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(α, β)-Z_b-GERAGHTY TYPE CONTRACTION IN b-METRIC-LIKE SPACES VIA b-SIMULATION FUNCTION

S. NEGI[1], S. ANTAL[1], U. C. GAIROLA[1*], §

ABSTRACT. The aim of this paper is to introduce the notion of (α, β)-Z_b-Geraghty type contraction via b-simulation function and use this contraction to establish a common fixed point theorem for a pair of self-mappings in the context of a b-metric-like space. Our result extends and generalizes the result of Matthews [21], Khojasteh et al. [20], Demma et al. [15], Chandok [12] and some others also. We deduce some corollaries from our main result and provide examples to illustrate our results. Moreover, we apply our result to obtain a solution of second order differential equation.

Keywords: Common fixed point, b-metric-like space, (α, β)-Z_b-Geraghty type contraction, b-simulation function.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory is an application oriented branch of mathematics which continually contributing in several disciplines of applied and engineering science. The inspiration behind all kinds of developments of this theory comes from the classical Banach contraction principle which was introduced in 1922 by Stephan Banach [10] and proved in the context of metric spaces. This principle guides researchers to prove useful fixed point theorems under different generalizations of the metric spaces. Some notable generalizations of metric spaces are b-metric spaces, partial metric spaces, metric-like spaces, S-metric spaces etc. The idea of b-metric was initiated from the works of Bourbaki [11] and Bakhtin [9]. Czerwik [14] defined a b-metric space with a view of generalizing the Banach contraction principle. In 1994, Matthews [21] introduced partial metric space which was further generalized by Amini-Harandi [4] to the notion of metric-like space. These spaces are specially used in logical programming. Recently, Alghamdi et al. [3] presented the idea of a b-metric-like space in 2013 which combined the idea of metric-like space and b-metric space. Later,

[1] Department of Mathematics, H.N.B. Garhwal University, BGR Campus Pauri, Pauri Garhwal, 246001, Uttarakhand, India.
smitanegi.sn@gmail.com; ORCID: https://orcid.org/0000-0002-9560-4485.
antalswati11@gmail.com; ORCID: https://orcid.org/0000-0001-5517-0021.
ucgairola@rediffmail.com; ORCID: https://orcid.org/0000-0002-3981-1033.
[*] Corresponding author.
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many fixed point results under different contractive conditions in such spaces have been obtained. For more details on this topic, we refer to ([7], [13] and [17]).

In 2015, Khojasteh et al. [20] gave the idea of simulation function to extend the class of mappings which under some conditions gives a fixed point and also provide a new technique to prove fixed point theorems. Further, generalizing the concept of simulation function Demma et al. [15] gave the notion of a b-simulation function and demonstrated some fixed point results in b-metric space. Later, simulation functions have been explored by many researchers for more general settings (see, for example [5], [18], [19], [23], [24], [25], [26] and references therein).

On the other hand, Geraghty generalized the Banach contraction principle and named it as Geraghty contraction. Recently, S. Chandok [12] presented the concept of (α,β)-admissible Geraghty type contractive mapping in metric space. Later, many authors worked in this direction and established many interesting results (see, for example [1], [2], [8] and references therein).

In this paper, inspired and motivated by the results of Demma et al. [15] and S. Chandok [12], we introduce the notion of (α,β)-Z_b-Geraghty type contraction for a pair of mappings via b-simulation function and establish a common fixed point theorem in the context of b-metric-like spaces. Moreover, as an application, we apply our results to solve a second order differential equation.

2. Preliminaries

In this section, we give some definitions and results used in the sequel. Throughout the paper, we use the symbol $\mathbb{R}$ for $(-\infty,\infty)$, $\mathbb{N}$ for $\{1,2,3,...\}$ and $\mathbb{N}_0$ for $\{0,1,2,3,...\}$.

Definition 2.1. [9] *Let V be a non-empty set and let $s \geq 1$ be a given real number. A function $b : V \times V \to [0,\infty)$ is said to be a b-metric if and only if for all $j,k,l \in V$, the following conditions are satisfied:*

(b_1) $b(j,k) = 0 \iff j = k$;

(b_2) $b(j,k) = b(k,j)$;

(b_3) $b(j,k) \leq s[b(j,l) + b(l,k)]$.

Then the triplet (V,b,s) is called a b-metric space.

Definition 2.2. [15] *Let (V,b,s) be a b-metric space. A b-simulation function is a function $\zeta_b^* : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_{1b}^*) $\zeta_b^*(j,k) < k - j$ *for all* $j,k > 0$;

(ζ_{2b}^*) *if $\{j_n\}$ and $\{k_n\}$ are sequences in $(0,\infty)$ such that*

$$0 < \lim_{n\to\infty} j_n \leq \liminf_{n\to\infty} k_n \leq \limsup_{n\to\infty} k_n \leq s \lim_{n\to\infty} j_n < +\infty,$$

then

$$\limsup_{n\to\infty} \zeta_b^*(sj_n, k_n) < 0.$$

Throughout the paper $\mathcal{Z}_b$ will represent the family of all b-simulation functions.

