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ON A SYSTEM INVOLVING AN INTEGRO-DIFFERENTIAL INCLUSION WITH SUBDIFFERENTIAL AND CAPUTO FRACTIONAL DERIVATIVE

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The current work is concerned with a new system involving an integro-differential inclusion of subdifferential type and Caputo fractional derivative, in Hilbert spaces. We use a discretization approach to deal with the integro-differential inclusion. Then, we proceed by a fixed point theorem to handle the considered system.

1. Introduction

We are interested, in this paper, in a new system governed by an integro-differential inclusion and Caputo fractional derivative as follows

$$\begin{cases} -\dot{u}(t) \in \partial\psi(t,u(t)) + g_1(t,x(t),u(t)) + \int_0^t g_2(t,s,x(s),u(s))ds \\ \quad \text{a.e. } t \in I := [0,T], \\ w\text{-}^cD^\alpha x(t) = u(t) \quad \text{a.e. } t \in I, \\ x(0) - w\text{-}\frac{dx}{dt}(0) = 0, \\ x(T) + w\text{-}\frac{dx}{dt}(T) = 0, \\ \quad u(0) = u_0 \in \operatorname{dom}\psi(0,\cdot), \end{cases} \tag{1}$$

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where $\alpha \in]1,2]$, w-${}^cD^\alpha$ denotes the weak Caputo fractional derivative, $\partial\psi(t,\cdot)$ stands for the subdifferential of a proper, lower semi-continuous, convex map $\psi(t,\cdot)$ from a real Hilbert space H to $[0,+\infty]$. The real-valued map $\psi(t,\cdot)$ (whose effective domain is denoted $\operatorname{dom}\psi(t,\cdot)$) satisfies an assumption expressed in term of its conjugate function (see (H^2_ψ) below).
The maps $g_1 : I \times H \times H \to H$ and $g_2 : I \times I \times H \times H \to H$ are single-valued maps satisfying suitable assumptions.

For this purpose, we first study the existence and uniqueness of absolutely continuous solution to a new class of integro-differential inclusions of the form

$$\begin{cases} -\dot{u}(t) \in \partial\psi(t,u(t)) + g_1(t,u(t)) + \displaystyle\int_0^t g_2(t,s,u(s))ds \quad \text{a.e. } t \in I, \\ u(0) = u_0 \in \operatorname{dom}\psi(0,\cdot), \end{cases} \tag{2}$$

The interest in (2) is motivated by the study of integro-differential sweeping processes (see [10], [11]), that is the problem above with $\partial\psi(t,\cdot) = N_{C(t)}(\cdot)$, i.e., the normal cone of a moving set $C(t)$ which is r-prox regular. Let us mention some related results in [9], [12], [24]. We also cite the recent papers [15], [18], dealing with integro-differential inclusions involving m-accretive (or maximal monotone) operators (instead of the subdifferentials).

The differential inclusion (2) involving only g_1 (resp. g_2) has been studied in [42] (resp. [8]). In the proof of the well-posedness theorem to (2), we proceed by a discretization approach. However in [8], a fixed point method is used there. In our development, we construct a sequence of solutions to perturbed differential inclusions with single-valued perturbations (depending only on time) in each subinterval (using the existence result in [39] and the estimates in [43]). Then, we prove its convergence to the solution of our problem (2).

Other variants of first-order differential inclusions of subdifferential type (with their applications) can be found in the scientific literature, see for instance [5], [6], [13], [14], [20], [21], [28], [29], [31], [39], [41], [43], [45], [47]. As examples of this class of evolution problems are sweeping processes, we refer the reader to some achievements on this topic (under different assumptions on the sets $C(t)$ or $C(t,x)$) in [1], [23], [26], [27], [32], [33], [34], [35], [36], [37], [38], [46], among others.

In the proof of the existence result to (1), we adopt a fixed point theorem, using the topological properties of the solution set to the Caputo fractional differential equation. Some coupled systems driven by evolution problems of subdifferential type and fractional differential equations have been considered in [8], [22] and [41].
Contributions on fractional differential theory have been discussed by many authors, see [2], [3], [4], [7], [16], [17], [19], [30], [40], [44], [48], and the references therein.

The paper consists of four sections. In Section 2, we introduce notation and preliminaries. In Section 3, we prove the well-posedness result concerning (2). Section 4 is devoted to study (1).

2. Notation and preliminaries

In all that comes, let $I := [0,T]$ denote an interval of $\mathbb{R}$ and let H be a real Hilbert space whose inner product is denoted by $\langle \cdot,\cdot \rangle$ and its associated norm by $\|\cdot\|$. We denote by $\overline{B}_H[u,r]$ the closed ball of center u and radius r on H, and by $\overline{B}_H$ the closed unit ball on H.
On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence on I, $\|x\|_\infty = \sup_{t\in I} \|x(t)\|$.
By $L^p_H(I)$ for $p \in [1,+\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|_{L^\infty_H(I)}$). Denote by $W^{1,2}_H(I)$, the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$.

Let ψ be a lower semi-continuous convex function from H into $\mathbb{R} \cup \{+\infty\}$ which is proper in the sense that its effective domain $(\operatorname{dom}\psi)$ defined by

$$\operatorname{dom}\psi = \{x \in H : \psi(x) < +\infty\}$$

is non-empty. As usual, its Fenchel conjugate is defined by

$$\psi^*(v) = \sup_{x\in H} [\langle v,x\rangle - \psi(x)].$$

The subdifferential $\partial\psi(x)$ of ψ at $x \in \operatorname{dom}\psi$ is the set

$$\partial\psi(x) = \{v \in H : \psi(y) \ge \langle v, y-x\rangle + \psi(x)\ \forall y \in \operatorname{dom}\psi\},$$

and its effective domain is $\operatorname{Dom}\partial\psi = \{x \in H : \partial\psi(x) \neq \emptyset\}$. It is well known (see, e.g., [13]) that if ψ is a proper lower semi-continuous convex function, then its subdifferential operator $\partial\psi$ is a maximal monotone operator.
Let S be a non-empty subset of H. Denote by $\mathbf{1}_S$ the indicator function of S, that is, $\mathbf{1}_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise.

A Gronwall-like differential inequality is proved in [10] as follows:

Lemma 2.1. *Let $y : I \to \mathbb{R}$ be a non-negative absolutely continuous function and let $h_1, h_2, g : I \to \mathbb{R}_+$ be non-negative integrable functions. Suppose for some $\varepsilon > 0$*

$$\dot{y}(t) \le g(t) + \varepsilon + h_1(t)y(t) + h_2(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds \text{ a.e. } t \in I.$$

Then, for all $t \in I$, one has

$$(y(t))^{\frac{1}{2}} \leq (y(0)+\varepsilon)^{\frac{1}{2}} \exp\left(\int_0^t (h(s)+1)ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2}\int_0^t \exp\left(\int_s^t (h(r)+1)dr\right)ds$$
$$+2\left[\left(\int_0^t g(s)ds+\varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}}\exp\left(\int_0^t (h(r)+1)dr\right)\right]$$
$$+2\int_0^t \Big(h(s)+1\Big)\exp\left(\int_s^t (h(r)+1)dr\right)\left(\int_0^s g(r)dr+\varepsilon\right)^{\frac{1}{2}} ds,$$

where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in I$.

We recall Schauder's fixed point theorem [25].

Theorem 2.2. *Let C be a non-empty closed bounded convex subset of a Banach space E. Let $f : C \to C$ be a continuous map. If $f(C)$ is relatively compact, then f has a fixed point.*

Let us recall an existence and uniqueness result from [39].

Theorem 2.3. *Let $\psi : I \times H \to [0,+\infty]$ be a map such that*

(H_ψ^1) *for each $t \in I$, the function $u \mapsto \psi(t,u)$ is proper, lower semi-continuous, and convex;*

(H_ψ^2) *there exist a ρ-Lipschitz function $k : H \longrightarrow \mathbb{R}_+$ and an absolutely continuous function $a : I \to \mathbb{R}$, with a derivative $\dot{a} \in L^2_{\mathbb{R}_+}(I)$, such that*

$$\psi^*(t,u) \leq \psi^*(s,u) + k(u)|a(t)-a(s)| \text{ for every } (t,s,u) \in I \times I \times H.$$

Let $u_0 \in \operatorname{dom}\psi(0,\cdot)$ be fixed. Then, the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\psi(t,u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0 \in \operatorname{dom}\psi(0,\cdot), \end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I such that $u(t) \in \operatorname{dom}\psi(t,\cdot)$ for all $t \in I$.

Now, denote by $A(t) := \partial\psi(t,\cdot)$ the maximal monotone operator in H associated with $\partial\psi(t,\cdot)$, $t \in I$ (ψ satisfies conditions (H_ψ^1)-(H_ψ^2)). Let us consider the operator $\mathcal{A} : L^2_H(I) \rightrightarrows L^2_H(I)$ defined by

$$\mathcal{A}u = \{v \in L^2_H(I) : v(t) \in A(t)u(t) \text{ a.e.}\}.$$

Then, $\mathcal{A}$ is well defined since by Theorem 2.3, the differential inclusion

$$-\dot{u}(t) \in A(t)u(t) = \partial\psi(t,u(t)) \text{ a.e. } t \in I,\ u(0) = u_0 \in \operatorname{dom}\psi(0,\cdot),$$

admits a unique absolutely continuous solution.

The operator $\mathcal{A}$ enjoys the following property, see [39] (see also [21] or [41]).

