Proceedings of SALT 33: 666–685, 2023

Donkey disjunctions and overlapping updates*

Patrick D. Elliott
Heinrich-Heine University Düsseldorf

Abstract This paper is devoted to an analysis of anaphoric dependencies in disjunctive sentences, and consequences for the understanding of the ∃/∀ ambiguity observed with donkey anaphora. The primary focus is on *donkey disjunctions*, which are sentences where a (negated) existential in an initial disjunct appears to bind a pronoun in a later disjunct, such as "Either there's no bathroom, or its upstairs". The main empirical focus is that donkey disjunctions, like donkey anaphora, exhibit the ∃/∀ ambiguity, and more generally oscillate between homogeneous and heterogeneous readings in a context-sensitive fashion. The paper then proceeds in two steps: first, a principled analysis of donkey disjunctions is developed in the context of a Bilateral Update Semantics (BUS). BUS, by default, generates heterogeneous readings for donkey anaphora/donkey disjunctions (i.e., ∃ readings, in a positive context). In order to account of homogeneous readings, the conjecture is that sentences may be interpreted *exhaustively* relative to their negations. This has non-trivial consequences due to the non-classicality of BUS — specifically, a failure of the *Law of Non-Contradiction*.

Keywords: Dynamic semantics, update semantics, donkey anaphora, homogeneity, negation, disjunction, non-classical logic, de Morgan's laws

1 Background

1.1 Donkey anaphora

Famously, an indefinite in a quantificational restrictor can seemingly *bind* a pronoun in the quantifier's scope, as illustrated by the universal example in (1). Donkey anaphora cannot be understand within the constraints of a classical *in-scope* approach to variable binding, and therefore constitutes one of the central empirical motivations for Dynamic Semantics (DS).

(1) Every farmer who owns a^x donkey treasures it$_x$.

* I'd like to thank participants in the SALT 33 poster session, the semantics group at Heinrich-Heine University Düsseldorf, as well as the audience at the SILT workshop (NYU). I'm especially grateful to Philippe Schlenker, Chris Barker, Simon Charlow, Keny Chatain, and Jacopo Romoli for their insightful comments and feedback.

First-generation dynamic theories are tailored to derive an equivalence known as *Egli's corrolary* (Groenendijk & Stokhof 1991), which says that an implication with an existential antecedent is equivalent to a universal scoping over the implication.

(2) **Egli's corrolary:**
$\exists_x \phi \to \psi \iff \forall_x(\phi \to \psi)$

Applying Egli's corrolary to (1), assuming a first-order semantics for universal quantification, derives the *universal* truth-conditions in (3).

(3) $\forall_y, \forall_x((O(y,x) \wedge D(x)) \to T(y,x))$
Every farmer-donkey pair (y,x) is s.t. if y owns x, y treasures x

So far so good. Interrogating the predictions of (3), (1) requires that, if a farmer owns donkeys, they treasure *every* donkey that they own. Concretely, in a scenario where a farmer *Giles* owns donkeys d_1, d_2, and only treasures d_1, *Giles* will falsify the universal statement. This seems somewhat plausible for the sentence in (1)

It was subsequently noticed that this simple and elegant approach to donkey anaphora has some serious issues. As pointed out in many previous works, it's possible to construct donkey sentences which receive weaker, *existential* truth-conditions (see, e.g., Chierchia 1995; Kanazawa 1994). Consider the following minimal pair (from Champollion, Bumford & Henderson 2019: p. 2; adapted from Yoon 1996).

(4) (Usually) If a man has a garage with a window…
a. …he keeps it open while he is away.
b. …she keeps it closed while he is away.

As noted by Champollion et al., the most contextually plausible reading for (4a) is that *one of the windows is kept open*; the most contexually plausible reading for (4b) is that *all of the windows are kept closed*. The sentences don't imply that the garage in question has more than one window, but our intuitions are clear about whether the sentence is true relative to garages with multiple windows.[1]. *Egli's corrolary* predicts all donkey sentences to have ∀-readings, which is falsified by (4a). This calls the logical foundation of the dynamic approach to donkey sentences into question.

It is typically assumed that the ∃/∀ ambiguity is a property of *donkey anaphora* proper, which encompasses anaphora from a conditional antecedent to conditional consequent, and more generally, from a quantificational restrictor to the scope. In next section, I'll suggest that the ∃/∀ ambiguity is more general than is typically assumed, and extends to certain instances of anaphoric dependencies in disjunctions.

1 It's important to consider scenarios involving garages with multiple windows, since if only scenarios involving garages with a single window are considered, the ∀ and ∃ readings collapse.

1.2 ∃/∀-readings beyond donkey anaphora

Alongside donkey anaphora, the other central motivation for DS is *discourse anaphora*, as illustrated by the example in (5a). A signature property of dynamic semantics is that it licenses the equivalence in (6), often referred to as *Egli's theorem*. According to Egli's theorem, (5a) is equivalent to (5b), which feels intuitively correct.

(5) a. Giles owns a donkey, and he treasures it.
b. A donkey owned by Giles is treasured by him.

(6) **Egli's theorem:**
$\exists_x \phi \wedge \psi \iff \exists_x (\phi \wedge \psi)$

Another kind of anaphora in complex sentences that first-generation dynamic semantics *fails* to capture is illustrated by the famous example in (7), often attributed to Barbara Partee (the original observation is due to Evans 1977). We'll refer to sentences like (7) as *donkey disjunctions*, to emphasize the parallel with classical donkey sentences.[2] Clearly, the possibility of anaphora in (7) is connected to patterns of *presupposition projection* in disjunctions. It is an old observation (Karttunen 1973) that the presupposition of a second disjunct may fail to project if it is entailed by the negation of the first disjunct, as illustrated in (8), where the presupposition that *Josie used to smoke* is entailed by the negation of the first disjunct. In (7) what is important is that (i) the pronoun in the second disjunct has an indefinite antecedent, and (ii) the *negation* of the first disjunct entails the existence of a bathroom in this house.