Theorem 2.1. [15] *Let (V,b,s) be a complete b-metric space and let $E : V \to V$ be a mapping. Suppose that there exists a $\zeta_b^* \in \mathcal{Z}_b$ such that*

$$\zeta_b^*\big(sb(Ej,Ek), b(j,k)\big) \geq 0, \tag{2.1}$$

holds for all $j,k \in V$. Then E has a unique fixed point.

Note that the Banach contraction can be obtained from contraction condition (2.1) by taking $\zeta_b^*(j,k) = \tau k - j$ for all $j,k \in [0,\infty)$ with $s = 1$ and $\tau \in [0,1)$.

Remark 2.1. *For the value $s = 1$, b-simulation function reduces to simulation function in the standard metric spaces.*

Definition 2.3. [3] *Let V be a non-empty set and let $s \geq 1$ be a given real number then a function $\omega : V \times V \to [0, \infty)$ is called b-metric-like if for all $j, k, l \in V$, the following conditions hold:*

(ω_1) $\omega(j, k) = 0 \implies j = k$;
(ω_2) $\omega(j, k) = \omega(k, j)$;
(ω_3) $\omega(j, k) \leq s[\omega(j, l) + \omega(l, k)]$.

Then the pair (V, ω) is called a b-metric-like space.

Remark 2.2. *Every b-metric space is b-metric-like space and every metric-like space is also b-metric-like space but converse need not be true.*

Example 2.1. *Let $V = \{1, 2, 3\}$ and $\omega : V \times V \to [0, \infty)$ is defined by*

$$\omega(j, k) = \begin{cases} 3, & \text{if } j = k, \\ 1, & \text{otherwise.} \end{cases}$$

Then (V, ω) is a b-metric-like space with coefficient $s = \frac{3}{2}$. Clearly, ω is not a b-metric as $\omega(2, 2) \neq 0$. Also ω is not a metric-like as $\omega(2, 2) \nleq \omega(2, 3) + \omega(3, 2)$.

Let (V, ω) be a b-metric-like space. Let $j \in V$ and $\nu > 0$, then the set $B(j, \nu) = \{k \in V : |\omega(j, k) - \omega(j, j)| < \nu\}$ is called an open ball with center at j and radius $\nu > 0$.

Definition 2.4. [3] *Let (V, ω) be a b-metric-like space and let $\{j_n\}$ be a sequence of points of V. A point $j^* \in V$ is said to be the limit of the sequence $\{j_n\}$ if $\lim_{n\to\infty} \omega(j_n, j_m) = \omega(j^*, j^*)$ and we say that the sequence $\{j_n\}$ is convergent to j^* and denote it by $j_n \to j^*$ as $n \to \infty$.*

Definition 2.5. [3] *Let (V, ω) be a b-metric-like space, then*

(i) *a sequence $\{j_n\}$ is called Cauchy if and only if $lim_{m,n\to\infty}\omega(j_n, j_m)$ exists and is finite;*
(ii) *a b-metric-like space (V, ω) is said to be complete if and only if every Cauchy sequence $\{j_n\}$ in V converges to $j^* \in V$ so that*

$$\lim_{m,n\to\infty} \omega(j_n, j_m) = \omega(j^*, j^*) = \lim_{n\to\infty} \omega(j_n, j^*).$$

Definition 2.6. [13] *Suppose that (V, ω) is a b-metric-like space. A mapping $E : V \to V$ is said to be continuous at $j \in V$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $E(B(j, \delta)) \subseteq B(Ej, \epsilon)$. We say that E is continuous on V if E is continuous at all $j \in V$.*

In 1973, Geraghty [16] generalized Banach contraction by introducing a new class of contractions and presented the existence and uniqueness theorem as follows:

Definition 2.7. [16] *Let (V, d) be a complete metric space and $E : V \to V$ be a mapping such that for all $j, k \in V$*

$$d(Ej, Ek) \leq \eta(d(j, k))d(j, k),$$

where $\eta : [0, \infty) \to [0, 1)$ is a function satisfying $\eta(j_n) \to 1$ implies $j_n \to 0$ as $n \to \infty$. Then E has a unique fixed point $j^* \in V$.

Definition 2.8. [6] *Let V be a non-empty set, $E,F : V \to V$ be two mappings and $\alpha,\beta : V \times V \to [0,\infty)$ be two functions then (E,F) is called a pair of (α,β)-admissible mappings, if for all $j,k \in V$*

$$\alpha(j,k) \geq 1 \text{ and } \beta(j,k) \geq 1$$

implies

$$\alpha(Ej,Fk) \geq 1, \alpha(Fj,Ek) \geq 1 \text{ and } \beta(Ej,Fk) \geq 1, \beta(Fj,Ek) \geq 1.$$

Definition 2.9. [22] *Let (V,ω) be a b-metric-like space, $E,F : V \to V$ be a pair of mappings and $\alpha,\beta : V \times V \to [0,\infty)$ be two functions then V is said to be (α,β)-regular, if $\{j_n\}$ is a sequence in V such that $j_n \to j^* \in V$ and $\alpha(j_n,j_{n+1}) \geq 1, \beta(j_n,j_{n+1}) \geq 1$, $\forall\ n \in \mathbb{N}$, then there exists a subsequence $\{j_{n_\iota}\}$ of $\{j_n\}$ such that $\alpha(j_{n_\iota},j_{n_\iota+1}) \geq 1$ and $\beta(j_{n_\iota},j_{n_\iota+1}) \geq 1$, $\forall\ \iota \in \mathbb{N}$. Also $\alpha(j^*,Ej^*) \geq 1$ and $\beta(j^*,Fj^*) \geq 1$.*

3. Main result

In this section, first we introduce (α,β)-Z_b-Geraghty type contraction and then prove our main result.