Proposition 2.4. *Assume that for any $t \in I$, $A(t) = \partial\psi(t,\cdot)$ where ψ satisfies conditions (H^1_ψ)-(H^2_ψ). Then, one has*
$(\mathcal{J})$ The operator $\mathcal{A}$ is maximal monotone.
$(\mathcal{J}\mathcal{J})$ If $(u_n)_n$ and $(v_n)_n$ are two sequences in $L^2_H(I)$ satisfying
(i) for a.e. *$t \in I$: $v_n(t) \in A(t)u_n(t)$, $\forall n \in \mathbb{N}$;*
(ii) the sequence $(u_n)_n$ strongly converges to u in $L^2_H(I)$;
(iii) the sequence $(v_n)_n$ weakly converges to v in $L^2_H(I)$.
Then, one has $v(t) \in A(t)u(t)$ a.e. *$t \in I$.*

We will need the following useful application of Theorem 2.3, regarding an evolution problem with single-valued perturbation depending only on time, see [41], [43].

Proposition 2.5. *Under the assumptions of Theorem* 2.3*, if $h \in L^2_H([T_0,T])$ and $u_0 \in \operatorname{dom}\psi(T_0,\cdot)$, then, the evolution problem*

$$\begin{cases} -\dot{u}(t) \in \partial\psi(t,u(t)) + h(t) \quad \text{a.e. } t \in [T_0,T], \\ u(T_0) = u_0 \in \operatorname{dom}\psi(T_0,\cdot), \end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ satisfying

$$\|\dot{u}\|^2_{L^2_H([T_0,T])} \leq \sigma\|h\|^2_{L^2_H([T_0,T])} + \gamma_*, \tag{3}$$

where γ_* and σ are the non-negative real constants defined by

$$\gamma_* = (k^2(0) + 3(\rho+1)^2)\int_0^T \dot{a}^2(t)dt + 2[T - T_0 + \psi(T_0,u_0) - \psi(T,u(T))], \tag{4}$$

$$\sigma = k^2(0) + 3(\rho+1)^2 + 4. \tag{5}$$

3. Well-posedness result to the integro-differential inclusion (2)

Our main result will be established under the following assumptions:
Let $g_1 : I \times H \to H$ be a map such that

$(H^1_{g_1})$ $g_1(\cdot,u)$ is measurable on I, for any $u \in H$;

$(H^2_{g_1})$ there exists a non-negative real constant M_{g_1} such that

$$\|g_1(t,u)\| \leq M_{g_1}(1+\|u\|) \text{ for all } (t,u) \in I \times H;$$

$(H^3_{g_1})$ there exists a non-negative real constant K_{g_1} such that

$$||g_1(t,u) - g_1(t,v)|| \leq K_{g_1}||u-v|| \text{ for all } (t,u,v) \in I \times H \times H.$$

Let $g_2 : I \times I \times H \to H$ be a map such that

$(H^1_{g_2})$ $g_2(\cdot,\cdot,u)$ is measurable on $I \times I$, for any $u \in H$;

$(H^2_{g_2})$ there exists a non-negative real constant M_{g_2} satisfying $M^2_{g_1} + 4T^2M^2_{g_2} < \frac{1}{8\sigma T^2}$ such that

$$\|g_2(t,s,u)\| \leq M_{g_2}(1+\|u\|) \text{ for all } (t,s,u) \in I \times I \times H;$$

$(H^3_{g_2})$ there exists a non-negative real constant K_{g_2} such that

$$||g_2(t,s,u) - g_2(t,s,v)|| \leq K_{g_2}||u-v|| \text{ for all } (t,s) \in I \times I \text{ and } u,v \in H.$$

Now, we are able to establish our new theorem regarding the integro-differential inclusion (2).

Theorem 3.1. *Let $\psi : I \times H \to [0,+\infty]$ be a map satisfying (H^1_ψ)-(H^2_ψ) of Theorem* 2.3. *Let $g_1 : I \times H \to H$ be a map satisfying $(H^1_{g_1})$-$(H^2_{g_1})$-$(H^3_{g_1})$. Let $g_2 : I \times I \times H \to H$ be a map satisfying $(H^1_{g_2})$-$(H^2_{g_2})$-$(H^3_{g_2})$. Then, for any $u_0 \in$* dom $\psi(0,\cdot)$*, there is a unique absolutely continuous solution $u(\cdot)$ to* (2)*. Moreover, the following inequalities*

$$\|g_1(t,u(t))\| \leq M_{g_1}(1+L), \quad \|g_2(t,s,u(t))\| \leq M_{g_2}(1+L) \quad \text{for all } (t,s) \in I \times I,$$

and

$$\int_0^T ||\dot{u}(t)||^2 dt \leq \gamma + \sigma \int_0^T ||g_1(t,u(t)) + \int_0^t g_2(t,s,u(s))ds||^2 dt, \tag{6}$$

hold true, with the same non-negative real constants σ, γ, L defined by (5)*,* (14)*,* (17)*, respectively.*

Proof. **Part 1: Existence.** We proceed by a discretization method.
(A) Construction of the sequence $(u_n(\cdot))$.
For any $n \in \mathbb{N}^*$, consider a partition of I such that

$$t_0^n = 0 < t_1^n < \cdots < t_i^n = i\frac{T}{n} < \cdots < t_n^n = T.$$

Put $u_0^n = u_0$. Then, consider the following problem on $I_0^n = [t_0^n, t_1^n]$

$$\begin{cases} -\dot{u}(t) \in \partial\psi(t,u(t)) + g_1(t,u_0^n) + \int_0^t g_2(t,s,u_0^n)ds \quad \text{a.e. } t \in I_0^n, \\ u(t_0^n) = u_0^n \in \operatorname{dom}\psi(t_0^n,\cdot). \end{cases}$$

Define the map $h^{n,0}$ for any $t \in I_0^n$ by

$$h^{n,0}(t) = g_1(t,u_0^n) + \int_0^t g_2(t,s,u_0^n)ds.$$

Thanks to Proposition 2.5, there exists a unique absolutely continuous solution $u^{n,0}(\cdot) : I_0^n \to H$ to the latter problem since $h^{n,0}(\cdot) \in L_H^2(I_0^n)$. In fact, noting that $(a+b)^2 \le 2a^2 + 2b^2$ for $a,b \in \mathbb{R}$, by $(H_{g_1}^2)$ and $(H_{g_2}^2)$, one has for any $t \in I_0^n$

$$\begin{aligned}
\int_0^{t_1^n} \|h^{n,0}(t)\|^2 dt &= \int_0^{t_1^n} \|g_1(t,u_0^n) + \int_0^t g_2(t,s,u_0^n)ds\|^2 dt \\
&\le \int_0^{t_1^n} \left(\|g_1(t,u_0^n)\| + \int_0^t \|g_2(t,s,u_0^n)\|ds \right)^2 dt \\
&\le 2\int_0^{t_1^n} \left(\|g_1(t,u_0^n)\|^2 + \Big(\int_0^t \|g_2(t,s,u_0^n)\|ds\Big)^2 \right) dt \\
&\le 2\int_0^{t_1^n} \left(\|g_1(t,u_0^n)\|^2 + \Big(\int_0^t M_{g_2}(1+\|u_0^n\|)ds\Big)^2 \right) dt \\
&\le 2\int_0^{t_1^n} \left(M_{g_1}^2(1+\|u_0^n\|)^2 + M_{g_2}^2(1+\|u_0^n\|)^2 t^2 \right) dt \\
&\le 2\int_0^{t_1^n} \left(M_{g_1}^2(1+\|u_0^n\|)^2 + M_{g_2}^2(1+\|u_0^n\|)^2 T^2 \right) dt \\
&= 2\int_0^{t_1^n} (M_{g_1}^2 + T^2 M_{g_2}^2)(1+\|u_0^n\|)^2 dt \\
&\le 2T(M_{g_1}^2 + T^2 M_{g_2}^2)(1+\|u_0^n\|)^2 < +\infty.
\end{aligned}$$

Furthermore, $u^{n,0}(t) \in \operatorname{dom}\psi(t,\cdot)$ for any $t \in I_0^n$, and by (3)-(4), setting $u_1^n = u^{n,0}(t_1^n)$, this solution satisfies

$$\int_{t_0^n}^{t_1^n} \|\dot{u}^{n,0}(t)\|^2 dt \le \sigma \int_{t_0^n}^{t_1^n} \|h^{n,0}(t)\|^2 dt + \gamma_0^n,$$

where

$$\gamma_0^n = [k^2(0)+3(\rho+1)^2]\int_{t_0^n}^{t_1^n}\dot{a}^2(t)dt+2[(t_1^n-t_0^n)+\psi(t_0^n,u_0^n)-\psi(t_1^n,u_1^n)].$$

Then, consider the integro-differential inclusion on $I_1^n=[t_1^n,t_2^n]$

$$\begin{cases} -\dot{u}(t)\in\partial\psi(t,u(t))+g_1(t,u_1^n)+\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds \\ \quad \text{a.e. } t\in I_1^n, \\ u(t_1^n)=u_1^n\in \operatorname{dom}\psi(t_1^n,\cdot). \end{cases}$$

Define the map $h^{n,1}$ for any $t\in I_1^n$ by

$$h^{n,1}(t)=g_1(t,u_1^n)+\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds.$$

