ &\leq \limsup_{k\to\infty} f\left(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k-1})\right) + 2\alpha. \end{aligned} \tag{10}$$

Now, using (10), (7), and (9), we have

$$\begin{aligned} \psi(f(\epsilon)+2\alpha) &\leq \psi\left(\limsup_{k\to\infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k})) + 2\alpha + 2\alpha\right) \\ &\leq \psi\left(\limsup_{k\to\infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) + 4\alpha\right) \\ &\leq \psi\left(\limsup_{k\to\infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k}))\right) \\ &\quad - \liminf_{k\to\infty} \phi\left(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})\right) \\ &\leq \psi(f(\epsilon)+\alpha) - \phi\left(\liminf_{k\to\infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})\right) \\ &\leq \psi(f(\epsilon)+2\alpha) - \phi\left(\liminf_{k\to\infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})\right) \end{aligned}$$

This gives

$$\liminf_{k\to\infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) = 0,$$

therefore, we have

$$\liminf_{k\to\infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) = -\infty,$$

which gives a contradiction in view of (8) and (10). Thus $\{\zeta_n\}$ is a $G_{\mathcal{F}}$-Cauchy sequence in $(\mathcal{A}, G)$, but $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$-complete. Therefore, there exists $b \in \mathcal{A}$ such that $\lim_{n\to\infty} \zeta_n = b$.

Next, we prove that b is a fixed point of T. Suppose that $Tb \neq b$, then

$$\begin{aligned} &\psi\left(f(M(b, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha\right) \\ &\leq \psi\left(f(G(Tb, T\zeta_{n+1}, T\zeta_{n+2}))\right) - \phi\left(M(b, \zeta_{n+1}, \zeta_{n+2})\right) \\ &= \psi\left(f(G(Tb, \zeta_{n+2}, \zeta_{n+3}))\right) - \phi\left(M(b, \zeta_{n+1}, \zeta_{n+2})\right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} M(b, \zeta_{n+1}, \zeta_{n+2}) &= max\{G(b, \zeta_{n+1}, \zeta_{n+2}),\ G(b, Tb, T\zeta_{n+1}), \\ &\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}),\ G(\zeta_{n+2}, T\zeta_{n+2}, Tb)\} \\ &= max\{G(b, \zeta_{n+1}, \zeta_{n+2}),\ G(b, Tb, \zeta_{n+2}), \\ &\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}),\ G(\zeta_{n+2}, \zeta_{n+3}, Tb)\}. \end{aligned}$$

Thus, by Proposition 2.6 and Proposition 2.7, we have

$$\begin{aligned} f\left(G(Tb,b,b)\right)-2\alpha &\leq \liminf_{n\to\infty} f\left(M(b,\zeta_{n+1},\zeta_{n+2})\right) \\ &\leq \limsup_{n\to\infty} f\left(M(b,\zeta_{n+1},\zeta_{n+2})\right) \\ &\leq f\left(G(Tb,b,b)\right)+2\alpha. \end{aligned} \tag{12}$$

Now, using (11) and (12), we have

$$\begin{aligned} &\psi(f(G(Tb,b,b))+2\alpha) \\ &\leq \psi\left(\limsup_{n\to\infty} f(M(b,\zeta_{n+1},\zeta_{n+2}))+4\alpha\right) \\ &\leq \psi\left(\limsup_{n\to\infty} f(G(Tb,\zeta_{n+2},\zeta_{n+3}))\right)-\liminf_{n\to\infty}\phi\left(M(b,\zeta_{n+1},\zeta_{n+2})\right) \\ &\leq \psi(f(G(Tb,b,b))+2\alpha)-\phi\left(\liminf_{n\to\infty} M(b,\zeta_{n+1},\zeta_{n+2})\right). \end{aligned}$$

It gives that

$$\liminf_{n\to\infty} M(b,\zeta_{n+1},\zeta_{n+2})=0,$$

therefore, we have

$$\liminf_{n\to\infty} f(M(b,\zeta_{n+1},\zeta_{n+2}))=-\infty,$$

which gives a contradiction in (12); therefore, $Tb=b$. Next we prove that the fixed point of T is unique. For this, let c be another fixed point of T such that $c\neq b$. Then

$$\begin{aligned} M(b,b,c) &= max\{G(b,b,c),\ G(b,Tb,Tb),\ G(b,Tb,Tc),\ G(c,Tc,Tb)\} \\ &= max\{G(b,b,c),\ G(b,b,b),\ G(b,b,c),\ G(c,c,b)\} \\ &= max\{G(b,b,c),\ G(c,c,b)\} \\ &= M(c,c,b). \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} \psi(f(G(b,b,c))) &\leq \psi\left(f(M(b,b,c))+4\alpha\right) \\ &\leq \psi\left(f(G(Tb,Tb,Tc))\right)-\phi\left(M(b,b,c)\right) \\ &= \psi\left(f(G(b,b,c))\right)-\phi\left(M(b,b,c)\right). \end{aligned} \tag{14}$$