Definition 3.1. *Let (V,ω) be a b-metric-like space, $\alpha,\beta : V \times V \to [0,\infty)$ be two functions and $E,F : V \to V$ are two mappings. We call the pair of mappings (E,F) is (α,β)-Z_b-Geraghty type generalized contraction if for $\zeta_b^* \in \mathcal{Z}_b$, we have*

$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk), \eta(M(j,k))M(j,k)\big) \geq 0,\ \forall\ j,k \in V, \tag{3.1}$$

where $\eta : [0,\infty) \to [0,1)$ is a Geraghty function and

$$M(j,k) = \max\Big\{\omega(j,k), \omega(j,Ej), \omega(k,Fk), \frac{\omega(j,Fk) + \omega(k,Ej)}{4s}\Big\}. \tag{3.2}$$

Theorem 3.1. *Let (V,ω) be a complete b-metric-like space, $\alpha,\beta : V \times V \to [0,\infty)$ be two functions and $E,F : V \to V$ be two mappings with the following assumptions:*

(i) *(E,F) is a pair of (α,β)-admissible mappings;*
(ii) *(E,F) is a pair of (α,β)-Z_b-Geraghty type generalized contraction mappings;*
(iii) *there exists $j_0 \in V$ such that $\alpha(j_0,Ej_0) \geq 1$ and $\beta(j_0,Ej_0) \geq 1$;*
(iv) *either E and F are continuous or V is (α,β)-regular space.*

Then E and F have a unique common fixed point in V.

Proof. From (iii) hypotheses, there exists $j_0 \in V$ such that $\alpha(j_0,Ej_0) \geq 1$ and $\beta(j_0,Ej_0) \geq 1$. We construct a sequence $\{j_n\}$ in V such that $j_{2n+1} = Ej_{2n}$ and $j_{2n+2} = Fj_{2n+1}$, $n \in \mathbb{N}_0$. Since (E,F) is a pair of (α,β)-admissible mappings, then $\alpha(j_0,j_1) \geq 1$ implies $\alpha(j_1,j_2) \geq 1$ and $\beta(j_0,j_1) \geq 1$ implies $\beta(j_1,j_2) \geq 1$. By repeating similar process, we obtain that $\alpha(j_n,j_{n+1}) \geq 1$ and $\beta(j_n,j_{n+1}) \geq 1, \forall\ n \in \mathbb{N}$.
If we put $j = j_{2n}, k = j_{2n+1}$ in (3.1), then we have

$$\zeta_b^*\big(s\alpha(j_{2n},Ej_{2n})\beta(j_{2n+1},Fj_{2n+1})\omega(Ej_{2n},Fj_{2n+1}), \eta(M(j_{2n},j_{2n+1}))M(j_{2n},j_{2n+1})\big) \geq 0.$$

From (ζ_{1b}^*), we have

$$s\alpha(j_{2n},Ej_{2n})\beta(j_{2n+1},Fj_{2n+1})\omega(Ej_{2n},Fj_{2n+1}) < \eta(M(j_{2n},j_{2n+1}))M(j_{2n},j_{2n+1}).$$

Now from above inequality, we see that

$$\begin{aligned} \omega(j_{2n+1},j_{2n+2}) = \omega(Ej_{2n},Fj_{2n+1}) &\leq s\alpha(j_{2n},Ej_{2n})\beta(j_{2n+1},Fj_{2n+1})\omega(Ej_{2n},Fj_{2n+1}) \\ &< \eta(M(j_{2n},j_{2n+1}))M(j_{2n},j_{2n+1}) \\ &\leq M(j_{2n},j_{2n+1}), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(j_{2n},j_{2n+1}) &= \max\left\{ \begin{array}{c} \omega(j_{2n},j_{2n+1}),\omega(j_{2n},Ej_{2n}),\omega(j_{2n+1},Fj_{2n+1}), \\ \dfrac{\omega(j_{2n},Fj_{2n+1})+\omega(j_{2n+1},Ej_{2n})}{4s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \omega(j_{2n},j_{2n+1}),\omega(j_{2n},j_{2n+1}),\omega(j_{2n+1},j_{2n+2}), \\ \dfrac{\omega(j_{2n},j_{2n+2})+\omega(j_{2n+1},j_{2n+1})}{4s} \end{array} \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{\omega(j_{2n},j_{2n+2})+\omega(j_{2n+1},j_{2n+1})}{4s} &\leq \frac{s\big[\omega(j_{2n},j_{2n+1})+3\omega(j_{2n+1},j_{2n+2})\big]}{4s} \\ &\leq \max\big\{\omega(j_{2n},j_{2n+1}),\omega(j_{2n+1},j_{2n+2})\big\}. \end{aligned}$$