Thanks to Proposition 2.5, there exists a unique absolutely continuous solution $u^{n,1}(\cdot):I_1^n\to H$ to the latter problem with $u^{n,1}(t_1^n)=u^{n,0}(t_1^n)$, since $h^{n,1}(\cdot)\in L_H^2(I_1^n)$. Indeed, noting that $(a+b)^2\le 2a^2+2b^2$ for $a,b\in\mathbb{R}$, by $(H_{g_1}^2)$ and $(H_{g_2}^2)$, one has for any $t\in I_1^n$

$$\begin{aligned}
&\int_{t_1^n}^{t_2^n}\|h^{n,1}(t)\|^2dt=\int_{t_1^n}^{t_2^n}\|g_1(t,u_1^n)+\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds\|^2dt\\
&\le\int_{t_1^n}^{t_2^n}\left(\|g_1(t,u_1^n)\|+\|\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds\|\right)^2dt\\
&\le\int_{t_1^n}^{t_2^n}\left(2\|g_1(t,u_1^n)\|^2+2\|\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds\|^2\right)dt\\
&=2\int_{t_1^n}^{t_2^n}\|g_1(t,u_1^n)\|^2dt+2\int_{t_1^n}^{t_2^n}\|\int_0^{t_1^n}g_2(t,s,u_0^n)ds+\int_{t_1^n}^t g_2(t,s,u_1^n)ds\|^2dt\\
&\le 2\int_{t_1^n}^{t_2^n}\|g_1(t,u_1^n)\|^2dt+4\int_{t_1^n}^{t_2^n}\|\int_0^{t_1^n}g_2(t,s,u_0^n)ds\|^2dt\\
&+4\int_{t_1^n}^{t_2^n}\|\int_{t_1^n}^t g_2(t,s,u_1^n)ds\|^2dt\\
&\le 2\int_{t_1^n}^{t_2^n}\|g_1(t,u_1^n)\|^2dt+4\int_{t_1^n}^{t_2^n}\left(\int_0^{t_1^n}\|g_2(t,s,u_0^n)\|ds\right)^2dt\\
&+4\int_{t_1^n}^{t_2^n}\left(\int_{t_1^n}^t\|g_2(t,s,u_1^n)\|ds\right)^2dt
\end{aligned}$$

$$
\begin{aligned}
&\le 2M_{g_1}^2(1+\|u_1^n\|)^2(t_2^n-t_1^n)+4\int_{t_1^n}^{t_2^n} M_{g_2}^2(1+\|u_0^n\|)^2(t_1^n)^2dt\\
&+4\int_{t_1^n}^{t_2^n} M_{g_2}^2(1+\|u_1^n\|)^2(t-t_1^n)^2dt\\
&\le 2TM_{g_1}^2(1+\|u_1^n\|)^2+4T^2M_{g_2}^2(1+\|u_0^n\|)^2(t_2^n-t_1^n)+4T^2M_{g_2}^2(1+\|u_1^n\|)^2(t_2^n-t_1^n)\\
&\le 2TM_{g_1}^2(1+\|u_1^n\|)^2+4T^3M_{g_2}^2(1+\|u_0^n\|)^2+4T^3M_{g_2}^2(1+\|u_1^n\|)^2\\
&\le 2T(M_{g_1}^2+4T^2M_{g_2}^2)(1+\max(\|u_0^n\|,\|u_1^n\|))^2<+\infty.
\end{aligned}
$$

Furthermore, $u^{n,1}(t)\in \operatorname{dom}\psi(t,\cdot)$ for any $t\in I_1^n$, and by (3)-(4), setting $u_2^n = u^{n,1}(t_2^n)$, one gets

$$\int_{t_1^n}^{t_2^n}\|\dot u^{n,1}(t)\|^2dt\le \sigma\int_{t_1^n}^{t_2^n}\|h^{n,1}(t)\|^2dt+\gamma_1^n,$$

where

$$\gamma_1^n=[k^2(0)+3(\rho+1)^2]\int_{t_1^n}^{t_2^n}\dot a^2(t)dt+2[(t_2^n-t_1^n)+\psi(t_1^n,u_1^n)-\psi(t_2^n,u_2^n)].$$

In a similar way, for each $i\in\{2,\cdots,n-1\}$, putting $u_i^n=u^{n,i-1}(t_i^n)$, consider the integro-differential inclusion on $I_i^n=[t_i^n,t_{i+1}^n]$

$$
\begin{cases}
-\dot u(t)\in\partial\psi(t,u(t))+g_1(t,u_i^n)+\displaystyle\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds+\int_{t_i^n}^{t} g_2(t,s,u_i^n)ds\\
\quad \text{a.e. } t\in I_i^n,\\
u(t_i^n)=u_i^n\in\operatorname{dom}\psi(t_i^n,\cdot).
\end{cases}
$$

Define the map $h^{n,i}$ for any $t\in I_i^n$ by

$$h^{n,i}(t)=g_1(t,u_i^n)+\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds+\int_{t_i^n}^{t} g_2(t,s,u_i^n)ds.$$

Thanks to Proposition 2.5, there exists a unique absolutely continuous solution $u^{n,i}(\cdot):I_i^n\to H$ to the latter problem with $u^{n,i}(t_i^n)=u^{n,i-1}(t_i^n)$, since $h^{n,i}(\cdot)\in L_H^2(I_i^n)$. Indeed, noting that $(a+b)^2\le 2a^2+2b^2$ for $a,b\in\mathbb{R}$, by $(H_{g_1}^2)$ and

$(H^2_{g_2})$, one has for any $t \in I^n_i$

$$\int_{t_i^n}^{t_{i+1}^n} \|h^{n,i}(t)\|^2 dt = \int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n) + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds + \int_{t_i^n}^{t} g_2(t,s,u_i^n)ds\|^2 dt$$

$$\leq \int_{t_i^n}^{t_{i+1}^n} \left(\|g_1(t,u_i^n)\| + \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds + \int_{t_i^n}^{t} g_2(t,s,u_i^n)ds \right\| \right)^2 dt$$

$$\leq 2\int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n)\|^2 dt + 2\int_{t_i^n}^{t_{i+1}^n} \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds + \int_{t_i^n}^{t} g_2(t,s,u_i^n)ds \right\|^2 dt$$

$$\leq 2\int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n)\|^2 dt + 4\int_{t_i^n}^{t_{i+1}^n} \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds \right\|^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} \| \int_{t_i^n}^{t} g_2(t,s,u_i^n)ds\|^2 dt$$

$$\leq 2\int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n)\|^2 dt + 4\int_{t_i^n}^{t_{i+1}^n} \left(\sum_{j=0}^{i-1} \| \int_{t_j^n}^{t_{j+1}^n} g_2(t,s,u_j^n)ds\| \right)^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} \| \int_{t_i^n}^{t} g_2(t,s,u_i^n)ds\|^2 dt$$

$$\leq 2\int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n)\|^2 dt + 4\int_{t_i^n}^{t_{i+1}^n} \left(\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}} \|g_2(t,s,u_j^n)\|ds \right)^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} \left(\int_{t_i^n}^{t} \|g_2(t,s,u_i^n)\|ds \right)^2 dt$$

$$\leq 2\int_{t_i^n}^{t_{i+1}^n} \|g_1(t,u_i^n)\|^2 dt + 4\int_{t_i^n}^{t_{i+1}^n} \left(M_{g_2} \sum_{j=0}^{i-1} (1+\|u_j^n\|)(t_{j+1}^n - t_j^n) \right)^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} \left(M_{g_2}(1+\|u_i^n\|)(t-t_i^n) \right)^2 dt$$

$$\leq 2M_{g_1}^2(1+\|u_i^n\|)^2(t_{i+1}^n - t_i^n) + 4\int_{t_i^n}^{t_{i+1}^n} M_{g_2}^2 (1+\max_{0\leq j\leq i-1} \|u_j^n\|)^2 (t_i^n)^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} M_{g_2}^2 (1+\max_{0\leq j\leq i} \|u_j^n\|)^2 T^2 dt$$

$$\leq 2M_{g_1}^2(1+\|u_i^n\|)^2(t_{i+1}^n - t_i^n) + 4\int_{t_i^n}^{t_{i+1}^n} M_{g_2}^2 (1+\max_{0\leq j\leq i-1} \|u_j^n\|)^2 T^2 dt$$

$$+4\int_{t_i^n}^{t_{i+1}^n} M_{g_2}^2 (1+\max_{0\leq j\leq i} \|u_j^n\|)^2 T^2 dt$$

$$\leq 2(M_{g_1}^2 + 4T^2 M_{g_2}^2)(1+\max_{0\leq j\leq i} \|u_j^n\|)^2 (t_{i+1}^n - t_i^n) \tag{7}$$

$$\leq 2T(M_{g_1}^2 + 4T^2 M_{g_2}^2)(1+\max_{0\leq j\leq i} \|u_j^n\|)^2 < +\infty.$$

Furthermore, $u^{n,i}(t) \in \operatorname{dom} \psi(t,\cdot)$ for any $t \in I_i^n$, and by (3)-(4), setting $u_{i+1}^n = u^{n,i}(t_{i+1}^n)$, one gets

$$\int_{t_i^n}^{t_{i+1}^n} \|\dot{u}^{n,i}(t)\|^2 dt \leq \sigma \int_{t_i^n}^{t_{i+1}^n} \|h^{n,i}(t)\|^2 dt + \gamma_i^n, \tag{8}$$

where

$$\gamma_i^n = [k^2(0) + 3(\rho+1)^2] \int_{t_i^n}^{t_{i+1}^n} \dot{a}^2(t) dt + 2[(t_{i+1}^n - t_i^n) + \psi(t_i^n, u_i^n) - \psi(t_{i+1}^n, u_{i+1}^n)].$$

