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Perspective and open problems on birational properties and singularities of moduli scheme of sheaves on surfaces

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Abstract. For complex projective smooth surface X, let M be the coarse moduli scheme of rank-two stable sheaves with fixed Chern classes. Grasping the birational structure of M, for example its Kodaira dimension, is a fundamental problem. However, in the case where $\kappa(X) > 0$, the study of this problem has not necessarily been active in recent years. In this article we survey the study of this problem, especially for the case where $\kappa(X) = 1$ and $c_1 = 0$. We will also survey some research on the structure of singularities of M, and a minimal model program of M. While explaining motivations, we raise several unsolved problems.

Keywords: moduli of stable sheaves, elliptic surface, singularities, obstruction, Kodaira dimension, minimal model program.

1 Overview of singularities and Kodaira dimension of moduli of sheaves on elliptic surfaces with $c_1 = 0$

For a complex projective smooth surface X and an ample line bundle H on it, there is the coarse moduli scheme $M = M_H(\mathbf{v})$ of rank-two H-stable sheaves with fixed Chern classes $\mathbf{v} = (2, c_1, c_2)$, and the coarse moduli scheme $\bar{M} = \bar{M}_H(\mathbf{v})$ of S-equivalence classes of rank-two H-semistable sheaves with fixed Chern classes $\mathbf{v}$ by [4].

Problem 1.1. ([16, Question.1.1]) How is the birational structure of M, for example its Kodaira dimension $\kappa(M)$?

This problem has been actively studied when $\kappa(X) = -\infty,\ 0$. When $\kappa(X) = 2$, J. Li's work [11] is well known. [5, Section 11] is an excellent summary for the birational structure of M.

In this section, we consider the case where $\kappa(X) = 1$ and X is a minimal surface. In this case, X is an elliptic surface, that is, there is a fibration morphism $\pi : X \to C$ to a curve C such that its general fibers are elliptic. We denote the number of multiple fibers of π by $\Lambda(X)$, $d = \chi(\mathcal{O}_X)$, and the fiber class of π by $\mathfrak{f} \in \mathrm{Num}(X)$. For $\mathbf{v} = (2, c_1, c_2)$, we suppose that H is $\mathbf{v}$-suitable ([3, Definition 3.1.]). Roughly speaking, a $\mathbf{v}$-suitable ample line bundle H is not separated from $\mathfrak{f}$ by any wall of type $\mathbf{v}$ ([3, Definition 2.1.]) in the nef cone of X. When $c_1 \cdot \mathfrak{f}$ is odd, Friedman's work is well known ([3, Theorem 3.14], [17]).

It states that $M_H(\mathbf{v})$ is birationally equivalent to a symmetric product of the Jacobian surface $J^{e+1}(X)$ when X is simply connected elliptic surface with at most two multiple fibers.

From here we further assume that $c_1 = 0$. The reasons for the major differences between these two cases are as follows. In this case $C = \mathbf{P}^1$, and we have the generic fiber $X_{\overline{\eta}}$ over $\overline{\eta} = \operatorname{Spec} \overline{k(\mathbf{P}^1)}$. Let E be an H-stable sheaf on X. It induces a vector bundle $E_{\overline{\eta}}$ on $X_{\overline{\eta}}$. Thanks to the suitability of H, $E_{\overline{\eta}}$ is stable if $c_1 \cdot \mathfrak{f}$ is odd. On the other hand, $E_{\overline{\eta}}$ is semi-stable but not stable if $c_1 = 0$.

Here, we recall a few words. Let V be a projective normal variety such that K_V is $\mathbb{Q}$-Cartier. The K-dimension $\kappa(K, V)$ is defined to be

$$\max\{\dim \Phi_{|mK_V|} : V \dashrightarrow |mK_V| \mid m \in \mathbb{N}, mK_V \text{ is Cartier and } h^0(mK_V) \neq 0\}.$$

The Kodaira dimension $\kappa(V)$ is $\kappa(K_{\tilde{V}}, \tilde{V})$, where $\tilde{V}$ is a smooth complete variety birationally equivalent to V. $\kappa(V)$ is a birational invariant. If V has only canonical singularities, then $\kappa(K, V)$ and $\kappa(V)$ are equal. For the definition of canonical singularities, refer to [6, Def. 6.2.4.] for example.