(7) Either there's no bathroom in this house, or it's in a funny place.

(8) Either Josie has never smoked, or she just stopped.

Sentences like (7) are problematic for classical DS — this stems from the fact that in classical DS, *negating* a negative existential statement can't introduce a Discourse Referent (DR). Subsequently, some variants of DS have been developed which aim to address this problem; see, e.g., (Krahmer & Muskens 1995; Gotham 2019; Hofmann 2019; Elliott 2020, 2023).

The primary concern of this paper will be the *truth-conditions* of donkey disjunctions; an issue which has received relatively little attention in the literature. Existing claims can be summarized rather succinctly: (i) Krahmer & Muskens (1995) suggest that donkey disjunctions have *universal* truth conditions; for them, (7) is equivalent to the universal statement *every bathroom (if there are any) is in a funny place*. Note that this predicts that (7) is *false* in what Champollion et al. (2019) describe as a

2 In the literature, these are often referred to as *bathroom disjunctions*.

mixed scenario: this house has a bathroom in a funny place, and a different bathroom in a non-funny place. Universal truth-conditions seem intuitively plausible, especially in light of logical considerations. Classically, $\neg\phi \vee \psi$ is equivalent to $\phi \rightarrow \psi$ — in this instance, *if there's a bathroom in this house, then it's in a funny place*. By Egli's corrolary, the implication is expected to receive universal truth-condititons. Krahmer & Muskens's account is tailored to derive this result.

More recently, Gotham (2019) discusses readings of donkey disjunctions in light of *uniqueness inferences*. Gotham claims that donkey disjunctions entail conditional uniqueness and tailors his account in order to make this prediction. Concretely, (7) is predicted to mean that *if there's a bathroom in this house, then there's exactly one and it's in a funny place*. A straightforward argument against conditional uniqueness can be constructed after Mandelkern & Rothschild's (2020) *generalized sage plant sentences*. Consider the example in (9). If conditional uniqueness were invariably associated with donkey disjunctions, then (9) should be felt to be infelicitous.

(9) Either Sue didn't buy a sage plant, or she bought eight others along with it.

Elliott (2023) discusses another reading, which intuitively can be taken to correspond to the $\exists$-reading of a donkey sentence. Elliott contends that bathroom disjunctions like (7) can have (conditional) *existential* readings. Note that this predicts that (7) could be *true* in Champollion et al.'s mixed scenario, mentioned earlier. We'll spend a little time here motivating the existence of both $\forall$- and $\exists$-readings of donkey disjunctions, arguing that they exhibit context-sensitivity in a way that is strikingly reminiscent of donkey sentences more generally.

In order to motivate the existence of the $\exists$-reading, Elliott uses the example in (10). This is intended to be read in a context where we're wondering how Gabe paid for dinner. Clearly, what is relevant here is whether Gabe has at least one credit card that he paid with. The existence of other credit cards that weren't used to pay is completely irrelevant, hence (10) can be true if there is a credit card that Gabe paid with, and one that he didn't (a mixed scenario).

(10) Either Gabe doesn't have a credit card with him, or he paid with it.

It's also possible to construct more minimal pairs in which donkey disjunctions oscillate between $\forall$- and $\exists$-readings depending on the choice of a particular lexical item. Consider the following pair of sentences in the provided context: in (11a), what's relevant is whether Gabe remembered to bring *any* of his umbrellas (the $\exists$-reading). In (11b), what's relevant is whether Gabe forgot to bring *all* of his umbrellas (the $\forall$-reading); if rememberd any, that will be sufficient to save him from the rain.

(11) Context: *It has been raining all day. We're wondering whether Gabe got wet on the way here.*

a. Either Gabe doesn't have an umbrella, or he remembered to bring it.

b. Either Gabe doesn't have an umbrella, or he forgot to bring it.

An additional parallel between donkey sentences and donkey disjunctions is that, in a negative context the $\exists$-reading is felt to be the most salient. To the extent to which judgements come easily, (12) is felt to be false as soon as there is a bathroom in this house in a funny place. This follows immediately from existential truth-conditions under negation.

(12) Neither is there no bathroom in this house, nor is it in a funny place.

1.3 Intermediate summary

$\exists$- and $\forall$-readings of donkey disjunctions, to my knowledge, have neither been discussed nor even noticed in the existing literature. The ultimate goal of this paper will be to develop an analysis, inspired by Champollion et al. (2019), which provides a uniform account of this ambiguity. The ultimate theory proposed will be rather different however. Champollion et al. model this ambiguity by ascribing donkey sentences "gappy" truth-conditions, leading to the possibility that a donkey sentences may be neither true nor false in certain scenarios. Here, we'll develop a logic where a donkey disjunction $\neg\phi \vee \psi$ and its *negation* $\neg(\neg\phi \vee \psi)$ can both be true in certain scenarios, leading to violations of the classical law of non-contradiction. I'll ultimately suggest that this unusual property is an inevitable consequence of a logic that validates both Egli's theorem and de Morgan's equivalence. The next step will be to exploit the logic of exhaustive interpretation (Groenendijk & Stokhof 1984) in order to derive stronger readings by strengthening donkey disjunctions relative to their negations.

The analysis will proceed as follows: in section 2 I'll introduce Bilateral Update Semantics (BUS) — presented as an *update semantics* for a simple first-order fragment. The twist is that, since BUS is a *bilateral* system, expressions are associated with both positive and negative updates.[3] The initial goal will be to account for donkey disjunctions, but BUS has some additional interesting logical properties which will be relevant, such as validating Double Negation Elimination (DNE) and de Morgan's equivalences. Initially, BUS will only derive weak readings of both donkey disjunctions and donkey sentences more generally.

3 For some precursors to this strategy, see, e.g., Krahmer & Muskens (1995); van den Berg (1996); Willer (2018, 2019b,a); Elliott (2020).