It gives that $M(b,b,c)=0$, and hence $G(b,b,c)=G(c,c,b)=0$. Thus $b=c$. □

4 Conclusion

With the aid of $\mathcal{F}$-metric space, we have introduced a new generalization of G-metric space, which we call $G_{\mathcal{F}}$-metric space. We have also shown a comparison between $G_{\mathcal{F}}$-metric space and several abstract spaces found in literature. This newly defined abstract space is also studied in terms of some fundamental concepts. In the framework of $G_{\mathcal{F}}$-metric space, we have demonstrated the Banach Contraction Principle and the fixed point result for (ψ, ϕ)-contractive mapping. In this newly defined abstract space, fixed point results for different mappings existing in the literature and for some new mappings can be studied.

Author Contributions: All the authors have equally contributed in the planning, execution, and analysis of the study.

Acknowledgements: Authors are very thankful to the reviewers for their valuable comments and suggestions.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Gähler, S.; (1963), 2 Metrische Räme und Ihr Topological Struktur, *Mathematische Nachrichten*, **26**, 115-148.

[2] Ha, K.S.; Cho, Y.J; and White, A; Strictly convex strictly 2-convex 2-normed spaces, *Bulletin of the Calcutta Mathematical Society*, **84**, (1992), 329-336.

[3] Dhage, B.C.; (1992), Generalized metric spaces and mappings with fixed point, *Bulletin of the Calcutta Mathematical Society*, **84**, 329-336.

[4] Mustafa, Z.; and Sims, B.; (2003), Some remarks concerning D-metric spaces, *Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia (Spain)*,189-198.

[5] Naidu, S.V.R; Rao, K.P.R; and Rao, N.S; (2005), On the concepts of Balls in a D-metric space, *International Journal of Mathematics and Mathematical Sciences*, **1**, 133-141.

[6] Naidu, S.V.R; Rao, K.P.R; and Rao, N.S; (2005), On convergent sequences and fixed point theorems in D-metric spaces, *International Journal of Mathematics and Mathematical Sciences*, 1969-1988.

[7] Mustafa, Z.; and Sims, B.; (2006), A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis*, **7**, 289-297.

[8] Mustafa, Z.; (2005), A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia.

[9] Mustafa, Z.; Obiedat, H.; and Awawdeh, F.; (2008), Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory and Applications*, article ID 189870, doi: 10.1155/2008/189870.

[10] Mustafa, Z.; Shatanawi, W.; and Bataineh, M.; (2009), Existence of fixed point results in G-metric spaces, *International Journal of Mathematics and Mathematical Sciences*, article ID 283028, 10 pages, doi: 10.1155/2009/283028.

[11] Mustafa, Z.; and Sims, B.; (2009), Fixed point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory and Applications*, article ID 917175, doi: 10.1155/2009/917175.

[12] Mustafa, Z.; Awawdeh, F.; and Shatanawi, W.; (2010), Fixed point theorem for expansive mappings in G-metric spaces, *International Journal of Contemporary Mathematical Sciences*, **5**, 2463-2472.

[13] Jleli, M.; and Samet, B.; (2012), Remarks on G-metric spaces and fixed point theorems, *Fixed Point Theory and Applications*, **2012:210**.

[14] Asadi, M.; Karapinar, E.; and Salimi, P.; (2013), A new approach to G-metric and related fixed point theorems, *Journal of Inequalities and Applications*, **2013:454**.

[15] Karapinar, E.; and Aggarwal, R.P.; (2013), Further fixed point results on G-metric spaces, *Fixed Point Theory and Applications*, **2013:154**.

[16] Mustafa, Z.; Arshad, M.; Khan, S.U.; Ahmad, J.; and Jaradat, M.M.M.; (2017), Common fixed points for multivalued mappings in G-metric spaces with applications, *Journal of Nonlinear Sciences and Applications*, doi: 10.22436/jnsa.010.05.23.

