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## Cover Page Footnote

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# On the Singular Pebbling Number of a Graph

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**Abstract.** In this paper, we define a new parameter of a connected graph as a spin-off of the pebbling number (which is the smallest $t$ such that every supply of $t$ pebbles can satisfy every demand of one pebble). This new parameter is the singular pebbling number, the smallest $t$ such that a player can be given any configuration of at least $t$ pebbles and any target vertex and can successfully move pebbles so that exactly one pebble ends on the target vertex. We also prove that the singular pebbling number of any graph on 3 or more vertices is equal to its pebbling number, and we find the singular pebbling numbers of the two remaining graphs, $K_1$ and $K_2$, which are not equal to their pebbling numbers.

## 1 Introduction

Graph pebbling is a game in which a player (termed the 'Pebbler' in this paper) moves pebbles from vertices to other vertices along edges in an attempt to end up with at least one pebble on the 'target vertex'. The Pebbler has an opponent (termed the 'Placer' in this paper) who decides the initial placement of the pebbles and chooses a vertex to be the target vertex. The Pebbler can move pebbles by taking two pebbles off of any vertex with two or more pebbles and moving one of them to an adjacent vertex, removing the other pebble from the game.

In this paper, all graphs are simple (loops and multiple edges are not allowed) and connected.

The definition of pebbling number that is typically used involves the common principles of supply and demand:

**Definition 1.1** ([3])**.** A graph's *pebbling number*, denoted $\pi(G)$, is the smallest $t$ such that every supply (configuration) of $t$ pebbles can satisfy every demand of one pebble.

In other words, the pebbling number is the smallest $t$ such that the Pebbler can be given any configuration of $t$ pebbles and any target vertex and can successfully move pebbles so that at least one pebble ends on the target vertex.

The problem of graph pebbling was initially studied in [1]. Two survey papers have been written on the various research done on this problem, [2] and [4]. Much research

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has been done not just on the pebbling number of a graph, but also on the $k$-fold pebbling number of a graph, the smallest $t$ such that the Pebbler can be given any configuration of $t$ pebbles and any target vertex and can successfully move pebbles so that at least $k$ pebbles end on the target vertex. What had not been studied until this paper, however, was the situation in which the Pebbler wants to get exactly $k$ pebbles onto the target vertex. This paper examines the case in which $k$ is 1 and the Pebbler wants to get precisely $k$ pebbles onto the target vertex.

Hence, this paper introduces a new concept, the singular pebbling number of a graph G.

**Definition 1.2.** A graph's *singular pebbling number*, denoted $\pi_s(G)$, is the smallest $t$ such that the Pebbler can be given any configuration of at least $t$ pebbles and any target vertex and can successfully move pebbles so that exactly one pebble ends on the target vertex.

Note that we use **at least** $t$ pebbles, rather than **exactly** $t$ pebbles, in this definition, which is an important distinction. In some cases, as we will see later on in the paper, the Pebbler may be able to singularly pebble a graph with a particular number of pebbles (say, $p$), but may not necessarily be able to singularly pebble that graph with more than the particular number of pebbles. In that case, the singular pebbling number of the graph is not $p$.

This paper also uses a different interpretation of pebbling number: the pebbling number of a graph is one more than the largest $t$ such that the Placer can configure $t$ pebbles and choose a target vertex so that the Pebbler cannot move pebbles to get at least one pebble on the target vertex. Thus, the singular pebbling number of a graph is one more than the largest $t$ such that the Placer can configure $t$ pebbles and choose a target vertex so that the Pebbler cannot move pebbles to get exactly one pebble on the target vertex.

In this paper, we thus assume the goal is for the Placer to configure $t$ pebbles and choose a target vertex so that the Pebbler cannot move pebbles to get exactly one pebble on the target vertex. Hence, the 'optimal configurations' of pebbles will be the configurations of pebbles that include the most possible pebbles without allowing the Pebbler to singularly pebble the graph.

A complete graph is denoted by $K_n$, with $n$ being the number of vertices.

## 2 Preliminary Results

**Proposition 2.1.** $\pi(G) \le \pi_s(G)$.

*Proof.* If the Pebbler is able to move exactly one pebble to the target vertex, they are certainly able to move at least one pebble to the target vertex. □

Because of **proposition 2.1**, for the remainder of the paper we only need to find upper bounds on $\pi_s(G)$.

**Proposition 2.2.** *Let* $G$ *be a graph on more than one vertex. Then if* $G$ *can be singularly pebbled with* $t$ *pebbles,* $G$ *can be singularly pebbled with any natural number greater than or equal to* $t$ *of pebbles.*

*Proof.* If the Pebbler begins with more than $t$ pebbles, and $G$ can be singularly pebbled with $t$ pebbles no matter what the configuration of the $t$ pebbles is, then the Pebbler can simply move pebbles around to get down to the number of pebbles that we know works, since each time they move two pebbles one is removed. Since $G$ has more than one vertex, moving pebbles is possible. □

Because of **proposition 2.2**, for graphs other than $K_1$ we only need to find the smallest number of pebbles such that any configuration of them allows the Pebbler to singularly pebble the graph.

**Lemma 2.3.** *In order for the Placer to win, they cannot put* $3$ *or more pebbles on the target vertex unless the graph is* $K_1$.

*Proof.* If there is an even number ($\geq 4$) of pebbles on the target vertex, all the pebbles can be moved to an adjacent vertex (which means the number of pebbles is divided by two, but since there were at least four pebbles to begin with at least two pebbles remain). There are therefore two pebbles on a vertex adjacent to the target vertex that can be moved to the target vertex, leaving precisely one pebble on the target vertex and allowing the Pebbler to win the game.