Now we denote $M_H((2, 0, c_2)) = M(c_2) = M$. Friedman showed the following.

Theorem 1.2. *([2, Sect. 7]) Suppose X is generic and $c_2 > \max(2(1+p_g), 2p_g(X) + (2/3)\Lambda(X))$. Then there is such a dense open set M_0 of $M(c_2)$ as follows. M_0 is contained in the good locus M_{gd} of $M(c_2)$ defined by*

$$M_{gd} = \left\{[E] \in M(c_2) \mid \operatorname{ext}^2(E, E)^0 = 0\right\}.$$

There is a morphism $\psi : M_0 \to \mathbf{P}^{2c_2 - 2p_g - 1}$ such that the general fiber of ψ is isomorphic to $\operatorname{Pic}^0(C)$, where C is some curve.

The map ψ in Theorem 1.2 and the prulicanonical maps of $M(c_2)$ are pretty similar:

Proposition 1.3. *([16, Cor.3.5.]) ψ and the pluricanonical map $\Phi : M(c_2) \dashrightarrow \mathbf{P}^N$ are coincident up to a quasi-finite map. In particular, $\dim(\psi) = 2c_2 - 2p_g - 1 = \{\dim M + 1\}/2$ equals the K-dimension $\kappa(K, \bar{M}(c_2))$.*

Thus, we can know the Kodaira dimension $\kappa(M(c_2))$ if $M(c_2)$ is projective (for example, when c_2 is odd) and if $M(c_2)$ admits only canonical singularities.

Problem 1.4. Are all singularities of $M(c_2)$ canonical or not?

Here we recall a classical and fundamental fact in the deformation theory of sheaves from [10]. If E is a singular point of $M(c_2)$, then for $b = \dim \operatorname{Hom}(E, E(K_X))^{\circ} \neq 0$ and $D = \dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Hom}(E, E(K_X))^{\circ}$, which is the expected dimension of $M(c_2)$, we have

$$\mathcal{O}_{M,E}^{\wedge} \simeq \mathbb{C}[[t_1, \cdots, t_{D+b}]]/(F_1, \ldots, F_b), \tag{1}$$

where F_i is a power series starting from degree-two terms.

Any sheaf E in $M(c_2)$ induces a rank-two vector bundle E_η with degree 0 on the generic fiber X_η, where $\eta = \operatorname{Spec}(k(\mathbf{P}^1))$. From [2, Fact 2.11.], this can be classified into three cases:

Case I : E_η has no sub line bundle with fiber degree 0.
Case II : E_η has a sub line bundle with fiber degree 0, but E_η is not decomposable.
Case III : E_η is decomposable into line bundles with fiber degree 0.

In Case I, we have the following theorem. In the proof, we use a sufficient condition for singularities to be canonical [16, Theorem 4.1.].

Theorem 1.5. *([16, Thm. 1.3.]) Suppose that E is a singular point of $M(c_2)$ applying to Case I. If $7(d+2)/4 \geq \Lambda(X)$ or $2 \geq \Lambda(X)$, then the following holds:*

(1) Let G be any nonzero $\mathbb{C}$-linear combination of $F_1, \cdots F_b$ in (1) *and we indicate G as*

$$G = t_1^2 + \cdots + t_R^2 + O(3), \tag{2}$$

where $O(3)$ are terms whose degrees are more than 2, and R is an integer depending on G. Then $R \geq 2b+1$.
(2) E is a canonical singularity of $M(c_2)$.
Moreover, there actually exist singularities meeting these conditions on $M(c_2)$ if $c_2 \gg 0$.