2 Bilateral Update Semantics (BUS)

Bilateral Update Semantics (BUS) is a dynamic logic of anaphora designed to meet a number of logical, empirical, and conceptual desiderata. Unlike first-generation varieties of DS (Heim 1982; Kamp 1981), BUS validates Double Negation Elimination (DNE), and de Morgan's equivalences. Empirically, this means that BUS provides an account of discourse anaphora from doubly-negated sentences, as well as varieties of disjunction that have been considered problematic for DS; namely, *donkey disjunctions*, and *program disjunctions* (Groenendijk & Stokhof 1991). Conceptually, BUS improves on first-generation DS by providing a concrete algorithm for deriving a dynamic semantics for the logical connectives, on the basis of the Strong Kleene logic of indeterminacy. In short, BUS is independently motivated on both empirical and conceptual grounds. Later on, we'll see that a somewhat more controversial feature of BUS is that it generates *weak* 'heterogeneous' readings for discourse and donkey anaphora (Champollion et al. 2019), which will be the main focus of this paper.

Here, BUS is presented as a semantics for a simple, first-fragment. In the remainder of this section, we'll provide a recursive definition for $s[.]^+$ and $s[.]^-$, which associate sentences of the first-order fragment with *positive* and *negative* updates respectively. Updates are Heimian — namely, an update maps an information state s to an output information state. The notion of information state we adopt here is that of a *file* (Heim 1982). A file is simply a set of world-assignment pairs. Importantly, assignments are assumed to be *partial* functions from variables to individuals. The way that this is implemented is by assuming the existence of a special value $\#_e$ which can be thought of as the *unknown/undefined individual* $\#_e$. We say that an assignment g is 'undefined' for a variable x, just in case g maps x to $\#_e$. We'll adopt a notational convention whereby variables which aren't explicitly mentioned in a mapping are assumed to be mapped to $\#_e$, as illustrated below.

(13) $[x \rightarrow a] := \begin{bmatrix} x \rightarrow a \\ y \rightarrow \#_e \\ z \rightarrow \#_e \\ \ldots \end{bmatrix}$

2.1 Atomic sentences

Atomic sentences in BUS don't introduce anaphoric information; rather, the positive update (14a) simply returns those possibilities in the input state at which any free variables are defined, and the sentence is true; the negative update (14b) returns those possibilities in the input state at which any free variables are defined, and the sentence is false.

(14) **Atomic sentences:**

a. $s[P(x)]^+ = \{\, (w,g) \in s \mid g(x) \neq \#_e, g(x) \in I(P)(w) \,\}$

b. $s[P(x)]^- = \{\, (w,g) \in s \mid g(x) \neq \#_e, g(x) \notin I(P)(w) \,\}$

Let's consider the effect of updating a concrete file. In the following, subscripts exhaustively indicate which individuals are P. The world w_a in which only a is P is paired with assignments that map x to either a or b; the world in which nobody is P is paired with the empty assignment (i.e., the one which maps every variable to $\#_e$).

(15) $\{\, (w_a, [x \to a]), (w_a, [x \to b]), (w_\emptyset, []) \,\}$

The positive update of $P(x)$ returns just those possibilities in which x is mapped to a P; (16a). The negative update returns just those possibilities in which x is *not* mapped to a P; (16b). Possibilities in which x is undefined are absent from the output of the positive/negative updates.

(16) a. $\{\, (w_a, [x \to a]), (w_a, [x \to b]), (w_\emptyset, []) \,\} \, [P(x)]^+ = \{\, (w_a, [x \to a]) \,\}$

b. $\{\, (w_a, [x \to a]), (w_a, [x \to b]), (w_\emptyset, []) \,\} \, [P(x)]^- = \{\, (w_a, [x \to b]) \,\}$

A salient difference from Heim's (1982) *File Change Semantics* is that updates associated with an expression containing a free variable x over a state s are defined even if x isn't defined at every possibility in s (in Heim's terms, even if x isn't *familiar*). Later on, I'll introduce a notion of presupposition failure in order to capture Heimian familiarity. First, I turn to negation, which han extremely simple definition in a bilateral setting.

2.2 Negation

As is standard in a bilateral setting, negation is defined as an operator that 'flip-flops' between positive and negative outputs (17).

(17) **Negated sentences:**

a. $s[\neg\phi]^+ = s[\phi]^-$

b. $s[\neg\phi]^- = s[\phi]^+$

Note that the semantics for negation guarantees that Double Negation Elimination (DNE) is valid. This has desirable empirical consequences which permeate through the system.[4]

4 Because $s[\neg\neg\phi]^+ = s[\neg\phi]^- = s[\phi]^+$, and $s[\neg\neg\phi]^- = s[\neg\phi]^+ = s[\phi]^-$.

2.3 Existential quantification in two steps

Existential quantification is defined in two steps in BUS. First, I define *random assignment* (Groenendijk & Stokhof 1991; van den Berg 1996), as an operation on information states. The definition in (18) serves to open up referential possibilities associated with a particular variable throughout the input state. Note that random assignment isn't itself part of the logic, but will serve as an auxiliary operation in the definition of the existential.

(18) **Random assignment:**
$s + x = \{\, (w,h) \mid g[x]h, (w,g) \in s \,\}$

An illustration of the effects of random assignment relative to a fixed domain $\{\, a, b \,\}$ is given in (19). Referential possibilities are opened up in a way that is insensitive to the worldly information encoded in the input state.

(19) $$\{\, (w_a, []), (w_b, []) \,\} + x = \left\{ \begin{matrix} (w_a, [x \to a]), (w_a, [x \to b]) \\ (w_b, [x \to a]), (w_b, [x \to b]) \end{matrix} \right\}$$

Existential quantification can now be defined in terms of random assignment. The positive update associated with $\exists x \phi$ simply opens up referential possibilities for x, and positively updates the output with ϕ. This is effectively identical to how existential quantification is ordinarily defined in DS. The negative update (20b) is a little more involved, and leans on the notion of *subsistence*. First, note that the negative update induced by an existential statement can never introduce any anaphoric information, it simply picks out possibilities in the input state. Concretely, it picks out those possibilities that don't 'survive' (*subsist*) in the positive update, but do survive random assignment followed by the negative update of the scope.