If there is an odd number ($\geq 3$) of pebbles on the target vertex, all but one of the pebbles (an even number) can be moved to an adjacent vertex, leaving exactly one pebble on the target vertex.

Thus, the Placer cannot win if the target vertex begins with 3 or more pebbles unless the graph is $K_1$.

□

**Lemma 2.4.** $\pi_s(G) \neq \pi(G)$ *implies that any of the Placer's optimal configurations of pebbles on* $G$ *must involve exactly 2 pebbles being placed on the target vertex, unless the graph is* $K_1$.

*Proof.* By **lemma 2.3**, the optimal configurations cannot include three or more pebbles beginning on the target vertex, and if one pebble begins on the target vertex the Pebbler automatically wins. If there are no pebbles beginning on the target vertex, the Pebbler will never be able to move more than one pebble onto the target vertex without being able to move exactly one pebble onto the target vertex. This is true because if there are more than two pebbles on a vertex adjacent to the target vertex, the Pebbler can choose to move just two pebbles toward the target vertex. Thus, if an optimal configuration can be achieved without any pebbles being placed on the target vertex, $\pi_s(G) = \pi(G)$. □

**Lemma 2.5.** *If the Placer puts* 2 *pebbles on the target vertex, and the Pebbler is able to move any pebbles onto a vertex adjacent to the target vertex while maintaining the* 2 *pebbles on the target vertex, the Pebbler wins.*

*Proof.* If there is ever a pebble on a vertex adjacent to the target vertex, the two pebbles on the target vertex can be moved off the target vertex toward that vertex. There are then two pebbles on a vertex adjacent to the target vertex, which can subsequently be moved to leave exactly one pebble on the target vertex. □

## 3 Main Results

**Theorem 3.1.** *For all graphs on three or more vertices,* $\pi_s(G) = \pi(G)$.

*Proof.* Let $v$ be the target vertex and let $\deg(v)$ denote the degree of $v$.

Since we are dealing only with connected graphs, we can assume that $\deg(v) > 0$.

Additionally, by **lemma 2.4**, the only way $\pi_s(G)$ can be different from $\pi(G)$ on a graph with two or more vertices is if the Placer's optimal configuration of pebbles must involve 2 pebbles being placed on $v$. By **lemma 2.5**, 2 pebbles being on $v$ means for the Placer to win, no pebbles can ever be moved onto the vertices adjacent to $v$. Given any pebble configuration that prevents the Pebbler from getting a single pebble to $v$ after starting with two pebbles on $v$, one can simply add a pebble to each vertex adjacent to the $v$ (which must be at least two vertices if $\deg(v) \geq 2$) and remove the pebbles from $v$, allowing pebbles on the vertices adjacent to $v$ but still not allowing a pebble to reach $v$. Thus, placing two pebbles on $v$ is not necessary to achieve an optimal configuration of pebbles for the Placer if $\deg(v) \geq 2$. Hence, if $\deg(v) \geq 2$, $\pi_s(G) = \pi(G)$.

This means we can also assume that for $\pi_s(G) \neq \pi(G)$, $\deg(v) < 2$. Thus, since $2 > \deg(v) > 0$, $\deg(v) = 1$ for $\pi_s(G) \neq \pi(G)$ if G is a graph on three or more vertices.

By **lemma 2.4**, for the singular pebbling number of a graph on two or more vertices to be different from the pebbling number, the optimal pebble configuration must have two pebbles on $v$. Thus, to prove that $\pi_s(G) = \pi(G)$ it suffices to show that it is equivalent or better not to have any pebbles on $v$.

By **lemma 2.5**, in order for the Placer to win, if they begin with 2 pebbles on $v$, they cannot allow the Pebbler to move any pebbles to vertices adjacent to $v$. In a graph on three or more vertices in which $\deg(v) = 1$, there is at least one vertex at distance two or more from $v$. Given any pebble configuration that prevents the Pebbler from getting a single pebble to $v$ after starting with two pebbles on $v$, one can simply add two pebbles to a vertex at distance 2 or more from $v$ and remove the pebbles from $v$, allowing a pebble to get to the vertex adjacent to $v$ but still not allowing a pebble to get to $v$. Thus, we have shown that it is always equivalent or better for the Placer not to place 2 pebbles on $v$ for graphs on at least three vertices if $\deg(v) = 1$ and we have therefore concluded the proof of this theorem.

□

**Example 3.2.** $\pi(K_1) = 1$, whereas $\pi_s(K_1)$ is infinite.

*Proof.* First, let us determine the pebbling number. If there are 0 pebbles on any graph, it is impossible for the Pebbler to move a pebble to the target vertex. If there is one pebble on a graph with only one vertex, it is impossible not to have at least one pebble on the target vertex. The pebbling number of $K_1$ is thus one.

Next, we determine the singular pebbling number. The Placer can put as many pebbles as they want (at least two) onto the vertex of $K_1$ in finding the singular pebbling number, since there can never be exactly one pebble on the vertex if the vertex begins with more than one pebble. Thus, the singular pebbling number of $K_1$ is infinite. □

**Example 3.3.** $\pi(K_2) = 2$, whereas $\pi_s(K_2) = 3$.

*Proof.* First, let us determine the pebbling number. The Placer can only place pebbles on the vertex adjacent to the target vertex in order to win. If the Placer places more than one pebble on the adjacent vertex, the Pebbler will win by moving two pebbles toward the target vertex. The pebbling number of $K_2$ is thus two.

Next, we determine the singular pebbling number. The Placer can either place one pebble on the vertex adjacent to the target vertex or can place two pebbles on the target vertex in order to win. The singular pebbling number of $K_2$ is thus three. □

## 4 Problems for Future Research

Since this paper solved the problem only when $k = 1$, future research could consider the singular $k$-fold pebbling number of graphs.

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