From (3.3), we obtain that

$$\omega(j_{2n+1},j_{2n+2}) \leq M(j_{2n},j_{2n+1}) \leq \max\big\{\omega(j_{2n},j_{2n+1}),\omega(j_{2n+1},j_{2n+2})\big\}. \tag{3.4}$$

If we take $\max\big\{\omega(j_{2n},j_{2n+1}),\omega(j_{2n+1},j_{2n+2})\big\} = \omega(j_{2n+1},j_{2n+2})$, then (3.4) gives a contradiction.
Thus, we obtain

$$\omega(j_{2n+1},j_{2n+2}) < \omega(j_{2n},j_{2n+1}).$$

This implies that $\{\omega(j_n,j_{n+1})\}$ is strictly decreasing sequence of non-negative real numbers, so it converges to some $\varsigma \geq 0$. Now we claim that $\varsigma = 0$. On the contrary, suppose $\varsigma > 0$, then from (3.3), we have

$$\frac{\omega(j_{2n+1},j_{2n+2})}{\omega(j_{2n},j_{2n+1})} < \eta(\omega(j_{2n},j_{2n+1})) < 1.$$

Taking $n \to \infty$, we get $\eta(\omega(j_{2n},j_{2n+1})) \to 1$ as $n \to \infty$, which contradict the fact that $\omega(j_n,j_{n+1}) \to \varsigma > 0$. Thus, we have $\lim_{n\to\infty}\omega(j_n,j_{n+1}) = 0$.

Now, we show that $\{j_n\}$ is a Cauchy sequence that is similar to show that $\lim_{n,m\to\infty} \omega(j_n,j_m) = 0$. Consider the sequence $U_\iota = \sup\{\omega(j_n,j_m) : m \geq n \geq \iota\}$, $\forall\ \iota \in \mathbb{N}$. We observe that $U_1 \geq U_2 \geq U_3 \geq ... \geq 0$, i.e., the sequence $\{U_\iota\}$ is decreasing sequence of non-negative real numbers bounded below by 0. Hence, there exists $\varsigma \geq 0$ such that

$$\lim_{\iota\to\infty} U_\iota = \varsigma.$$

From the last expression, we see that for each $\iota \in \mathbb{N}$, there exists $m_\iota \geq n_\iota \geq \iota$ such that

$$U_\iota - \frac{1}{\iota} < \omega(j_{n_\iota},j_{m_\iota}) \leq U_\iota.$$

Taking limit as $\iota \to \infty$, by Sandwitch theorem, we get

$$\lim_{\iota\to\infty} \omega(j_{n_\iota},j_{m_\iota}) = \lim_{\iota\to\infty} U_\iota = \varsigma. \tag{3.5}$$

Now, put $j = j_{n_\iota-1}$ and $k = j_{m_\iota-1}$ in (3.1), we get

$$0 \leq \zeta_b^* \left(\begin{array}{r} s\alpha(j_{n_\iota-1},Ej_{n_\iota-1})\beta(j_{m_\iota-1},Fj_{m_\iota-1})\omega(Ej_{n_\iota-1},Fj_{m_\iota-1}), \\ \eta(M(j_{n_\iota-1},j_{m_\iota-1}))M(j_{n_\iota-1},j_{m_\iota-1}) \end{array} \right). \tag{3.6}$$

From (ζ_{1b}^*), we have

$$\begin{aligned} &s\alpha(j_{n_\iota-1},Ej_{n_\iota-1})\beta(j_{m_\iota-1},Fj_{m_\iota-1})\omega(Ej_{n_\iota-1},Fj_{m_\iota-1}) \\ &\qquad < \eta(M(j_{n_\iota-1},j_{m_\iota-1}))M(j_{n_\iota-1},j_{m_\iota-1}). \end{aligned}$$

Using above inequality, we see that

$$\begin{aligned} s\omega(j_{n_\iota}, j_{m_\iota}) &= s\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &\leq s\alpha(j_{n_\iota-1}, Ej_{n_\iota-1})\beta(j_{m_\iota-1}, Fj_{m_\iota-1})\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &< \eta(M(j_{n_\iota-1}, j_{m_\iota-1}))M(j_{n_\iota-1}, j_{m_\iota-1}) \\ &\leq M(j_{n_\iota-1}, j_{m_\iota-1}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} M(j_{n_\iota-1}, j_{m_\iota-1}) &= \max\left\{\begin{array}{c} \omega(j_{n_\iota-1}, j_{m_\iota-1}), \omega(j_{n_\iota-1}, Ej_{n_\iota-1}), \omega(j_{m_\iota-1}, Fj_{m_\iota-1}), \\ \dfrac{\omega(j_{n_\iota-1}, Fj_{m_\iota-1}) + \omega(j_{m_\iota-1}, Ej_{n_\iota-1})}{4s} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} \omega(j_{n_\iota-1}, j_{m_\iota-1}), \omega(j_{n_\iota-1}, j_{n_\iota}), \omega(j_{m_\iota-1}, j_{m_\iota}), \\ \dfrac{\omega(j_{n_\iota-1}, j_{m_\iota}) + \omega(j_{m_{k-1}}, j_{n_\iota})}{4s} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} \omega(j_{n_\iota-1}, j_{m_\iota-1}), \omega(j_{n_\iota-1}, j_{n_\iota}), \omega(j_{m_\iota-1}, j_{m_\iota}), \\ \dfrac{s[\omega(j_{n_\iota-1}, j_{n_\iota}) + \omega(j_{n_\iota}, j_{m_\iota})] + s[\omega(j_{m_\iota-1}, j_{m_\iota}) + \omega(j_{m_\iota}, j_{n_\iota})]}{4s} \end{array}\right\}. \end{aligned}$$