Now, define the maps $u_n, h_n : I \to H$, for each n by

$$h_n(t) = h^{n,i}(t),\ u_n(t) = u^{n,i}(t)\ \ \forall t \in I_i^n\ ,\ i \in \{0,\cdots,n-1\}.$$

It is clear that $u_n(\cdot)$ is absolutely continuous on I, and putting

$$\begin{cases} \theta_n(0) = 0 \\ \theta_n(t) = t_i^n \quad \text{if } t \in]t_i^n, t_{i+1}^n],\ i \in \{0,\cdots,n-1\}, \end{cases}$$

one has

$$\begin{cases} -\dot{u}_n(t) \in \partial\psi(t, u_n(t)) + g_1(t, u_n(\theta_n(t))) + \displaystyle\int_0^t g_2(t,s,u_n(\theta_n(s)))ds \\ \text{a.e. } t \in I, \\ u_n(0) = u_0. \end{cases} \tag{9}$$

From (8), one gets

$$\int_0^T \|\dot{u}_n(t)\|^2 dt \leq \sigma \int_0^T \|h_n(t)\|^2 dt + \gamma_n^n, \tag{10}$$

where

$$h_n(t) = g_1(t, u_n(\theta_n(t))) + \int_0^t g_2(t,s,u_n(\theta_n(s)))ds, \forall t \in I, \tag{11}$$

and

$$\gamma_n^n = [k^2(0) + 3(\rho+1)^2] \int_0^T \dot{a}^2(t) dt + 2[T + \psi(0,u_0) - \psi(T, u_n(T))],$$

noting that $\displaystyle\int_0^T \|h_n(t)\|^2 dt = \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|h^{n,i}(t)\|^2 dt$.
From (7), one writes

$$\begin{aligned} \int_0^T \|h_n(t)\|^2 dt &\leq \sum_{i=0}^{n-1} 2(M_{g_1}^2 + 4T^2 M_{g_2}^2)(1 + \max_{0 \leq j \leq i} \|u_j^n\|)^2 (t_{i+1}^n - t_i^n) \\ &\leq 2T(M_{g_1}^2 + 4T^2 M_{g_2}^2)(1 + \max_{0 \leq i \leq n} \|u_i^n\|)^2. \end{aligned} \tag{12}$$

As the function ψ takes non-negative real values, one has $-\psi(T,u_n(T)) \leq 0$, then, coming back to (10), it follows

$$\int_0^T \|\dot{u}_n(t)\|^2 dt \leq \sigma \int_0^T \|h_n(t)\|^2 dt + \gamma, \tag{13}$$

where

$$\gamma = [k^2(0) + 3(\rho+1)^2] \int_0^T \dot{a}^2(t)dt + 2[T + \psi(0,u_0)]. \tag{14}$$

Combining (12) and (13), it results

$$\int_0^T \|\dot{u}_n(t)\|^2 dt \leq 4T\sigma(M_{g_1}^2 + 4T^2 M_{g_2}^2)(1 + \|u_n(\cdot)\|_\infty^2) + \gamma.$$

Simplifying yields

$$\int_0^T \|\dot{u}_n(t)\|^2 dt \quad \leq \quad \kappa + \eta \|u_n(\cdot)\|_\infty^2, \tag{15}$$

where the real non-negative constants η and κ are given by

$$\eta = 4T\sigma(M_{g_1}^2 + 4T^2 M_{g_2}^2) \quad \text{and} \quad \kappa = \eta + \gamma.$$

Since $(u_n(\cdot))$ is absolutely continuous, then, using the Cauchy-Schwarz inequality and (15), one has for all $\tau \in I$

$$\begin{aligned} \|u_n(\tau) - u_0\|^2 &\leq \tau \left(\int_0^\tau \|\dot{u}_n(t)\|^2 dt \right) \\ &\leq T(\kappa + \eta \|u_n(\cdot)\|_\infty^2), \end{aligned}$$

which gives

$$\begin{aligned} \|u_n(\tau)\|^2 &\leq 2\|u_0\|^2 + 2\|u_n(\tau) - u_0\|^2 \\ &\leq 2\|u_0\|^2 + 2T(\kappa + \eta \|u_n(\cdot)\|_\infty^2). \end{aligned}$$

As a result for each n, one gets

$$(1 - 2T\eta)\|u_n(\cdot)\|_\infty^2 \leq 2\|u_0\|^2 + 2T\kappa.$$

By the choice of the constants M_{g_1} and M_{g_2} in $(H_{g_2}^2)$, one has $1 - 2T\eta > 0$, then for any n

$$\|u_n(\cdot)\|_\infty \leq L, \tag{16}$$

where

$$L := \left(\frac{2(\|u_0\|^2 + T\kappa)}{1 - 2T\eta} \right)^{\frac{1}{2}}. \tag{17}$$

This along with (11), (16) and $(H^2_{g_1})$-$(H^2_{g_2})$, one obtains

$$\begin{aligned}\|h_n(t)\| &= \|g_1(t,u_n(\theta_n(t))) + \int_0^t g_2(t,s,u_n(\theta_n(s)))ds\| \\ &\leq \|g_1(t,u_n(\theta_n(t)))\| + \int_0^t \|g_2(t,s,u_n(\theta_n(s)))\|ds \\ &\leq (M_{g_1} + TM_{g_2})(1+L). \end{aligned} \tag{18}$$

Combining (15) and (16), it results

$$\sup_{n\in\mathbb{N}} \int_0^T \|\dot{u}_n(t)\|^2 dt \leq \kappa + \eta L^2 := L_1. \tag{19}$$

(B) Convergence of the sequence $(u_n(\cdot))$.
Let us prove that $(u_n(\cdot))_n$ is a Cauchy sequence on I.
From (9), one writes for p,q arbitrary integers and a.e. $t \in I$

$$\begin{aligned} &-\dot{u}_p(t) - g_1(t,u_p(\theta_p(t))) - \int_0^t g_2(t,s,u_p(\theta_p(s)))ds \in \partial\psi(t,u_p(t)), \\ &-\dot{u}_q(t) - g_1(t,u_q(\theta_q(t))) - \int_0^t g_2(t,s,u_q(\theta_q(s)))ds \in \partial\psi(t,u_q(t)). \end{aligned}$$

Since $\partial\psi(t,\cdot)$ is monotone, then, one obtains

$$\begin{aligned} &\langle \dot{u}_p(t) - \dot{u}_q(t), u_p(t) - u_q(t)\rangle \\ &\leq \langle g_1(t,u_q(\theta_q(t))) - g_1(t,u_p(\theta_p(t))), u_p(t) - u_q(t)\rangle \\ &+ \left\langle \int_0^t g_2(t,s,u_q(\theta_q(s)))ds - \int_0^t g_2(t,s,u_p(\theta_p(s)))ds, u_p(t) - u_q(t)\right\rangle. \end{aligned} \tag{20}$$

According to $(H^3_{g_1})$ and $(H^3_{g_2})$, one has for a.e. $t \in I$,

$$\begin{aligned} &\langle g_1(t,u_q(\theta_q(t))) - g_1(t,u_p(\theta_p(t))), u_p(t) - u_q(t)\rangle \\ &+ \left\langle \int_0^t g_2(t,s,u_q(\theta_q(s)))ds - \int_0^t g_2(t,s,u_p(\theta_p(s)))ds, u_p(t) - u_q(t)\right\rangle \\ &\leq \|u_p(t) - u_q(t)\| \times \|g_1(t,u_q(\theta_q(t))) - g_1(t,u_p(\theta_p(t)))\| \\ &+ \|u_p(t) - u_q(t)\| \times \int_0^t \|g_2(t,s,u_q(\theta_q(s))) - g_2(t,s,u_p(\theta_p(s)))\|ds \\ &\leq \|u_p(t) - u_q(t)\| \times \\ &\left(K_{g_1}\|u_q(\theta_q(t)) - u_p(\theta_p(t))\| + K_{g_2}\int_0^t \|u_q(\theta_q(s)) - u_p(\theta_p(s))\|ds\right) \end{aligned}$$

$$\begin{aligned}
&\le K_{g_1}\|u_p(t)-u_q(t)\|\times\\
&\big(\|u_q(\theta_q(t))-u_q(t)\|+\|u_q(t)-u_p(t)\|+\|u_p(t)-u_p(\theta_p(t))\|\big)\\
&+K_{g_2}\|u_p(t)-u_q(t)\|\times\\
&\int_0^t\Big(\|u_q(\theta_q(s))-u_q(s)\|+\|u_q(s)-u_p(s)\|+\|u_p(s)-u_p(\theta_p(s))\|\Big)ds. \quad (21)
\end{aligned}$$

By the absolute continuity of u_p for each p, one gets any $t\in I$

$$\|u_p(t)-u_p(\theta_p(t))\|=\|\int_{\theta_p(t)}^t \dot{u}_p(s)ds\|\le\int_{\theta_p(t)}^t\|\dot{u}_p(s)\|ds.$$