[17] Matthews, G.S.; (1992), Partial metric topology, Research Report 212, Deptt. of Computer Science, University of Warwick.

[18] Zand, M.R.A.; and Nezhad, A.D.; (2011), A generalization of partial metric spaces, *Journal of Contemporary Applied Mathematics*, **24**, 86-93.

[19] Parvaneh, V.; Roshan, J.R.; and Kadelburg, Z.; (2013), On generalized weakly G_p-contractive mappings in ordered G_p-metric spaces, *Gulf Journal of Mathematics*, **1**, 78-97.

[20] Aydi, H.; Karapinar, E.; and Salimi, P.; (2012), Some fixed point results in G_p-metric spaces, *Journal of Applied Mathematics*, **2012**, 16 pages, article ID 891713.

[21] Bilgili, N.; Karapinar, E.; and Salimi, P.; (2013), Fixed point theorems for generalized contractions on G_p-metric spaces, *Fixed Point Theory and Applications*, **2013:317**.

[22] Ciric, Lj.; Alsulami, S.M.; Parvaneh, V.; and Roshan, J.R.; (2013), Some fixed point results in ordered G_p-metric spaces, *Fixed Point Theory and Applications*, **2013:317**.

[23] Barakat, M.A.; and Zidan, A.M.; (2015), A common fixed point theorem for weak contractive maps in G_p-metric spaces, *Journal of Egptian Mathematical Society*, **23**, 309-314.

[24] Popa, V.; and Patriciu, A.M.; (2015), Two general fixed point theorems for a sequence of mappings satisfying implicit relation in G_p-metric spaces, *Applied General Topology*, **16(2)**, 225-231.

[25] Ansari, A.H.; Vetro, P.; and Radenovic, S.; (2017), Existence of fixed point for $GP_{(\Lambda,\Theta)}$-contractive mappings in G_p-metric spaces, *Filomat*, **31(8)**, 2211-2218. doi: 10.2298/FIL1708211A.

[26] Gajic, L.; Kadelburg, Z.; and Radenovic, S.; (2017), G_p-metric spaces$-$symmetric and asymmetric, *University of Novi Paza Series A Applied Mathematics Informatics and Mechanics*, **9(1)**, 37-46.

[27] Yazdi, H.G.; Zand, M.R.A.; and Radenovic, S.; (2018), Coupled fixed point on G_p-metric spaces$-$symmetric and asymmetric, *Advances and Applications of Mathematical Sciences*, **17(10)**, 681-692.

[28] Popa, V.; (2020), A general fixed point theorem for two pairs of absorbing mappings in G_p-metric spaces, *Annales Mathematicae Silesianae*, doi: 10.2478/amsil-2020-0004.

[29] Aghajani, A.; Abbas, M.; and Roshan, J.R.; (2014), Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, *Filomat*, **28:6**, 1087-1101, doi: 10.2298/Fil1406087A.

[30] Czerwik, S.; (1998), Nonlinear set-valued contraction mapping in b-metric spaces, *Atti del Seminaro Matematico e Fisico dell'Universita di Modena e Reeggio Emilia*, **46**, 263-276.

[31] Mustafa, Z.; Roshan, J.R.; and Parvaneh, V.; (2013), Coupled coincidence point results for (ψ,ϕ)-weakly contractive mappings in partially ordered Gb-metric spaces, *Fixed Point Theory and Applications*, **2013**, 206.

[32] Roshan, J.R.; Shobkolaei, N.; Sedghi, S.; Parvaneh V. and Radenovic, S.; (2014), Common fixed point theorems for three maps in discontinuous G_b-metric spaces, *Acta Mathematica Scientia*, **34B(5)**, 1643-1654.

[33] Sedghi, S.; Shobkolaei, N.; Roshan J.R.; and Shatanawi, W.; (2014), Coupled fixed point theorems in G_b-metric spaces, *Matematicki Vesnik*, **66(2)**:190-201.

[34] Shahkoohi, R.J.; Kazemipour, S.A.; and Eyvali, A.R.; (2014), Tripled coincidence point under ϕ-contractions in ordered G_b-metric spaces, *Journal of Linear and Topological Algebra*, **03(03)**, 131-147.

[35] Khomdram, B.; Rohen, Y.; and Singh, T.C.; (2016), Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions, *SpringerPlus*, **5**, 1261.