As a result, we can know $\kappa(M(c_2))$ in some situation where the structure of X is rather simple:

Theorem 1.6. *([16, Thm. 1.6.]) We suppose that every fiber of $X \to \mathbf{P}^1$ is irreducible. Also we suppose that X has just two multiple fibers with multiplicities $m_1 = 2$ and $m_2 = m \geq 3$, and $d = \chi(\mathcal{O}_X) = 1$. Then all singularities of M apply to Case I. As a result, all singularities of M are always canonical singularities from Theorem 1.5. Thus the Kodaira dimension $\kappa(M)$ equals to $\kappa(K, M) = 2c_2 - 2p_g - 1 = (\dim M(c_2) + 1)/2$ from Proposition 1.3.*

On the other hand, there are cases where it is not possible to know whether $M(c_2)$ has only canonical singularities or not only from the evaluation of the second-order terms in the defining equation:

Theorem 1.7. *([16, Thm.1.4.]) There is an example of an elliptic surface X satisfying the following. For every obstructed sheaf E in $M(c_2)$, $(M(c_2), E)$ is always a hypersurface singularity, and so*

$$\mathcal{O}^{\wedge}_{M,E} \simeq \mathbb{C}[[t_1, \cdots, t_{D+1}]]/(F), \tag{3}$$

where $F = t_1^2 + \cdots + t_R^2 + O(3)$. There actually exist locally-free obstructed stable sheaves of Case II satisfying $R = 1$ in (2) *for every $c_2 \gg 0$. In this case, we cannot judge if (M, E) is a canonical singularity or not only from the second-order terms in the defining equation from [6, Example 7.4.2., Proposition 5.3.12.].*

2 Problems to be solved in the future –Singularities and Kodaira dimension–

Ongoingly, we suppose that $\kappa(X) = 1$ and $c_1 = 0$ in this section. We posed the problem of finding the Kodaira dimension of M in Problem 1.1. The Kodaira dimension of M was obtained by Theorem 1.6 because we were able to show by Theorem 1.5 that all singularities of M are canonical. In this case, the problem was settled by the evaluation of the second-order terms in the defining equation of the moduli scheme. On the other hand, from Theorem 1.7, it may not be known from the second-order terms alone whether the singularity of M is canonical or not. Therefore the following problem can be raised.

Problem 2.1. Let E be a singularity of the moduli space of stable sheaves on surfaces. Suppose that we can not judge whether it is a canonical singularity or not only from the second-order terms in the defining equation. Is there a way to evaluate third order or higher terms in a defining equation? By using this method, can we judge it is a canonical singularity or not?

We can examine the degree-two part of the defining equation of the moduli space from linear map between Ext-modules of sheaves ([10]. cf. [16, Fact 2.2.]). However, examining terms of third order or higher would be more difficult than that. Some related keywords include Massey product ([10]) and DG algebra $R\operatorname{Hom}^{\bullet}(E, E)$ (See e.g. [1], [7, p.3]).

In building a theory using moduli spaces, one may successfully avoid facing singularities at the front. Thereby it seems worth considering not only Problem 2.1, but also the following problem.

Problem 2.2. Can we find a way to evaluate the Kodaira dimension $\kappa(M)$ without necessarily facing singularities of M?

As for this direction, there is a work by J. Li [11]. In this paper, it is shown that M is of general type when the base surface X is of general type, $p_g(X) \neq 0$, $c_2 \gg 0$ and $\dim(M)$ is an even number. Can we come up with some anwer for Problem 2.2 in the above-mentioned case where $\kappa(X) = 1$ and $c_1 = 0$?

3 Problems to be solved in the future –Minimal model program–

Next, for a normal projective variety V such that K_V is $\mathbb{Q}$-Cartier and the singularities of V are log-terminal, the minimal model program (MMP) of V is proposed and researched. (For the definition of log-terminal singularities, refer to [6, Definition 5.2.7.]. We remark that canonical singularities are log-terminal.) Let us describe a very rough idea of what the MMP of V is. If the Kodaira dimension $\kappa(V)$ is positive, we can contract an extremal ray of V to get a small contraction or flip. After performing these birational transformations finite times, we obtain a minimal model V' of V, that is, $K_{V'}$ becomes nef, and all singularities of V' are log-terminal. For a detailed explanation, see for example [9] and [12], where Flowchart 3-1-15 is very useful.