By construction, for any $t\in I$ and any $p\in\mathbb{N}$, one has $0\le t-\theta_p(t)\le T/p$. By Cauchy-Schwarz inequality, then using (19), one gets for any $0\le t\le T$

$$\|u_p(t)-u_p(\theta_p(t))\|\le(t-\theta_p(t))^{\frac{1}{2}}\left(\int_0^T\|\dot{u}_p(s)\|^2ds\right)^{\frac{1}{2}}\le\left(\frac{T}{p}\right)^{\frac{1}{2}}L_1^{\frac{1}{2}}. \quad (22)$$

We return to (21) with the help of (16) and (22), it results

$$\begin{aligned}
&\langle g_1(t,u_q(\theta_q(t)))-g_1(t,u_p(\theta_p(t))),u_p(t)-u_q(t)\rangle\\
&+\left\langle\int_0^t g_2(t,s,u_q(\theta_q(s)))ds-\int_0^t g_2(t,s,u_p(\theta_p(s)))ds,u_p(t)-u_q(t)\right\rangle\\
&\le K_{g_1}\|u_p(t)-u_q(t)\|^2+K_{g_2}\|u_p(t)-u_q(t)\|\int_0^t\|u_p(s)-u_q(s)\|ds+G_{p,q}(t),
\end{aligned}$$

where $G_{p,q}$ is defined on I by

$$G_{p,q}(t)=2LL_1^{\frac{1}{2}}(K_{g_1}+TK_{g_2})\left[\left(\frac{T}{p}\right)^{\frac{1}{2}}+\left(\frac{T}{q}\right)^{\frac{1}{2}}\right],\ t\in I.$$

This along with (20), it follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_p(t)-u_q(t)\|^2=\langle\dot{u}_p(t)-\dot{u}_q(t),u_p(t)-u_q(t)\rangle\\
&\le K_{g_1}\|u_p(t)-u_q(t)\|^2+K_{g_2}\|u_p(t)-u_q(t)\|\int_0^t\|u_p(s)-u_q(s)\|ds+G_{p,q}(t).
\end{aligned} \quad (23)$$

Remark that

$$\lim_{p,q\to\infty}G_{p,q}(t)=0 \text{ a.e. } t\in I.$$

Moreover, since $|G_{p,q}(t)|\le 4LL_1^{\frac{1}{2}}T^{\frac{1}{2}}(K_{g_1}+TK_{g_2})$ for all $t\in I$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{p,q\to\infty}\int_0^T G_{p,q}(t)dt=0. \quad (24)$$

By (23), recall that

$$\frac{d}{dt}\|u_p(t)-u_q(t)\|^2 \leq 2K_{g_1}\|u_p(t)-u_q(t)\|^2+2G_{p,q}(t)$$
$$+2K_{g_2}\|u_p(t)-u_q(t)\|\int_0^t \|u_p(s)-u_q(s)\|ds.$$

Let us apply Lemma 2.1. Take $\varepsilon > 0$ and set

$$y(t)=\|u_p(t)-u_q(t)\|^2,\ h_1(t)=2K_{g_1},\ h_2(t)=2K_{g_2},$$
$$g(t)=2G_{p,q}(t),\ h(t)=\max\left\{\frac{h_1(t)}{2},\frac{h_2(t)}{2}\right\} \text{ a.e. } t\in I,$$

then, one gets for all $t \in I$

$$\|u_p(t)-u_q(t)\| \leq (\|u_p(0)-u_q(0)\|^2+\varepsilon)^{\frac{1}{2}}\exp\left(\int_0^t (h(s)+1)ds\right)$$
$$+\frac{\varepsilon^{\frac{1}{2}}}{2}\int_0^t \exp\left(\int_s^t (h(r)+1)dr\right)ds$$
$$+2\left[\left(2\int_0^t G_{p,q}(s)ds+\varepsilon\right)^{\frac{1}{2}}-\varepsilon^{\frac{1}{2}}\exp\left(\int_0^t (h(r)+1)dr\right)\right]$$
$$+2\int_0^t \Big(h(s)+1\Big)\exp\left(\int_s^t (h(r)+1)dr\right)\left(2\int_0^s G_{p,q}(r)dr+\varepsilon\right)^{\frac{1}{2}}ds.$$

This along with (24) and the fact that $\|u_p(0)-u_q(0)\|=0$, letting $p,q\to\infty$ and $\varepsilon\to 0$, it results that $\|u_p(t)-u_q(t)\|\to 0$ for all $t\in I$. Hence

$$\lim_{p,q\to\infty}\|u_p(\cdot)-u_q(\cdot)\|_\infty=0.$$

The uniform Cauchy's criterion therefore ensures the existence of some map $u(\cdot)\in\mathcal{C}_H(I)$ such that

$$(u_n(\cdot)) \text{ uniformly converges to } u(\cdot). \tag{25}$$

Observe that

$$\|u_n(\theta_n(t))-u(t)\| \leq \|u_n(\theta_n(t))-u_n(t)\|+\|u_n(t)-u(t)\|.$$

Then, from (22) and (25), we infer that for any $t\in I$

$$\|u_n(\theta_n(t))-u(t)\|\to 0 \text{ as } n\to\infty. \tag{26}$$

From $(H^2_{g_2})$ and (16), one gets for any $(t,s)\in I\times I$

$$\|g_2(t,s,u_n(\theta_n(s)))\| \leq M_{g_2}(1+L).$$

Moreover, by $(H^3_{g_2})$ one has for any $(t,s) \in I \times I$

$$\begin{aligned}
&\left\| \int_0^t g_2(t,s,u_n(\theta_n(s)))ds - \int_0^t g_2(t,s,u(s))ds \right\| \\
&= \left\| \int_0^t \Big(g_2(t,s,u_n(\theta_n(s))) - g_2(t,s,u(s)) \Big) ds \right\| \\
&\leq \int_0^t \|g_2(t,s,u_n(\theta_n(s))) - g_2(t,s,u(s))\| ds \\
&\leq K_{g_2} \int_0^t \|u_n(\theta_n(s)) - u(s)\| ds,
\end{aligned}$$

along with (16) and (26), it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \left\| \int_0^t g_2(t,s,u_n(\theta_n(s)))ds - \int_0^t g_2(t,s,u(s))ds \right\| = 0.$$

Combining this with (18), then, again by the Lebesgue dominated convergence theorem, it results

$$\lim_{n\to\infty} \int_0^T \left\| \int_0^t g_2(t,s,u_n(\theta_n(s)))ds - \int_0^t g_2(t,s,u(s))ds \right\|^2 dt = 0. \tag{27}$$

Furthermore, note by $(H^2_{g_1})$ and (16) that for any $t \in I$

$$\|g_1(t,u_n(\theta_n(t)))\| \leq M_{g_1}(1+L),$$

along with $(H^3_{g_1})$ and (26), one deduces

$$\|g_1(t,u_n(\theta_n(t))) - g_1(t,u(t))\| \leq K_{g_1} \|u_n(\theta_n(t)) - u(t)\| \to 0 \text{ as } n \to \infty.$$

An application of the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \int_0^T \|g_1(t,u_n(\theta_n(t))) - g_1(t,u(t))\|^2 dt = 0. \tag{28}$$

Hence, from the preceding modes of convergence in $L^2_H(I)$ (see (27)-(28)), one deduces

$$h_n(\cdot) \longrightarrow h(\cdot) \text{ in } L^2_H(I), \tag{29}$$

where $h : I \to H$ is defined by $h(t) = g_1(t,u(t)) + \displaystyle\int_0^t g_2(t,s,u(s))ds$, $t \in I$.

Moreover, in view of (19), the sequence $(\dot{u}_n)$ is bounded in $L^2_H(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{u}_n(\cdot))_n$ weakly converges in $L^2_H(I)$ to some $w(\cdot) \in L^2_H(I)$.

For any integer n and any $v \in H$ and for $0 \le \tau \le t \le T$, from the absolute continuity of $(u_n(\cdot))_n$, one writes

$$\int_0^T \langle v\mathbf{1}_{[\tau,t]}(s), \dot{u}_n(s)\rangle ds = \langle v, u_n(t) - u_n(\tau)\rangle.$$

A passage to the limit in the equality as n tends to $+\infty$ leads to

$$\langle v, \int_\tau^t w(s)ds\rangle = \langle v, u(t) - u(\tau)\rangle.$$

Thus, given any $\tau, t \in I$ with $\tau \le t$, one gets $\int_\tau^t w(s)ds = u(t) - u(\tau)$, and $u(\cdot)$ is absolutely continuous such that $w(\cdot)$ coincides almost everywhere on I with $\dot{u}(\cdot)$. As a result, $\dot{u} \in L^2_H(I)$ and

$$\dot{u}_n(\cdot) \to \dot{u}(\cdot) \quad \text{weakly in } L^2_H(I). \tag{30}$$

(C) Statement of the integro-differential inclusion.
Recall that $\mathcal{A}$ is a maximal monotone operator (see Proposition 2.4), along with (9) and the preceding modes of convergence (25), (29)-(30), then, the integro-differential inclusion

$$-\dot{u}(t) \in \partial\psi(t, u(t)) + g_1(t, u(t)) + \int_0^t g_2(t, s, u(s))ds \quad \text{a.e. } t \in I,$$

holds true.
Let us now verify (6).
Passing to lim inf as n tends to $+\infty$ in (13) (see (11)), using (29)-(30), it results

$$\int_0^T \|\dot{u}(t)\|^2 dt \le \gamma + \sigma \int_0^T \|g_1(t, u(t)) + \int_0^t g_2(t, s, u(s))ds\|^2 dt.$$

Part 2: Uniqueness.
Let $u_1(\cdot)$, $u_2(\cdot)$ be two solutions to (2). Since $\partial\psi(t,\cdot)$ is monotone then, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 &\le \langle g_1(t, u_1(t)) - g_1(t, u_2(t)), u_2(t) - u_1(t)\rangle \\ &+ \left\langle \int_0^t g_2(t, s, u_1(s))ds - \int_0^t g_2(t, s, u_2(s))ds, u_2(t) - u_1(t)\right\rangle.\end{aligned}$$

By $(H^3_{g_1})$ and $(H^3_{g_2})$, it results

$$\begin{aligned}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 \le& 2K_{g_1}\|u_2(t) - u_1(t)\|^2 \\ &+ 2K_{g_2}\|u_2(t) - u_1(t)\| \int_0^t \|u_2(s) - u_1(s)\|ds.\end{aligned}$$

Applying Lemma 2.1 with $\varepsilon > 0$ arbitrary yields $u_1 = u_2$ and ensures the uniqueness of the solution to the considered problem, namely (2).
The proof of the theorem is then finished. □

The following remark ends this section.