[36] Kumar, J.; and Vashistha, S.; (2016), Coupled fixed point theorems in complex-valued G_b-metric spaces, *Advances in Fixed Point Theory*, **6**:341-351.

[37] Jaradat, M.M.M.; Mustafa, Z.; Khan, S.U.; Arshad, M.; and Ahmad, J.; (2017) Some fixed point results on G-metric and G_b-metric spaces, *Demonstratio Mathematica*, **207**: 190-207.

[38] Mustafa, Z.; Jaradat, M.M.M.; Aydi, H.; and Alrhayyel, A.; (2018), Some common fixed points of six mappings on G_b-metric spaces using (E.A) property, *European Journal of Pure and Applied Mathematics*, **11**:90-109.

[39] Aydi, H.; Rakic, D.; Aghajani, A.; Dosenovic, T.; Noorani, M.S.M.; and Qawaqneh, H.; (2019), On fixed point results in G_b-metric spaces, *Mathematics*, **7(7)**:617.

[40] Ege, O.; Park, C.; and Ansari, A.H.; (2020), A different approach to complex valued G_b-metric spaces, *Advances in Difference Equations*, **2020**:152.

[41] Gupta, V.; Ege, O.; Saini, R.; and Sen, M.D.L.; (2021), Various fixed point results in complete G_b-metric spaces, *Dynamic Systems and Applications*, **30(2)**, 277-293.

[42] Wangwe, L.; (2022), Coincidence fixed point theorems for p-hybrid contraction mappings in G_b-metric space with application, *Abstract and Applied Analysis*, Article ID 7688168, 13 pages.

[43] Jain, K.; and Kaur, J.; (2019), A generalization of G-metric spaces and related fixed point theorems, *Mathematical Inequalities and Applications*, **22**, 1145-1160.

[44] Jain, K.; Kaur, J.; and Bhatia, S.S.; (2022), A generalization of GP-metric space and generalized G_b-metric space and related fixed point results, *Journal of Mathematical and Computational Science*, **12**, 27 pages, Article ID 132.

[45] Jleli, M.; and Samet, B.; (2018), On a new generalization of metric spaces, *Journal of Fixed Point Theory and Application*, **20**, 128.

[46] Alnaser, L.A.; Lateef, D., Fouad, H.A., and Ahmad, J.; (2019), Relational theoretic contraction results in $\mathcal{F}$-metric spaces, *Journal of Nonlinear Sciences and Applications*, **12**, 337-344, doi: 10.22436/jnsa.012.05.06.

[47] Bera, A., Garai, H., Damjanović, B., and Chanda, A.; (2019), Some interesing results on $\mathcal{F}$-metric spaces, *Filomat*, **33:10**, 3257-3268, doi: 10.2298/FIL1910257B.

[48] Lateef, D., and Ahmad, J.; (2019), Dass and Gupta's fixed point theorem in $\mathcal{F}$-metric spaces, *Journal of Nonlinear Sciences and Applications*, **12**, 405-411, doi: 10.22436/jnsa.012.06.06.

[49] Mitrović, Z.D., Aydi, H., Hussain, N., and Mukheimer, A.; (2019), Reich, Jungck, and Berinde common fixed point results in $\mathcal{F}$-metric spaces and an application, *Mathematics*, **7**, 387, 10 pages, doi: 10.3390/math7050387.

[50] Altun, I., and Erduran, A.; (2022), Two fixed point results on $\mathcal{F}$-metric spaces, *Topological Algebra and its Applications*, **10**, 61-67, doi: 10.1515/taa-2022-0114.

[51] Dutta, P.N.; and Choudhary, B.S.; (2008), A generalization of contraction principle in metric spaces, *Fixed Point Theory and Applications*, Article ID 406368.

[52] Doric, D.; (2009), Common fixed point for generalized (ψ, ϕ)-weak contraction, *Applied Mathematics Letters*, Article ID 406368.

[53] Aghajani, A.; Abbas, M.; and Roshan, J.R.; (2014), Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Mathematica Slovaca*, **64(4)**, 941-960, doi: 10.2478/s12175-014-0250-6.

[54] Rao, N.S; and Kalyani, K; (2022), Fixed point results for (ψ, ϕ)-weak contractions in ordered b-metric spaces, *CUBO, A Mathematical Journal*, **24(2)**, 343-368, doi: 10.56754/0719-0646.2402.0343.