Now, taking limit as $\iota \to \infty$ in (3.7), we get

$$s \lim_{\iota\to\infty} \omega(j_{n_\iota}, j_{m_\iota}) \leq \lim_{\iota\to\infty} M(j_{n_\iota-1}, j_{m_\iota-1}) \leq \lim_{\iota\to\infty} \max\left\{\omega(j_{n_\iota-1}, j_{m_\iota-1}), \frac{\omega(j_{n_\iota}, j_{m_\iota})}{2}\right\}.$$

If we take $\max\left\{\omega(j_{n_\iota-1}, j_{m_\iota-1}), \frac{\omega(j_{n_\iota}, j_{m_\iota})}{2}\right\} = \frac{\omega(j_{n_\iota}, j_{m_\iota})}{2}$, then above inequality leads to a contradiction. Hence, we have $\max\left\{\omega(j_{n_\iota-1}, j_{m_\iota-1}), \frac{\omega(j_{n_\iota}, j_{m_\iota})}{2}\right\} = \omega(j_{n_\iota-1}, j_{m_\iota-1})$.
From (3.7), we have

$$\begin{aligned} s\omega(j_{n_\iota}, j_{m_\iota}) &= s\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &\leq s\alpha(j_{n_\iota-1}, Ej_{n_\iota-1})\beta(j_{m_\iota-1}, Fj_{m_\iota-1})\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &< \eta(M(j_{n_\iota-1}, j_{m_\iota-1}))M(j_{n_\iota-1}, j_{m_\iota-1}) \\ &\leq \omega(j_{n_\iota-1}, j_{m_\iota-1}) \leq U_{\iota-1}. \end{aligned}$$

Again taking limit and using (3.5), we get

$$\begin{aligned} s\varsigma &= \lim_{\iota\to\infty} s\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &\leq \lim_{\iota\to\infty} s\alpha(j_{n_\iota-1}, Ej_{n_\iota-1})\beta(j_{m_\iota-1}, Fj_{m_\iota-1})\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) \\ &< \lim_{\iota\to\infty} \eta(M(j_{n_\iota-1}, j_{m_\iota-1}))M(j_{n_\iota-1}, j_{m_\iota-1}) \\ &\leq \lim_{\iota\to\infty} U_{\iota-1} = \varsigma. \end{aligned} \tag{3.8}$$

Now, here two case arises. Firstly, if $s > 1$ then above inequality implies that $\varsigma = 0$. Secondly, if $s = 1$, then using Sandwitch theorem, inequality (3.8) implies that

$$\lim_{\iota\to\infty} \alpha(j_{n_\iota-1}, Ej_{n_\iota-1})\beta(j_{m_\iota-1}, Fj_{m_\iota-1})\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1}) = \varsigma \tag{3.9}$$

and

$$\lim_{\iota\to\infty} \eta(M(j_{n_\iota-1}, j_{m_\iota-1}))M(j_{n_\iota-1}, j_{m_\iota-1}) = \varsigma. \tag{3.10}$$

Now, if we take $a_n = \alpha(j_{n_\iota-1}, Ej_{n_\iota-1})\beta(j_{m_\iota-1}, Fj_{m_\iota-1})\omega(Ej_{n_\iota-1}, Fj_{m_\iota-1})$ and $b_n = \eta(M(j_{n_\iota-1}, j_{m_\iota-1}))M(j_{n_\iota-1}, j_{m_\iota-1})$ then from (3.9) and (3.10) and using (ζ^*_{2b}), we get

$$\limsup_{n\to\infty} \zeta^*_b(sa_n, b_n) < 0,$$

which leads to a contradiction due to equation (3.6). Therefore, $\varsigma = 0$ and this shows that $\{j_n\}$ is a Cauchy sequence in V. Since V is complete, there exists $j^* \in V$ such that $j_n \to j^*$ i.e., $j_{n_\iota+1} \to j^*$ and $j_{n_\iota+2} \to j^*$ and

$$\lim_{n\to\infty} \omega(j_n, j^*) = \omega(j^*, j^*) = \lim_{m,n\to\infty} \omega(j_n, j_m) = 0. \tag{3.11}$$

Now, we shall show that $j^* = Fj^* = Ej^*$.

By hypotheses (iv), First we assume that E and F are continuous, then using (3.11), we have

$$\lim_{\iota\to\infty} \omega(j_{2\iota+1}, j^*) = \lim_{\iota\to\infty} \omega(Ej_{2\iota}, j^*) = \omega(Ej^*, j^*) = 0.$$

Similarly,

$$\lim_{\iota\to\infty} \omega(j_{2\iota+2}, j^*) = \lim_{\iota\to\infty} \omega(Fj_{2\iota+1}, j^*) = \omega(Fj^*, j^*) = 0.$$

This implies that $Ej^* = Fj^* = j^*$. Hence, the pair (E, F) has a common fixed point $j^* \in V$.