Remark 3.2. Theorem 3.1 ensures the existence and the uniqueness of the solution to the corresponding integro-differential sweeping process (see Remark 3.1 [41]). We refer to [11], for some concrete examples in this direction.

4. Existence result to (1)

We are interested, here, in the study of a differential equation involving Caputo fractional derivative coupled with an integro-differential inclusion of subdifferential type, namely (1).

Assume that $\alpha \in]1,2]$.

Definition 4.1. Let $f \in L^1_H(I)$, we define the Caputo fractional derivative of order $\beta > 0$ by

$$^cD^\beta f(t) := \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1-n+\beta}} ds,$$

with $n = [\beta] + 1$.

Definition 4.2. A mapping $u : I \to H$ is w-derivable if there exists a function $w\text{-}\frac{du}{dt} : I \to H$ such that, for each $\phi \in H$, the real valued function $t \mapsto \langle \phi, u(t) \rangle$ is derivable on I and that

$$\frac{d}{dt}\langle \phi, u(t) \rangle = \langle \phi, w\text{-}\frac{du}{dt}(t) \rangle, \qquad \forall t \in I$$

$w\text{-}\frac{du}{dt}$ is the w-derivative of u.

Definition 4.3. Let $u : I \to H$. The w-Caputo fractional derivative of order $\alpha > 0$ of the function u is the mapping $w\text{-}^cD^\alpha u : I \to H$ such that for each $\phi \in H$,

$$\langle \phi, w\text{-}^cD^\alpha u(t) \rangle = {}^cD^\alpha \langle \phi, u \rangle (t), \ \forall t \in I$$

where $^cD^\alpha \langle \phi, u \rangle$ denotes the Caputo fractional derivative of order α of the real-valued function $\langle \phi, u \rangle : t \mapsto \langle \phi, u(t) \rangle$.

Denote by

$$W^{\alpha,\infty}_H(I) = \{u \in \mathcal{C}_H(I),\ \dot{u} \in \mathcal{C}_H(I),\ {}^cD^{\alpha-1}u \in \mathcal{C}_H(I),\ {}^cD^\alpha u \in L^\infty_H(I)\},$$

where $^cD^{\alpha-1}u$, $^cD^\alpha u$ are the Caputo fractional derivatives of order $\alpha - 1$ and α respectively. Denote by

$$w\text{-}W^{\alpha,\infty}_H(I) = \{u \in \mathcal{C}_H(I),\ w\text{-}\dot{u} \in \mathcal{C}_H(I),\ w\text{-}{}^cD^{\alpha-1}u \in \mathcal{C}_H(I),\ w\text{-}{}^cD^\alpha u \in L^\infty_H(I)\},$$

where $w\text{-}^cD^{\alpha-1}u, w\text{-}^cD^{\alpha}u$ are the weak Caputo fractional derivatives of order $\alpha-1$ and α respectively.

We recall useful lemmas from [16].

Lemma 4.4. *Let $G : I \times I \to \mathbb{R}$ be the function defined by*

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1+t}{T+2}\left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right], & \text{if } 0 \le s < t, \\ -\frac{1+t}{T+2}\left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] & \text{if } t \le s < T. \end{cases}$$

Then,
(1) *there exists a non-negative real constant M_G such that*

$$|G(t,s)| \le M_G.$$

(2) *Let $f \in L_H^\infty(I)$ and let $u_f : I \to H$ be the function defined by*

$$u_f(t) = \int_0^T G(t,s)f(s)ds \ \ \forall t \in I.$$

Then, the following hold

$$u_f \in w\text{-}W_H^{\alpha,\infty}(I),$$

$$u_f(0) - w\text{-}\frac{du_f}{dt}(0) = 0,$$

$$u_f(T) + w\text{-}\frac{du_f}{dt}(T) = 0,$$

$$w\text{-}^cD^{\alpha}u_f(t) = f(t), \ \ \forall t \in I.$$

(3) *Let $f \in L_H^\infty(I)$. Then, $u(t) = \int_0^T G(t,s)f(s)ds$ is a $w\text{-}W_H^{\alpha,\infty}(I)$ solution to*

$$\begin{cases} w\text{-}^cD^{\alpha}u(t) = f(t) \quad \text{a.e. } t \in I, \\ u(0) - w\text{-}\frac{du}{dt}(0) = 0, \\ u(T) + w\text{-}\frac{du}{dt}(T) = 0. \end{cases}$$

Lemma 4.5. *Let $X : I \rightrightarrows H$ be a convex weakly compact valued measurable set-valued map such that $X(t) \subset \xi\overline{B}_H$ for all $t \in I$. Then, the $w\text{-}W_H^{\alpha,\infty}(I)$ solutions set of*

$$\begin{cases} w\text{-}^cD^{\alpha}u(t) \in X(t) \quad \text{a.e. } t \in I, \\ u(0) - w\text{-}\frac{du}{dt}(0) = 0, \\ u(T) + w\text{-}\frac{du}{dt}(T) = 0, \end{cases}$$

is bounded convex equicontinuous and weakly compact in $\mathcal{C}_H(I)$.

In this section we adopt the following assumptions:
Let $\psi : I \times H \to [0,+\infty]$ be a map such that
(H^3_ψ) For each $t \in I$, $\operatorname{dom}\psi(t,\cdot) \subset Y(t) \subset \xi\overline{B}_H$, where $Y : I \rightrightarrows H$ is a measurable set-valued map with convex compact values, and ξ is a non-negative real constant.
Let $g_1 : I \times H \times H \to H$ be a map such that
(j) $g_1(\cdot,x,u)$ is measurable on I, for any $(x,u) \in H \times H$;
(jj) there exists a non-negative real constant m_{g_1} such that

$$\|g_1(t,x,u)\| \le m_{g_1}(1+\|x\|+\|u\|) \quad \text{for all } (t,x,u) \in I \times H \times H;$$

(jjj) there exists a non-negative real constant k_{g_1} such that

$$\|g_1(t,u,x) - g_1(t,v,y)\| \le k_{g_1}(\|u-v\|+\|x-y\|) \quad \text{for all } t \in I \text{ and } u,v,x,y \in H.$$

Let $g_2 : I \times I \times H \times H \to H$ be a map such that

(i) $g_2(\cdot,\cdot,u,x)$ is measurable on $I \times I$, for any $(u,x) \in H \times H$;

(ii) there exists a non-negative real constant m_{g_2} satisfying
$m^2_{g_1} + 4T^2 m^2_{g_2} < \frac{1}{8\sigma T^2(1+TM_G\xi)^2}$ such that

$$\|g_2(t,s,u,x)\| \le m_{g_2}(1+\|u\|+\|x\|) \quad \text{for all } (t,s,u,x) \in I \times I \times H \times H;$$

(iii) there exists a non-negative real constant k_{g_2} such that

$$\|g_2(t,s,u,x) - g_2(t,s,v,y)\| \le k_{g_2}(\|u-v\|+\|x-y\|)$$

for all $(t,s) \in I \times I$ and $u,v,x,y \in H$.

Now, we prove the existence result to a new class of systems governed by integro-differential inclusions of time-dependent subdifferential type and Caputo fractional derivative, namely (1).

Theorem 4.6. *Let $\psi : I \times H \to [0,+\infty]$ be a map satisfying (H^1_ψ)-(H^2_ψ)-(H^3_ψ). Let $g_1 : I \times H \times H \to H$ be a map satisfying (j)-(jj)-(jjj). Let $g_2 : I \times I \times H \times H \to H$ be a map satisfying (i)-(ii)-(iii). Then, there is a w-$W^{\alpha,\infty}_H(I)$ mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying* (1).