(20) **Existential quantification:**
a. $s[\exists x \phi]^+ = (s + x)[\phi]^+$
b. $s[\exists x \phi]^- = \{\, i \in s \mid i \not\prec (s + x)[\phi]^+, i \prec (s + x)[\phi]^- \,\}$

Let's unpack the negative update a little bit. First, consider the formal definition for subsistence in (21) (Groenendijk, Stokhof & Veltman 1996). Essentially, this says that a possibility i *subsists* in a state s, just in case we can find a possibility i' in s that is just like i except for possibly providing values for more variables.

(21) **Subsistence**: $i \prec s$ (i *subsists* in s) iff there is a $i' \in s$ such that i' is a *descendant* of i. (w', g') is a descendant of (w, g) iff $w = w'$ and g' is defined an every variable that g is defined for.

Let's consider how this works in a concrete case: the negative update of $\exists x P(x)$ picks out just those possibilities in s that can't be extended to ones where x is a P, and can be extended to ones where x isn't a P; this amounts to just those possibilities in s in which there are no Ps, as indicated in (22).

(22) a. $s[\exists x P(x)]^- = \{\, i \in s \mid i \not\prec (s+x)[P(x)]^+, i \prec (s+x)[P(x)]^- \,\}$

b. $= \left\{ i \in s \;\middle|\; \begin{array}{l} i \not\prec \{\, (w,h) \mid g[x]h, (w,g) \in s, h(x) \in I(P)(w) \,\} \\ , i \prec \{\, (w,h) \mid g[x]h, (w,g) \in s, h(x) \notin I(P)(w) \,\} \end{array} \right\}$

c. $= \{\, (w,g) \in s \mid I(P)(w) = \emptyset \,\}$

Since $P(x)$ is bivalent once x is defined, the requirement that i be extendable to a possibility in which x isn't a P becomes vacuous, so why do we impose this condition in the semantics? This is to guarantee that anaphoric presuppositions associated with free variables project. Consider, e.g., a case where the scope of the existential contains a globally free variable: $\exists x R(x,y)$. This preserves a possibility i just in case i isn't extendable (at x) to one where x R-ed y, and is extendable to one where x didn't R y. The latter requirement will only be satisfiable by a possibility that already provides a value for y, hence the anaphoric presupposition associated with y is inherited by the entire existential statement, as shown in (23).

(23) a. $s[\exists x R(x,y)]^- = \{\, i \in s \mid i \not\prec (s+x)[R(x,y)]^+, i \prec (s+x)[R(x,y)]^- \,\}$

b. $= \{\, (w,g) \in s \mid g(y) \neq \#_e, \{\, x \mid I(R)(x, g(y))(w) \,\} = \emptyset \,\}$

To close out this section, recall that we adopted a 'flip-flop' entry for negation in section 2.2. This provides a straightforward resolution for a well-known problem in the dynamic literature — doubly-negated sentences may introduce DRS just like their positive counterparts.[5]

(24) Steven doesn't own no car — it's a ferrari.

2.4 The logical connectives

2.4.1 Presupposition failure and assertion

In order to define the logical connectives, it is important to define a dynamic notion of *presupposition failure* — this is because the account of anaphoric accessibility is layered on a trivalent account of presupposition projection. Relative to a sentence ϕ, presupposition failure picks out the possibilities in s which neither subsist in the positive update of ϕ, nor the negative update of ϕ (i.e., the possibilities at which ϕ is

5 This is effectively the same strategy used by Krahmer & Muskens (1995).

undefined). Crucially for the semantics of the logical connectives 'presupposition failure' is defined as an update; specifically, an update that is incapable of introducing anaphoric information.

(25) **Presupposition failure:**
$s[\phi]^? = \{\, i \in s \mid i \not\prec s[\phi]^+, i \not\prec s[\phi]^- \,\}$

A by-product of defining presupposition failure is that it is now straightforward to define a pragmatic notion of assertion, stated below in (26). This corresponds straightforwardly to the classical notion of *Stalnaker's bridge* (von Fintel 2008).[6] Note that one immediate consequence of (26) is that Heimian familiarity follows straightforwardly from the requirement that the failure update returns the empty set — in other words, for a sentence containing a free variable x to be assertable at a file context, x must be defined at every possibility in the file context.

(26) **Assertion**
Given a sentence ϕ, and a file context c:

a. Assertion of ϕ against c is defined if $c[\phi]^? \neq \emptyset$.

b. If defined, assertion of ϕ against c results in a new file context $c[\phi]^+$.

Having defined a dynamic notion of presupposition failure, I'll now introduce the general strategy for defining logical connectives in BUS, which is based on the Strong Kleene logic of indeterminacy. In fact, we straightforwardly translate the Strong Kleene truth-tables into recipes for composing updates. Let's begin with conjunction.

2.4.2 Conjunction

Consider the Strong Kleene truth-table for conjunction in fig. 1, where $+$ corresponds to *true*, $-$ corresponds to *false*, and ? corresponds to *maybe true and maybe false*. We'll use the truth-table to determine the positive/negative updates of conjunction in the following way: the positive update of a complex sentence is the union of all the ways in which the sentence can be dynamically verified (i.e., the $+$ cells); the negative update is the union of all the ways in which the sentence can be dynamically falsified (i.e., the $-$ cells).

6 Stalnaker's bridge is a way of connecting sentences carrying presuppositions qua partial propositions to the pragmatic requirements they place on the discourse context. Classically, a partial proposition must be defined at every world in the context set for assertion to be felicitous. Here, every possibility in the context set must subsist in a non-failure state.