Now, consider that V is (α, β)-regular space then there exists a subsequence $\{j_{n_\iota}\}$ of $\{j_n\}$ such that $\alpha(j_{n_\iota}, j_{n_\iota+1}) \geq 1$ and $\beta(j_{n_\iota}, j_{n_\iota+1}) \geq 1$ for each $\iota \in \mathbb{N}$ and $\alpha(j^*, Ej^*) \geq 1$ and $\beta(j^*, Fj^*) \geq 1$.

Putting $j = j_{n_\iota}, k = j^*$ in (3.1), we get

$$\zeta^*_b\big(s\alpha(j_{n_\iota}, Ej_{n_\iota})\beta(j^*, Fj^*)\omega(Ej_{n_\iota}, Fj^*), \eta(M(j_{n_\iota}, j^*))M(j_{n_\iota}, j^*)\big) \geq 0.$$

From (ζ^*_{2b}), we get

$$s\alpha(j_{n_\iota}, Ej_{n_\iota})\beta(j^*, Fj^*)\omega(Ej_{n_\iota}, Fj^*) < \eta(M(j_{n_\iota}, j^*))M(j_{n_\iota}, j^*).$$

This further implies that

$$\begin{aligned}
\omega(j_{n_\iota+1}, Fj^*) = \omega(Ej_{n_\iota}, Fj^*) &\leq s\alpha(j_{n_\iota}, Ej_{n_\iota})\beta(j^*, Fj^*)\omega(Ej_{n_\iota}, Fj^*) \\
&< \eta(M(j_{n_\iota}, j^*))M(j_{n_\iota}, j^*) \\
&\leq M(j_{n_\iota}, j^*),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
M(j_{n_\iota}, j^*) &= \max\left\{\omega(j_{n_\iota}, j^*), \omega(j_{n_\iota}, Ej_{n_\iota}), \omega(j^*, Fj^*), \frac{\omega(j_{n_\iota}, Fj^*) + \omega(j^*, Ej_{n_\iota})}{4s}\right\} \\
&= \max\left\{\omega(j_{n_\iota}, j^*), \omega(j_{n_\iota}, j_{n_\iota+1}), \omega(j^*, Fj^*), \frac{\omega(j_{n_\iota}, Fj^*) + \omega(j^*, j_{n_\iota+1})}{4s}\right\}.
\end{aligned}$$

Taking limit as $\iota \to \infty$ in above expression, we get

$$\lim_{\iota\to\infty} M(j_{n_\iota}, j^*) = \max\left\{\omega(j^*, Fj^*), \frac{\omega(j^*, Fj^*)}{4s}\right\} = \omega(j^*, Fj^*). \tag{3.13}$$

Therefore, taking limit as $\iota \to \infty$ in (3.12) and using (3.13), we get

$$\lim_{\iota\to\infty} \omega(j_{n_\iota+1}, Fj^*) = \omega(j^*, Fj^*) \leq \lim_{\iota\to\infty} \eta(M(j_{n_\iota}, j^*))M(j_{n_\iota}, j^*) \leq \omega(j^*, Fj^*).$$

This implies that $\lim_{\iota\to\infty} \eta(M(j_{n_\iota}, j^*)) = 1$ and therefore, $\lim_{\iota\to\infty} M(j_{n_\iota}, j^*) = 0$. Thus, we obtain $\omega(j^*, Fj^*) = 0$ i.e., $j^* = Fj^*$. Similarly, we get $j^* = Ej^*$ and we conclude that

j^* is a common fixed point of E and F.
Now, for the uniqueness part, let j^* and k^* are two common fixed points of E and F and $j^* \neq k^*$. Also $\alpha(j^*, Ej^*) \geq 1, \alpha(k^*, Ek^*) \geq 1$ and $\beta(j^*, Fj^*) \geq 1, \beta(k^*, Fk^*) \geq 1$. By (3.1), we have

$$0 \leq \zeta_b^*\big(s\alpha(j^*, Ej^*)\beta(k^*, Fk^*)\omega(Ej^*, Fk^*), \eta(M(j^*, k^*))M(j^*, k^*)\big).$$

From (ζ_{2b}^*), we get

$$s\alpha(j^*, Ej^*)\beta(k^*, Fk^*)\omega(Ej^*, Fk^*) < \eta(M(j^*, k^*))M(j^*, k^*), \tag{3.14}$$

where

$$\begin{aligned} M(j^*, k^*) &= \max\left\{\omega(j^*, k^*), \omega(j^*, Ej^*), \omega(k^*, Fk^*), \frac{\omega(j^*, Fk^*) + \omega(k^*, Ej^*)}{4s}\right\} \\ &= \omega(j^*, k^*). \end{aligned}$$

Thus, by using (3.14), we have

$$\begin{aligned} \omega(j^*, k^*) = \omega(Ej^*, Fk^*) &\leq s\alpha(j^*, Ej^*)\beta(k^*, Fk^*) \\ &< \eta(M(j^*, k^*))M(j^*, k^*) \\ &\leq M(j^*, k^*) = \omega(j^*, k^*), \end{aligned}$$

which is a contradiction. This shows that $j^* = k^*$ and this completes the proof. □

We illustrate Theorem 3.1 by the following example.