Proof. Consider the weak-$W^{\alpha,\infty}_H(I)$ solutions set

$$\mathcal{X} := \{u_f : I \to H,\ u_f(t) = \int_0^T G(t,s)f(s)ds,\ f \in S^\infty_{\xi\overline{B}_H}\},$$

where $S^{\infty}_{\xi\overline{B}_H}$ denotes the set of all $L^{\infty}_H(I)$-selections of the convex weakly compact set-valued map $t \mapsto \xi\overline{B}_H$. In view of Lemma 4.5, the set $\mathcal{X}$ is convex, bounded, equicontinuous, and weakly compact in $\mathcal{C}_H(I)$.
For any $h \in \mathcal{X}$, let us define the maps

$$g_1^h(t,x) = g_1(t,h(t),x), \text{ for each } (t,x) \in I \times H$$

and

$$g_2^h(t,s,x) = g_2(t,s,h(s),x), \text{ for each } (t,s,x) \in I \times I \times H.$$

It's clear that by (j) that $g_1^h(\cdot,x)$ is measurable on I, for any $x \in H$. Furthermore, by (jj) for all $(t,x) \in I \times H$, one has

$$\|g_1^h(t,x)\| \leq m_{g_1}(1+\|h(t)\|+\|x\|),$$

so that there exists a non-negative real constant M_{g_1} such that

$$\|g_1^h(t,x)\| \leq M_{g_1}(1+\|x\|),$$

where

$$M_{g_1} := m_{g_1}(1+TM_G\xi),$$

using Lemma 4.4 (1). Moreover, by (jjj) for all $(t,x,y) \in I \times H \times H$, one has

$$\|g_1^h(t,x) - g_1^h(t,y)\| = \|g_1(t,h(t),x) - g_1(t,h(t),y)\| \leq k_{g_1}\|x-y\|.$$

Similarly, note by (i) that $g_2^h(\cdot,\cdot,x)$ is measurable on $I \times I$, for any $x \in H$. Furthermore, by (ii) for all $(t,s,x) \in I \times I \times H$, one has

$$\|g_2^h(t,s,x)\| \leq m_{g_2}(1+\|h(s)\|+\|x\|)$$

so that, there exists a non-negative real constant M_{g_2} such that

$$\|g_2^h(t,s,x)\| \leq M_{g_2}(1+\|x\|),$$

where

$$M_{g_2} := m_{g_2}(1+TM_G\xi),$$

using Lemma 4.4 (1). Moreover, by (iii) for all $(t,s,x,y) \in I \times I \times H \times H$, one has

$$\|g_2^h(t,s,x) - g_2^h(t,s,y)\| = \|g_2(t,s,h(s),x) - g_2(t,s,h(s),y)\| \leq k_{g_2}\|x-y\|.$$

Hence, by Theorem 3.1 there is a unique absolutely continuous solution u_h to the integro-differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in \partial\psi(t,u_h(t)) + g_1(t,h(t),u_h(t)) + \int_0^t g_2(t,s,h(s),u_h(s))ds \\ \quad \text{a.e. } t \in I, \\ u_h(0) = u_0 \in \operatorname{dom}\psi(0,\cdot), \end{cases} \tag{31}$$

with $\|u_h(t)\| \leq \xi$ for all $t \in I$ and $\int_0^T \|\dot{u}_h(t)\|^2 dt \leq \zeta$ for non-negative real constants ξ and ζ (see Theorem 3.1 and (H^3_ψ)).
Now, for each $h \in \mathcal{X}$, define the map

$$\Phi(h)(t) = \int_0^T G(t,s)u_h(s)ds \ \ \forall t \in I,$$

where u_h denotes the unique absolutely continuous solution to problem (31).
Since $u_h(t) \in \operatorname{dom}\psi(t,\cdot)$, for each $t \in I$, then (H^3_ψ) entails that $u_h(t) \in Y(t) \subset \xi\overline{B}_H$, where $Y(t)$ is a convex compact subset of H, for each $t \in I$. Then, for any $h \in \mathcal{X}$, one has $\Phi(h) \in \mathcal{Z}$, where

$$\mathcal{Z} := \{u_f : I \to H,\ u_f(t) = \int_0^T G(t,s)f(s)ds,\ f \in S^\infty_Y\},$$

and $\mathcal{Z}$ is convex compact in $\mathcal{C}_H(I)$ by Lemma 4.5, with $\Phi(\mathcal{X}) \subset \mathcal{Z} \subset \mathcal{X}$. This proves that $\Phi(\mathcal{X})$ is relatively compact.
It remains to check that Φ is continuous on $\mathcal{X}$.
Let $(h_n)_n \subset \mathcal{X}$ be a sequence that uniformly converges to h in $\mathcal{X}$. Then, for each $n \in \mathbb{N}$, denote by u_{h_n} the unique absolutely solution to

$$\begin{cases} -\dot{u}_{h_n}(t) \in \partial\psi(t,u_{h_n}(t)) + g_1(t,h_n(t),u_{h_n}(t)) + \displaystyle\int_0^t g_2(t,s,h_n(s),u_{h_n}(s))ds \\ \quad \text{a.e. } t \in I, \\ u_{h_n}(0) = u_0 \in \operatorname{dom}\psi(0,\cdot), \end{cases} \tag{32}$$

such that $\int_0^T \|\dot{u}_{h_n}(t)\|^2 dt \leq \zeta$ and $\|u_{h_n}(t)\| \leq \xi$ for all $t \in I$.
Note that $(u_{h_n}(t))$ is relatively compact for any $t \in I$ (see (H^3_ψ)), and $(u_{h_n}(\cdot))$ is equicontinuous. Then, by Ascoli's theorem there is a map $u \in W^{1,2}_H(I)$, such that $(u_{h_n}(\cdot))$ (up to a subsequence that we do not relabel) uniformly converges in $\mathcal{C}_H(I)$ to u with $u(0) = u_0$. Moreover, since $\int_0^T \|\dot{u}_{h_n}(t)\|^2 dt \leq \zeta$, one gets $(\dot{u}_{h_n})$ weakly converges to $\dot{u}$ in $L^2_H(I)$.
From (jj) and Lemma 4.4 (1), one remarks that

$$\|g_1(t,h_n(t),u_{h_n}(t))\| \leq m_{g_1}(1 + TM_G\xi + \xi), \qquad \text{for all } t \in I.$$

Furthermore, in view of (jjj), one has

$$\|g_1(t,h_n(t),u_{h_n}(t)) - g_1(t,h(t),u(t))\| \leq k_{g_1}(\|h_n(t) - h(t)\| + \|u_{h_n}(t) - u(t)\|).$$

We know that $(u_{h_n})_n$ (resp. $(h_n)_n$) uniformly converges to $u(\cdot)$ (resp. $h(\cdot)$), then, applying the Lebesgue dominated convergence theorem, one gets

$$\lim_{n\to\infty} \int_0^T \|g_1(t,h_n(t),u_{h_n}(t)) - g_1(t,h(t),u(t))\|^2 dt = 0. \tag{33}$$

Next, from (ii) and Lemma 4.4 (1), one remarks that

$$\|g_2(t,s,h_n(s),u_{h_n}(s))\| \leq m_{g_2}(1+TM_G\xi+\xi), \qquad \text{for all } t \in I. \tag{34}$$

Furthermore, in view of (iii), one has

$$\begin{aligned}&\|g_2(t,s,h_n(s),u_{h_n}(s)) - g_2(t,s,h(s),u(s))\|\\ &\leq k_{g_2}(\|h_n(s)-h(s)\| + \|u_{h_n}(s)-u(s)\|).\end{aligned}$$

Since $(u_{h_n})_n$ (resp. $(h_n)_n$) uniformly converges to $u(\cdot)$ (resp. $h(\cdot)$), then applying the Lebesgue dominated convergence theorem gives

$$\begin{aligned}&\left\|\int_0^t g_2(t,s,h_n(s),u_{h_n}(s))ds - \int_0^t g_2(t,s,h(s),u(s))ds\right\|\\ &\leq \int_0^t \|g_2(t,s,h_n(s),u_{h_n}(s)) - g_2(t,s,h(s),u(s))\|ds\\ &\leq k_{g_2}\int_0^t (\|h_n(s)-h(s)\| + \|u_{h_n}(s)-u(s)\|)ds \to 0 \text{ as } n\to\infty.\end{aligned} \tag{35}$$

From (34), one writes

$$\left\|\int_0^t g_2(t,s,h_n(s),u_{h_n}(s))ds\right\| \leq Tm_{g_2}(1+TM_G\xi+\xi),$$

along with (35), the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^T \left\|\int_0^t g_2(t,s,h_n(s),u_{h_n}(s))ds - \int_0^t g_2(t,s,h(s),u(s))ds\right\|^2 dt = 0. \tag{36}$$

Combining (32), the uniform convergence (resp. weak convergence in $L^2_H(I)$) of (u_{h_n}) (resp. $(\dot{u}_{h_n})$), along with (33) and (36), then, Proposition 2.4, yields

$$-\dot{u}(t) \in \partial\psi(t,u(t)) + g_1(t,h(t),u(t)) + \int_0^t g_2(t,s,h(s),u(s))ds \quad \text{a.e. } t \in I.$$

Consequently, by uniqueness, one deduces $u_h = u$.
Now, we come back to Φ, one has for all $t \in I$

$$\begin{aligned}\|\Phi(h_n)(t) - \Phi(h)(t)\| &\leq \left\|\int_0^T G(t,s)u_{h_n}(s)ds - \int_0^T G(t,s)u_h(s)ds\right\|\\ &\leq M_G\int_0^T \|u_{h_n}(s)-u_h(s)\|ds,\end{aligned}$$

using Lemma 4.4 (1). Since (u_{h_n}) is uniformly bounded and uniformly converges to $u_h(\cdot)$, then,

$$\sup_{t\in I}\|\Phi(h_n)(t) - \Phi(h)(t)\| \to 0 \text{ as } n\to\infty.$$

Thus, $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. An application of Schauder's fixed point theorem ensures that Φ admits a fixed point h

$$h(t) = \Phi(h)(t) = \int_0^T G(t,s)u_h(s)ds, \text{ for all } t \in I,$$

which means that, there exists a w-$W_H^{\alpha,\infty}(I)$ map h and an absolutely continuous solution u_h such that

$$\begin{cases} -\dot{u}_h(t) \in \partial\psi(t,u_h(t)) + g_1(t,h(t),u_h(t)) + \displaystyle\int_0^t g_2(t,s,h(s),u_h(s))ds \\ \quad \text{a.e. } t \in I, \\ w\text{-}^c D^\alpha h(t) = u_h(t) \quad \text{a.e. } t \in I, \\ h(0) - w\text{-}\frac{dh}{dt}(0) = 0, \\ h(T) + w\text{-}\frac{dh}{dt}(T) = 0, \\ \quad u_h(0) = u_0 \in \operatorname{dom}\psi(0,\cdot). \end{cases}$$

□

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REFERENCES

[1] S. Adly - F. Nacry - L. Thibault, *Discontinuous sweeping process with prox-regular sets*, ESAIM Control Optim. Calc. Var. 23 (2017), 1293–1329.