$\phi \vee \psi$	ψ_{+}	ψ_{-}	$\psi_{?}$
ϕ_{+}	$+$	$-$	$?$
ϕ_{-}	$-$	$-$	$-$
$\phi_{?}$	$?$	$-$	$?$

Figure 1 Strong Kleene conjunction

Of course, the only way to verify a conjunction is to verify both conjuncts; in BUS this corresponds to successively processing the *positive* update of each conjunct.

(27) **Conjunction** (pos.)
$s[\phi \wedge \psi]^{+} = s[\phi]^{+}[\psi]^{+}$

Falsifying a conjunctive statement is more complex in strong Kleene trivalent semantics, but it follows straightforwardly from principles of reasoning about indeterminacy. To falsify a conjunction, it's enough to know if one conjunct or the other is false, which means that the entire conjunction may be false even if one of the conjuncts is a presupposition failure. In BUS, we use this as a recipe for computing the negative update of a conjunction — concretely, we gather up the different ways of dynamically falsifying a conjunctive statement. The notion of presupposition failure defined in (25) is used to specify the output state in the event of a presupposition failure. The resulting semantics for conjunction (negative) is given below:

(28) **Conjunction** (neg.)

$$s[\phi \wedge \psi]^{-} = s[\phi]^{-}[\psi]^{+} \cup s[\phi]^{-}[\psi]^{-} \cup s[\phi]^{-}[\psi]^{?}$$
$$s[\phi]^{+}[\psi]^{-} \cup s[\phi]^{?}[\psi]^{-}$$

At this stage, we won't explore the predictions for anaphora and negated conjunctions in detail. Rather, we'll move on to the semantics of disjunction, which will allow us to start thinking about the central datapoints discussed at the beginning of the paper.

2.4.3 Donkey disjunctions in BUS

The positive update of a disjunctive sentence follows from the Strong Kleene truth-table for disjunction. What's pertinent here is that it's possible to verify the truth of a disjunction, just so long as one of the disjuncts is true — unknown truth/falsity of the other disjunct is tolerated.

$\phi \vee \psi$	ψ_+	ψ_-	$\psi_?$
ϕ_+	$+$	$+$	$+$
ϕ_-	$+$	$-$	$?$
$\phi_?$	$+$	$?$	$?$

Figure 2 Strong Kleene disjunction

(29) **Disjunction** (pos.)

$$s[\phi \vee \psi]^+ = s[\phi]^+[\psi]^+ \cup s[\phi]^+[\psi]^- \cup s[\phi]^+[\psi]^?$$
$$s[\phi]^-[\psi]^+ \cup s[\phi]^?[\psi]^+$$

Let's see how this accounts for anaphora in donkey disjunctions, and the predicted truth-conditions.

(30) Either there's no^x bathroom, or it$_x$'s upstairs.
$\neg\exists x B(x) \vee U(x)$

It will be helpful to begin by focusing on the first line of the positive update in (29), which intuitively corresponds to verifying the disjunction via the truth of the first disjunct. Since the first disjunct is a negative existential statement, it fails to introduce a DR x. The positive/negative update and presupposition failure of $U(x)$ are guaranteed to be a partition of the input state, therefore the first line simply corresponds to the negative update of $\exists x B(x)$, i.e., the possibilities in the input state in which there are no bathrooms.

(31) **Donkey disjunction: verification via the first disjunct**

$$s[\exists x B(x)]^-[U(x)]^+ \cup s[\exists x B(x)]^-[U(x)]^- \cup s[\exists x B(x)]^-[U(x)]^?$$
$$= s[\exists x B(x)]^-$$
$$= \{\, (w,g) \in s \mid I(B)(w) = \emptyset \,\}$$

Let's now consider the second line of the update outlined in (29), which intuitively corresponds to verifying the disjunction via the truth of the second disjunct. Since DNE is valid in BUS, the negative update of *there is no bathroom* introduces a bathroom DR x. Since the first disjunct is an existential statement, it can never be a presupposition failure, so we can ignore this case and simply compute the update corresponding to the information that *there is a bathroom upstairs*, as below.

(32) **Donkey disjunction: verification via the second disjunct**

$$s[\neg\exists x B(x)]^-[U(x)]^+ \cup s[\neg\exists x B(x)]^?[U(x)]^+$$
$$= s[\exists x B(x)]^+[U(x)]^+$$
$$= \{\, (w,h) \mid (w,g) \in s, g[x]h, h(x) \in I(B)(w) \cap I(U)(w) \,\}$$

The positive update of disjunction is simply the union of (31) and (32); possibilities in which (i) *there are no bathrooms*, and (ii) *there is a bathroom upstairs*, are both preserved, but DRS are only introduced at those possibilities in which there is a bathroom upstairs. Possibilities in which *there are bathrooms but all are not upstairs* are eliminated.

(33) **Bathroom disjunctions: positive update**

$$s[\neg\exists x B(x) \vee U(x)]^+ = \{\, (w,g) \in s \mid I(B)(w) = \emptyset \,\} \cup \{\, (w,h) \mid (w,g) \in s, g[x]h, h(x) \in I(B)(w) \cap I(U)(w) \,\}$$

Note that this corresponds to a heterogeneous $\exists$-*reading*: possibilities in which *there is a bathroom upstairs and a bathroom not upstairs* are preserved by the positive update, and paired with *bathroom upstairs* DRS. We'll now move on to the *negative* update of disjunction (which is much simpler), and how it applies to bathroom disjunctions specifically. The only way to falsify a disjunction is for both disjuncts to be false — this corresponds to the negative update in (34).

(34) **Disjunction** (neg.)

$$s[\phi \vee \psi]^- = s[\phi]^-[\psi]^-$$

Applying this recipe to a bathroom disjunction is very straightforward. Since DNE is valid, it basically expresses that *there is a bathroom that isn't upstairs*.