Example 3.1. *Let $V = \{0, 1, 2\}$ and a b-metric-like is defined on V with the values given as $\omega(0,0) = 0$, $\omega(1,1) = \frac{3}{4}$, $\omega(2,2) = 3$, $\omega(1,0) = \omega(0,1) = \frac{1}{4}$, $\omega(0,2) = \omega(2,0) = 1$, $\omega(1,2) = \omega(2,1) = 2$. Clearly, (V, ω) is a complete b-metric-like space with coefficient $s = \frac{8}{5}$. Note that $\omega(2,2) \neq 0$, so ω is not a b-metric. Also $\omega(1,2) \nleq \omega(1,0) + \omega(0,2)$, so ω is not a metric-like.*

Let $\zeta_b^* : V \times V \to \mathbb{R}$ defined by $\zeta_b^*(j,k) = \frac{3}{4}k - j$. We define mappings $E, F : V \to V$ and Geraghty function $\eta : [0,\infty) \to [0,1)$ as follows:

$$Ej = \begin{cases} 0, & j \in \{0,1\}, \\ 2, & j = 2, \end{cases} \qquad Fj = \begin{cases} 0, & j \in \{0,2\}, \\ 1, & j = 1, \end{cases} \qquad \text{and} \quad \eta(j) = \frac{5}{6}.$$

Also, we define $\alpha, \beta : V \times V \to [0, \infty]$ by

$$\alpha(j,k) = \beta(j,k) = \begin{cases} 0, & \text{if } (j,k) \in \{(1,2),(2,1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Now, for $\alpha(j,k) \geq 1$ and $\beta(j,k) \geq 1$, we have $(j,k) \in \big\{(0,0),(1,1),(2,2),(0,2),(2,0),(1,0),(0,1)\big\}$, then it follows that $\alpha(Ej,Fk) \geq 1$, $\alpha(Fj,Ek) \geq 1$ and $\beta(Ej,Fk) \geq 1, \beta(Fj,Ek) \geq 1$. Therefore, (E,F) is a pair of (α,β)-admissible mapping. Also, for all $j \in V$, we have $\alpha(j,Ej) = 1$ and $\beta(j,Fj) = 1$.
Now, from inequality (3.1)

$$\begin{aligned} 0 &\leq \zeta_b^*\big(s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk), \eta(M(j,k))M(j,k)\big) \\ &= \frac{3}{4}\eta(M(j,k))M(j,k) - s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk), \end{aligned}$$

which implies that

$$\frac{8}{5}\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk) \leq \frac{3}{4}\eta(M(j,k))M(j,k) = \frac{5}{8}M(j,k).$$

Now, we consider the following cases.

1. *If $j=0$ and $k=0$, then*
$$\frac{8}{5}\alpha(0,E0)\beta(0,F0)\omega(E0,F0)=0\leq\frac{5}{8}M(0,0)=0.$$
2. *If $j=1$ and $k=1$, then*
$$\frac{8}{5}\alpha(1,E1)\beta(1,F1)\omega(E1,F1)=\frac{2}{5}\leq\frac{5}{8}M(1,1)=\frac{15}{32}.$$
3. *If $j=2$ and $k=2$, then*
$$\frac{8}{5}\alpha(2,E2)\beta(2,F2)\omega(E2,F2)=\frac{8}{5}\leq\frac{5}{8}M(2,2)=\frac{15}{8}.$$
4. *If $j=0$ and $k=1$, then*
$$\frac{8}{5}\alpha(0,E0)\beta(1,F1)\omega(E0,F1)=\frac{2}{5}\leq\frac{5}{8}M(0,1)=\frac{15}{32}.$$
5. *If $j=1$ and $k=0$, then*
$$\frac{8}{5}\alpha(1,E1)\beta(0,F0)\omega(E1,F0)=0\leq\frac{5}{8}M(1,0)=\frac{5}{32}.$$
6. *If $j=0$ and $k=2$, then*
$$\frac{8}{5}\alpha(0,E0)\beta(2,F2)\omega(E0,F2)=0\leq\frac{5}{8}M(0,2)=\frac{5}{8}.$$
7. *If $j=2$ and $k=0$, then*
$$\frac{8}{5}\alpha(2,E2)\beta(0,F0)\omega(E2,F0)=\frac{8}{5}\leq\frac{5}{8}M(2,0)=\frac{15}{8}.$$

Thus, all conditions of Theorem 3.1 are satisfied and therefore E and F have a unique common fixed point (namely, $j=0$) in V.

Now, we state some corollaries of Theorem 3.1.

Corollary 3.1. *Let (V,ω) be a complete b-metric-like space, $\alpha,\beta: V\times V\to[0,\infty)$ be two functions and $E,F:V\to V$ be two mappings with the following assumptions:*

(i) *(E,F) is a pair of (α,β)-admissible mappings;*

(ii) *there exist $j,k\in V$ such that*
$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk),\eta(\omega(j,k))\omega(j,k)\big)\geq 0,$$
where η is a Geraghty function and $\zeta_b^*\in Z_b$;

(iii) *there exists $j_0\in V$ such that $\alpha(j_0,Ej_0)\geq 1$ and $\beta(j_0,Ej_0)\geq 1$;*

(iv) *either E and F are continuous or V is (α,β)-regular space.*

Then E and F have a unique common fixed point in V.