[2] R. P. Agarwal - B. Ahmad - A. Alsaedi - N. Shahzad, *Dimension of the solution set for fractional differential inclusion*, J. Nonlinear Convex Anal. 14 (2) (2013), 314–323.

[3] R. P. Agarwal - S. Arshad - D. O'Regan - V. Lupulescu, *Fuzzy fractional integral equations under compactness type condition*, Fract. Calc. Appl. Anal. 15 (4) (2012), 572–590.

[4] B. Ahmad - J. Nieto, *Riemann-Liouville fractional differential equations with fractional boundary conditions*, Fixed Point Theory 13 (2) (2012), 329–336.

[5] H. Attouch - A. Damlamian, *On multivalued evolution equations in Hilbert spaces*, Israel J. Math. 12 (1972), 373–390.

[6] H. Benabdellah - C. Castaing - A. Salvadori, *Compactness and discretization methods for differential inclusions and evolution problems*, Atti. Sem. Math. Fis. Univ. Modena, XLV (1997), 9–51.

[7] M. Benchohra - J. Graef - F. Z. Mostefai, *Weak solutions for boundary-value problems with nonlinear fractional differential inclusions*, Nonlinear Dyn. Syst. Theory 11 (3) (2011), 227–237.

[8] A. Bouabsa - S. Saïdi, *Coupled systems of subdifferential type with integral perturbation and fractional differential equations*, Adv. Theory Nonlinear Anal. Appl. 7 (1) (2023), 253–271.

[9] A. Bouach - T. Haddad - B. S. Mordukhovich, *Optimal control of nonconvex integro-differential sweeping processes*, J. Differential Equations 329 (2022), 255–317.

[10] A. Bouach - T. Haddad - L. Thibault, *Nonconvex integro-differential sweeping process with applications*, SIAM J. Control Optim 60 (5) (2022), 2971–2995.

[11] A. Bouach - T. Haddad - L. Thibault, *On the discretization of truncated integro-differential sweeping process and optimal control*, J. Optim. Theory Appl. 193 (1) (2022), 785–830.

[12] Y. Brenier - W. Gangbo - G. Savare - M. Westdickenberg, *Sticky particle dynamics with interactions*, J. Math. Pures Appl. 99 (2013), 577–617.

[13] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Lecture Notes in Math., North-Holland, 1973.

[14] B. Brogliato, A. Tanwani, *Dynamical Systems Coupled with Monotone Set-Valued Operators: Formalisms, Applications, Well-Posedness and Stability*, SIAM Rev. 62 (1) (2020), 3–129.

[15] C. Castaing - C. Godet-Thobie - M. D. P. Monteiro Marques - A. Salvadori, *Evolution problems with m-accretive operators and perturbations*, Mathematics (2022).

[16] C. Castaing - C. Godet-Thobie - F .Z. Mostefai, *On a fractional differential inclusion with boundary conditions and application to subdifferential operators*, J. Nonlinear Convex Anal. 18 (9) (2017), 1717–1752.

[17] C. Castaing - C. Godet-Thobie - P. D. Phung - L. T. Truong, *On fractional differential inclusions with nonlocal boundary conditions*, Fract. Calc. Appl. Anal. 22 (2019), 444–478.

[18] C. Castaing - C.Godet-Thobie - S. Saïdi - M. D. P. Monteiro Marques, *Various perturbations of time dependent maximal monotone/accretive operators in evolution inclusions with applications*, Appl. Math. Optim. 87:24 (2023).

[19] C. Castaing - C. Godet-Thobie - L. T. Truong - F. Z. Mostefai, *On a fractional differential inclusion in Banach space under weak compactness condition*, Adv. Math. Econ. 20 (2016), 23–75.

[20] C. Castaing - S. Marcellin, Evolution inclusions with pln functions and application to viscosity and control, J. Nonlinear Convex Anal. 8 (2) (2007), 227–255.

[21] C. Castaing - M. D. P. Monteiro Marques - S. Saïdi, *Evolution problems with time-dependent subdifferential operators*, Adv. Math. Econ. 23 (2019), 1–39.

[22] C. Castaing - S. Saïdi, *Lipschitz perturbation to evolution inclusion driven by time-dependent maximal monotone operators*, Topol. Methods Nonlinear Anal. 58 (2) (2021), 677–712.

[23] C. Castaing - A. Salvadori - L. Thibault, *Functional evolution equations governed by nonconvex sweeeping process*, J. Nonlinear Convex Anal. 2 (2) (2001), 217–241.

[24] G. Colombo - C. Kozaily, *Existence and uniqueness of solutions for an integral perturbation of Moreau's sweeping process*, J. Convex Anal. 27 (2020), 227–236.

[25] K. Deimling, *Non Linear Functional Analysis*, Springer, Berlin, 1985.

[26] J. F. Edmond - L. Thibault, *Relaxation of an optimal control problem involving a perturbed sweeping process*, Math. Program., Ser. B 104 (2005), 347–373.

[27] J. F. Edmond - L. Thibault, *BV solution of nonconvex sweeping process differential inclusion with perturbation*, J. Differential Equations 226 (1) (2006), 135–159.

[28] S. Hu - N. S. Papageorgiou, *Handbook of multivalued analysis.* Vol. II, volume 500 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2000.

[29] N. Kenmochi, *Solvability of nonlinear evolution equations with time-dependent constraints and applications*, Bull. Fac. Educ., Chiba Univ. 30 (1981), 1–87.

[30] A. A. Kilbas - H. M. Srivastava - J. J. Trujillo, *Theory and applications of fractional differential equations*, Math. Studies 204, North Holland: Amsterdam, The Netherlands, 2006.

[31] M. Kubo, *Characterisation of a class of evolution operators generated by time dependent subdifferential*, Funkcial. Ekvac. 32 (1989), 301–321.

[32] M. D. P. Monteiro Marques, *Differential Inclusions in Nonsmooth Mechanical Problems-Shocks and Dry Friction*, Birkhauser Verlag, Basel, 1993.

[33] J. J. Moreau, *Rafle par un convexe variable*, I, Sém. Anal. Convexe, Montpellier, Vol. 1 (1971), Exposé No. 15.

[34] J. J. Moreau, *Rafle par un convexe variable*, II, Sém. Anal. Convexe, Montpellier, Vol. 2 (1972), Exposé No. 3.

[35] J. J. Moreau, *Rétraction d'une multiapplication*, Sém. Anal. convexe, Montpellier (1972), Exposé No. 13.

[36] J. J. Moreau, *An introduction to unilateral dynamics*, in "Approaches in Civil Engineering (M. Frémond, F. Maceri Eds.)", Springer, Berlin, 2002.

[37] F. Nacry, *Perturbed BV sweeping process involving prox-regular sets*, J. Nonlinear Convex Anal. 18 (2017), 1619–1651.

[38] F. Nacry - L. Thibault, *BV prox-regular sweeping process with bounded truncated variation*, Optimization 69 (2020), 1391–1437.

[39] J. C. Peralba, *Équations d'évolution dans un espace de Hilbert, associées à des opérateurs sous-différentiels*, Thèse de doctorat de spécialité, Montpellier, 1973.

[40] P. H. Phung - L. X. Truong, *On a fractional differential inclusion with integral boundary conditions in Banach space*, Fract. Calc. Appl. Anal. 16 (3) (2013), 538–558.

[41] S. Saïdi, *Some results associated to first-order set-valued evolution problems with subdifferentials*, J. Nonlinear Var. Anal. 5 (2) (2021), 227–250.

[42] S. Saïdi - L. Thibault - M. Yarou, *Relaxation of optimal control problems involving time dependent subdifferential operators*, Numer. Funct. Anal. Optim. 34(10) (2013), 1156–1186.

[43] S. Saïdi - M. F. Yarou, *Set-valued perturbation for time dependent subdifferential operator*, Topol. Methods Nonlinear Anal. 46 (1) (2015), 447–470.

[44] S. G. Samko - A. A. Kilbas - O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, New York, 1993.

[45] A. A. Tolstonogov, *Properties of attainable sets of evolution inclusions and control systems of subdifferential type*, Sib. Math. J. 45 (4) (2004), 763–784.

[46] M. Valadier, *Quelques résultats de base concernant le processus de la rafle*, Sém. Anal. Convexe, Montpellier, 3 (1988), 30 pp.

[47] Y. Yamada, *On evolution equations generated by subdifferential operators*, J. Math. Sci. Univ. Tokyo 23 (1976), 491–515.

[48] Y. Zhou, *Basic theory of fractional differential equations*, World Scientific Publishing, 2014.

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