(35)
$$\begin{aligned} & s[\neg\exists x B(x) \vee U(x)]^- \\ & = s[\exists x B(x)]^+[U(x)]^- \\ & = \{\, (w,h) \mid (w,g) \in s, g[x]h, h(x) \in I(B)(w) - I(U)(w) \,\} \end{aligned}$$

Note that again, this corresponds to a *heterogeneous* reading; this time, due to the presence of negation, it corresponds to a heterogeneous $\forall$-reading.

The semantics outlined for disjunction is, on the face of it, not obviously problematic. As we've seen, donkey disjunctions can readily receive heterogeneous $\exists$-readings, and it's even possible to generate *negated* donkey-disjunctions which arguably receive heterogeneous readings, with the right context. Consider the dialogue in (36), and most especially B's response. In the provided discourse context, B is most readily understood as asserting that *Gabe has an umbrella, and he remembered to bring one of his umbrellas, which turned out to have a leak.*

(36) A: It's been raining, and Gabe looks wet! I guess that either he doesn't own an umbrella, or he forgot to bring it.
B: Neither does Gabe have no umbrella, nor did he forget to bring it
— it has a leak.

So, these readings are attested, but nevertheless BUS has some properties which strike one as highly unusual. To begin with, since the heterogeneous reading of $\neg\exists x B(x) \vee U(x)$ is compatible with the hetereogeneous reading of $\neg(\neg\exists x B(x) \vee U(x))$, the expression in (37) should be non-contradictory (i.e., have a non-empty positive update). It will be non-empty just in case there is a bathroom which is upstairs, and a bathroom which isn't upstairs. This means that the classical Law of Non-Contradiction (LNC) doesn't hold in BUS.

(37) $(\neg\exists x B(x) \vee U(x)) \wedge \neg(\neg\exists x B(x) \vee U(x))$

In fact, this is exactly the property which I'll exploit in order to account for homogeneous readings of donkey disjunctions, but it will also be important to explain why an expression like (37) doesn't correspond to a felicitous assertion in natural language.

Another interesting property of the resulting logic is that de Morgan's equivalences are valid — to my knowledge, this is the only variant of DS in which this is so (except for the bilateral variant of Dynamic Predicate Logic (DPL) first developed in Elliott 2020). This is shown concretely in (38).

(38) **de Morgan's equivalences in BUS**

a. $s[\neg\phi \wedge \neg\psi]^{+} = s[\neg(\phi \vee \psi)]^{+} = s[\phi]^{-}[\psi]^{-}$

b. $s[\neg(\phi \wedge \psi)]^{+} = s[\neg\phi \vee \neg\psi]^{+} = s[\phi]^{-}[\psi]^{+} \cup s[\phi]^{-}[\psi]^{-} \cup s[\phi]^{-}[\psi]^{?}$
$s[\phi]^{+}[\psi]^{-} \cup s[\phi]^{?}[\psi]^{-}$

This equivalence can be used to see the predictions made for negated conjunctions, i.e., (40), which receives a heterogeneous reading just like its donkey disjunction counterpart in (39).

(39) Either there's no bathroom, or it's not upstairs.
If there's a bathroom, then there's a bathroom which isn't upstairs

(40) It's not the case that there's a bathroom and it's upstairs.
If there's a bathroom, then there's a bathroom which isn't upstairs

This fact is particularly interesting, as it relates to a weakening of the principle known as *Egli's theorem*. Although its obvious that Egli's theorem holds in the positive instance, as in (41b), the negative updates are not necessarily equivalent (41b). This is because negated discourse anaphora is equivalent to a donkey disjunction, whereas a negated existential statement isn't.

(41) **Egli's theorem in BUS**

a. $s[\exists x\phi \wedge \psi]^{+} = s[\exists x(\phi \wedge \psi)]^{+}, \forall s$

b. $s[\exists x\phi \wedge \psi]^- \neq s[\exists x(\phi \wedge \psi)]^-, \exists s$

In the next section, I turn to the question of how to derive *homogeneous* readings of donkey disjunctions, via the logic of exhaustive interpretation. The initial theory will be too strong, deriving homogeneous readings everywhere. It will subsequently be weakened relative to a Question under Discussion (QUD), following Champollion et al. (2019).

3 Homogeneous readings

3.1 Relation to exhaustive interpretation

Polar questions such as "Is it raining?" can be understood as inducing an equivalence relation between worlds, thereby partitioning logical space (Groenendijk & Stokhof 1984). In the literature on the semantics of questions, it's common to derive this equivalence relation from a set of alternatives introduced by the question (Heim 1994). For example, the alternatives introduced by "Is it raining?" correspond to possible answers to the polar question:

(42) $\{\textit{it's raining}, \textit{it's not raining}\}$

An equivalence relation is derived based on the following recipe. Together with (42), this means that two worlds are equivalent with respect to the question of *whether it's raining* just in case they agree on whether it's raining.

(43) $w \sim w' \iff \forall p \in Q[p(w) = p(w')]$

Due to the classical LNC, polar questions typically give rise to *two cells*, corresponding to the positive answer and the negative answer. The insight behind our account of homogeneous readings is that since a donkey disjunction and its negation can simultaneously be true, the resulting equivalence relation partitions logical space into *three* cells. Consider, the following set of alternatives, corresponding to the heterogeneous reading of a donkey disjunction, and its negation:

(44) $\left\{\begin{array}{l}\textit{if there's a bathroom, then there's a bathroom upstairs},\\ \textit{there's a non-upstairs bathroom}\end{array}\right\}$

Combined with the recipe in (43), this gives rise to three cells:

- **Cell 1:** *every bathroom (if any) is upstairs* (postitive alternative is true; negative alternative isn't true).
- **Cell 2:** *there's a bathroom, and every bathroom is not upstairs* (positive alternative is false, negative alternative is true).

- **Cell 3:** *there's a bathroom upstairs, and a bathroom not upstairs* (both alternatives are true).