Definition 3.1. Let $f : W \to Y$ be a birational proper morphism such that K_W is $\mathbb{Q}$-Cartier and $-K_W$ is f-ample, the codimension of the exceptional set $\mathrm{Ex}(f)$ of f is more than 1, and the relative Picard number of f is 1. We say a birational proper morphism $f_+ : W_+ \to Y$ is a *flip* of f if (1) K_{W_+} is $\mathbb{Q}$-Cartier, (2) K_{W_+} is f_+-ample, (3) the codimension of the exceptional set $\mathrm{Ex}(f_+)$ is more than 1, and (4) the relative Picard number of f_+ is 1.

From here, we suppose that $\kappa(X) > 0$ and that X is a minimal surface, and consider the moduli of stable sheaves. In this case, K_X is contained in the closure of the ample cone $\mathrm{Amp}(X)$ in $\mathrm{Num}(X) \otimes \mathbb{R}$. We have the following theorem on the birational structure of the moduli scheme $M(H)$ of H-stable sheaves of type $\mathbf{v}$, and the moduli scheme $\bar{M}(H)$ of S-equilavence classes of H-semistable sheaves of type $\mathbf{v}$.

Theorem 3.2 ([14], [15])**.** *Suppose c_2 is sufficiently large.*
(1) Let H and H' be ample line bundles on X devided by just one $\mathbf{v}$-wall W. Assume that K_X does not lie in W, and that K_X and H' lie in the same side with respect to W. For $a \in (0,1)$, we can define a-stability of sheaves on X using H and H', and there is a moduli scheme $M(a)$ of a-stable sheaves of type $\mathbf{v}$, and a moduli scheme $\bar{M}(a)$ of S-equivalence classes of a-semistable sheaves of type $\mathbf{v}$. The wall-crossing problem of a-stability induces the sequences of flips in the sence of Definition 3.1

$$M(H) = M(a_0) \dashrightarrow M(a_1) \dashrightarrow \cdots \dashrightarrow M(a_{N-1}) \dashrightarrow M(a_N) = M(H'), \quad \text{and}$$
$$\bar{M}(H) = \bar{M}(a_0) \dashrightarrow \bar{M}(a_1) \dashrightarrow \cdots \dashrightarrow \bar{M}(a_{N-1}) \dashrightarrow \bar{M}(a_N) = \bar{M}(H')$$

in a moduli-theoretic way.
(2) When H varies in $\mathrm{Amp}(X)$ and gets closer to K_X, after crossing finitely many $\mathbf{v}$-walls, it reaches an ample line bundle $\tilde{H}$ such that no $\mathbf{v}$-wall devides K_X and $\tilde{H}$. Then the canonical class $K_{\bar{M}(\tilde{H})}$ of $\bar{M}(\tilde{H})$ becomes nef.

Thus one can regard this natural process described in a moduli-theoretic way as an analogy of the minimal model program of $M(H)$. However it is unknown whether $M(\tilde{H})$ admits only log-terminal singularities in general. Thereby we should notice that we need to verify that all singularities of $M(\tilde{H})$ are log-terminal, in order to say that this sequence is a genuine MMP of $M(H)$. Here Problem 1.4 appears again. From Theorem 3.2, we can raise the following problem.

Problem 3.3. We pick an ample line bundle H on X and move it closer to K_X, then we obtain an analogy of MMP of $M(H)$ by Theorem 3.2.
(1) Can we investigate how $M(H)$ is improved by this moduli-theoretic sequence of flips? Especially, the

case where $M(H)$ admits only log-terminal singularities is more desirable, because the above-mentioned birational maps give the genuine MMP in this case. Also it is more desirable that the structure of the starting point $M(H)$ is easy to understand.
(2) It can be interesting if one starts H near K_X and gradually moves H away from K_X, and one observes flips occuring from wall-crossing as in Theorem 3.2.

Remark 3.4. We remark that flips in Theorem 3.2 are also Thaddeus flips, that are birational transformations appearing from the variation of GIT quotients and linearizations ([13]). Also, we remark that Thaddeus flips are not necessarily flips in the sense of Definition 3.1 in general.

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