The intuition behind the account of homogeneous readings of donkey disjunctions is that a sentence ϕ, when asserted, is by default interpreted exhaustively relative to the polar question *whether* ϕ. In a classical setting, this is trivial — if ϕ is true, then $\neg\phi$ isn't, due to the LNC, so ϕ simply identifies the ϕ cells. Since the LNC fails in BUS, this becomes non-trivial — interpreting ϕ exhaustively amounts to identifying the cell in which ϕ is true, and $\neg\phi$ isn't.[7]

In the context of a concrete donkey disjunction "Either there's no bathroom, or its upstairs", this has the following effect: interpreting this sentence exhaustively relative to its negation picks out **Cell 1**, where every bathroom (if any) is upstairs. This is because (a) the sentence is true in worlds where there's no bathroom, and worlds where there is at least one bathroom upstairs, and (b) its negation *Neither is there no bathroom nor is it upstairs* is true in worlds where there is at least one bathroom not upstairs. Focusing on the a-not-b worlds gives us the exhaustive interpretation, which corresponds to the homogeneous reading of the donkey disjunction.

As the reader can verify for themselves, interpreting a negated donkey disjunction "Neither is there no bathroom, nor is it upstairs" exhaustively relative to *its* negation picks out **Cell 2**, so homogeneous readings for negated donkey disjunctions are also derived.

How to implement this technically? This can be done in a number of different ways, including via an *exhaustification operator*.[8] In the interests of expository simplicity, the simplest approach is to define a notion of exhaustive assertion, ultimately intended to be replaced by some more general notion of exhaustivity.

(45) **Exhaustive assertion (ver. 1)**
Given a sentence ϕ, and a file context c:

a. Exhaustive assertion of ϕ against c is defined if $c[\phi]^{?} \neq \emptyset$.

b. If defined, exhaustive assertion of ϕ against c results in a new file context $\{\, (w,g) \in c[\phi]^{+} \mid (w,*) \notin c[\phi]^{-} \,\}$.

7 The logic behind the account is similar to ideas developed in Spector (2021), who develops a static semantics for discourse and donkey anaphora. In Spector's static semantics, weak $\exists$-readings for donkey sentences are generated by default, and strong $\forall$-readings are derived via a a pragmatic strengthening principle. I leave a more detailed consideration of static approaches to donkey anaphora to future work.

8 This is perhaps not so trivial as is implied here. An important assumption underlying the account is that, given a sentence ϕ, $\neg\phi$ is an *alternative* to ϕ. This goes against the prevailing narrative in the literature on alternatives that alternatives to a sentence ϕ only include those sentences that are *at most as complex as* ϕ, given some suitable notion of structural complexity (Fox & Katzir 2011).

Exhaustive interpretation essentially guarantees that a sentence ϕ is interpreted as being "just true" relative to its negation.

3.2 Weakening relative to the QuD

The initial version of exhaustive assertion simply generates homogeneous readings everywhere — we're already seen that this is certainly *not* a desirable result. We need a way of generating heterogeneous readings, in a way that is sensitive to the discourse context. In this final seciton of the paper, I'll show how to incorporate the solution proposed by Champollion et al. (2019) into BUS.

Champollion et al.'s idea is that the availability of a heterogeneous reading is dependent on a salient QUD. The simplest implementation of the QUD is as a contextually-provided question Q, which partitions the context set into cells using the induced equivalence relation $\sim_Q$. Using this notion, I weaken the definition of exhaustive assertion relative to a QUD Q in (46).

(46) **Exhaustive assertion (final)**
Given a sentence ϕ, a file context c, and a QUD Q:

a. Exhaustive assertion of ϕ against c is defined if $c[\phi]^? \neq \emptyset$.

b. If defined, exhaustive assertion of ϕ against c results in a new file context $\{\, (w,g) \in c[\phi]^+ \mid \exists w', w \sim_Q w', (w',*) \notin c[\phi]^- \,\}$.

This is substantially weaker than the initial formulation of exhaustive assertion. Instead of retaining possibilities only in the positive update, (46) demands that possibilities from the positive update be retained which have a *cell-mate* not in the negative update.

Following Champollion et al. (2019), I assume that if no specific question is contextually salient, an assertion is interpreted in light of the *fact-finding question*, according to which no two worlds are Q-equivalent. The fact-finding question simply asks how things actually are. Relative to the fact-finding question, the final definition of exhaustive assertion (46) simply amounts to the initial definition (45). This is because, relative to the fact-finding question, a world is only Q-equivalent to itself. This means that, relative to the fact-finding question, (46) derives homogeneous readings across the board, as laid out in the previous section. In order to derive heterogeneous readings, the flexibility of question-sensitive exhaustivity will be crucial.

To illustrate how heterogeneous readings may be derived, imagine that we're in a context where the salient question Q is "How did Gabe pay for dinner?", which corresponds to the alternatives $\{$*Gabe paid with a credit card*, *Gabe paid with cash*$\}$. This will induce an equivalence relation $\sim_Q$ which partitions the context set into possibilities which agree on how Gabe paid (I'll assume that it's contextually entailed that Gabe didn't use both methods of payment, and ignore this cell). We have a **Credit card cell**, which includes worlds in which Gabe paid with all of his credit cards, and worlds in which he paid with only some, and a **Cash cell**, which includes worlds in which Gabe paid with cash. Crucially, the question renders the distinction between mixed and non-mixed scenarios with respect to credit cards irrelevant. The revised notion of assertion in (46), when applied to the sentence "Either Gabe has no credit card, or he paid with it" in this context will leave mixed possibilities in which Gabe has multiple credit cards and just paid with one of them, as long as its contextually possible that there's no credit card which wasn't used for paying. This is because possibilities in which Gabe paid with at least one credit card are all Q-equivalent.

4 Conclusion

In this paper, I've primarily focused on novel data coming from *donkey disjunctions*, but the analysis outlined here generalizes readily to more familiar donkey sentences too. Due to the use of the Strong Kleene truth-tables in BUS, many classical equivalences which don't go through in other variants of DS do go through in BUS. For example, the equivalence in (47), assuming a Strong Kleene semantics for material implication. Via the equivalence in (47) and DNE, its easy to see the predictions made for donkey sentences like that in (48b) — it is predicted to have the same truth-conditions as the donkey disjunction in (48a), and $\forall$-readings can be derived using the same mechanism.

(47) $\phi \to \psi \iff \neg\phi \vee \psi$

(48) a. Either there isn't a bathroom, or its upstairs.

b. If there's a bathroom, then its upstairs.

However, material implication is of course not sufficient as an analysis of natural language conditionals, and this account won't extend to donkey anaphora from a quantificational restrictor to its scope more generally. In future work, I plan to examine the viability of extending the BUS account of $\exists/\forall$ ambiguities via a Strong Kleene semantics for generalized quantifiers.

References

Champollion, Lucas, Dylan Bumford & Robert Henderson. 2019. Donkeys under discussion. *Semantics and Pragmatics* 12(0). 1. doi:10.3765/sp.12.1.

Chierchia, Gennaro. 1995. *Dynamics of Meaning - Anaphora, Presupposition, and the Theory of Grammar*. Chicago: University of Chicago Press.

Elliott, Patrick D. 2020. Towards a principled logic of anaphora. Lingbuzz/005562.

Elliott, Patrick D. 2023. Disjunction in a Predictive Theory of Anaphora. In Dun Deng, Mingming Liu, Dag Westerståhl & Kaibo Xie (eds.), *Dynamics in Logic and Language*, vol. 13524, 76–98. Cham: Springer International Publishing. doi:10.1007/978-3-031-25894-7_4.

Evans, Gareth. 1977. Pronouns, Quantifiers, and Relative Clauses (I). *Canadian Journal of Philosophy* 7(3). 467–536.

Fox, Danny & Roni Katzir. 2011. On the characterization of alternatives. *Natural Language Semantics* 19(1). 87–107. doi:10.1007/s11050-010-9065-3.

Gotham, Matthew. 2019. Double negation, excluded middle and accessibility in dynamic semanttics. In Julian J. Schlöder, Dean McHugh & Floris Roelofsen (eds.), *Amsterdam Colloquium 22*, 142–151.

Groenendijk, Jeroen & Martin Stokhof. 1984. *Studies on the Semantics of Questions and the Pragmatics of Answers*: Institute for Logic, Language and Computation (ILLC) PhD dissertation.

Groenendijk, Jeroen & Martin Stokhof. 1991. Dynamic predicate logic. *Linguistics and Philosophy* 14(1). 39–100.

Groenendijk, Jeroen a. G., Martin J. B. Stokhof & Frank J. M. M. Veltman. 1996. Coreference and modality. In *The Handbook of Contemporary Semantic Theory* Blackwell Handbooks in Linguistics, 176–216. Oxford: Blackwell.

Heim, Irene. 1982. *The Semantics of Definite and Indefinite Noun Phrases*: University of Massachusetts - Amherst PhD dissertation.

Heim, Irene. 1994. Interrogative semantics and Karttunen's semantics for 'know'. In R. Buchalla & A. Mittwock (eds.), *Israeli Association for Theoretical Linguistics (IATL)*, Jerusalem.

Hofmann, Lisa. 2019. The anaphoric potential of indefinites under negation and disjunction. In Julian J. Schlöder, Dean McHugh & Floris Roelofsen (eds.), *Amsterdam Colloquium 22*, 181–190.

Kamp, Hans. 1981. A theory of truth and semantic representation. In Paul Portner & Barbara H. Partee (eds.), *Formal Semantics*, 189–222. Blackwell.

Kanazawa, Makoto. 1994. Weak vs. Strong Readings of Donkey Sentences and Monotonicity Inference in a Dynamic Setting. *Linguistics and Philosophy* 17(2). 109–158.

Karttunen, Lauri. 1973. Presuppositions of Compound Sentences. *Linguistic Inquiry* 4(2). 169–193.

Krahmer, Emiel & Reinhard Muskens. 1995. Negation and Disjunction in Discourse Representation Theory. *Journal of Semantics* 12(4). 357–376. doi:10.1093/jos/12.4.357.

Mandelkern, Matthew & Daniel Rothschild. 2020. Definiteness projection. *Natural Language Semantics* 28(2). 77–109. doi:10.1007/s11050-019-09159-2.

Spector, Benjamin. 2021. Trivalence and transparency: A non-dynamic approach to anaphora. Slides from a talk given at the Linguistics in Göttingen Colloquium.

van den Berg, M. H. 1996. *Some Aspects of the Internal Structure of Discourse. The Dynamics of Nominal Anaphora*: AmsterdamILLC PhD dissertation.

von Fintel, Kai. 2008. What Is Presupposition Accommodation, Again?*. *Philosophical Perspectives* 22(1). 137–170. doi:10.1111/j.1520-8583.2008.00144.x.

Willer, Malte. 2018. Simplifying with Free Choice. *Topoi. Orient – Occident* 37(3). 379–392. doi:10.1007/s11245-016-9437-5.

Willer, Malte. 2019a. Negating conditionals in bilateral semantics. In Julian J. Schlöder, Dean McHugh & Floris Roelofsen (eds.), *Amsterdam Colloquium 22*, 181–190.

Willer, Malte. 2019b. Widening free choice. In *Amsterdam Colloquium 22*, 407–416.

Yoon, Youngeun. 1996. Total and partial predicates and the weak and strong interpretations. *Natural Language Semantics* 4(3). 217–236. doi:10.1007/BF00372820.

Patrick D. Elliott
Heinrich-Heine Universität Düsseldorf
Universitätsstraße 1
40225 Düsseldorf
Germany
patrick.d.elliott@gmail.com