Proof. The proof follows from Theorem 3.1 by taking $M(j,k)=\omega(j,k)$. □

We illustrate Corollary 3.1 by the following example.

Example 3.2. *Let $V=[0,\infty)$ and $\omega: V\times V\to[0,\infty)$ be defined by*
$$\omega(j,k)=(j+k)^2.$$
Clearly, (V,ω) is a complete b-metric-like space with coefficient $s=2$. Let $\zeta_b^*: V\times V\to\mathbb{R}$ defined by $\zeta_b^*(j,k)=\frac{1}{2}k-j$. We define mappings $E,F:V\to V$ and Geraghty function $\eta:[0,\infty)\to[0,1)$ as follows:
$$Ej=\frac{j}{20},\ Fj=\frac{j}{50},\ \forall\ j,k\in V \text{ and } \eta(j)=\begin{cases}\frac{1}{1+j}, & j>0,\\ \frac{1}{2}, & j=0.\end{cases}$$

Also, we define $\alpha,\beta : V\times V \to [0,\infty]$ *by*

$$\alpha(j,k)=\beta(j,k)=\begin{cases}1, & \text{if } j,k\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

Now, for $\alpha(j,k)\ge 1$ *and* $\beta(j,k)\ge 1$*, we have* $j,k\in[0,1]$*, then it follows that* $\alpha(Ej,Fk)\ge 1,\alpha(Fj,Ek)\ge 1$ *and* $\beta(Ej,Fk)\ge 1,\beta(Fj,Ek)\ge 1$*. Therefore,* (E,F) *is a pair of* (α,β)*-admissible mapping. Furthermore, if* $\{j_n\}$ *is a sequence in* V *such that* $\alpha(j_n,j_{n+1})\ge 1,\beta(j_n,j_{n+1})\ge 1$*, then* $j_n\subseteq[0,1]$*. Suppose* $j_n\to j^*$ *then* $j^*\in[0,1]$ *and this implies that* $\alpha(j^*,Ej^*)=\beta(j^*,Fj^*)=1$*.*
Now, for $j,k\in[0,1]$*, we have*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk)=2\omega(Ej,Fk)=2\left[\frac{j}{20}+\frac{k}{50}\right]^2=\frac{(5j+2k)^2}{5000}.$$

When $j,k>0$ *or* $j=0,k>0$ *or* $j>0,k=0$*, in each cases we have*

$$\eta(\omega(j,k))\omega(j,k)=\frac{\omega(j,k)}{1+\omega(j,k)}=\frac{(j+k)^2}{1+(j+k)^2}.$$

For $j=k=0$*, we have*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk)=\eta(\omega(j,k))\omega(j,k)=0.$$

So, for all $j,k\in V$*, we get*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk)\le\frac{1}{2}\eta(\omega(j,k))\omega(j,k),$$

and this shows that the contraction condition

$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk),\eta(\omega(j,k))\omega(j,k)\big)\ge 0,$$

is satisfied for all $j,k\in V$*. Hence, all conditions of Corollary 3.1 are satisfied and therefore* E *and* F *have a unique common fixed point (namely,* $j=0$*) in* V*.*

Corollary 3.2. *Let* (V,ω) *be a complete b-metric-like space,* $\alpha,\beta:V\times V\to[0,\infty)$ *be two functions and assuming the mapping* $E:V\to V$ *with the following assumptions:*

(i) E *is* (α,β)*-admissible mapping;*
(ii) *there exist* $j,k\in V$ *such that*

$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek),\eta(M'(j,k))M'(j,k)\big)\ge 0, \tag{3.15}$$

where $M'(j,k)=\max\left\{\omega(j,k),\omega(j,Ej),\omega(k,Ek),\dfrac{\omega(j,Ek)+\omega(k,Ej)}{4s}\right\}$ *and* η *is a Geraghty function and* $\zeta_b^*\in Z_b$*;*

(iii) *there exists* $j_0\in V$ *such that* $\alpha(j_0,Ej_0)\ge 1$ *and* $\beta(j_0,Ej_0)\ge 1$*;*
(iv) *either* E *is continuous or* V *is* (α,β)*-regular space.*

Then E *has a unique fixed point in* V*.*

Proof. The proof follows from Theorem 3.1 by taking $E=F$. □

Example 3.3. *Let* $V=\{1,2,3\}$ *and a b-metric-like is defined on* V *with the values given as* $\omega(1,1)=\frac{1}{2},\ \omega(2,2)=0,\ \omega(3,3)=2,\ \omega(1,2)=\omega(2,1)=1,\ \omega(1,3)=\omega(3,1)=4,\ \omega(2,3)=\omega(3,2)=2$*. Clearly,* (V,ω) *is a complete b-metric-like space with coefficient* $s=\frac{4}{3}$*.*

Let $\zeta_b^* : V \times V \to \mathbb{R}$ be a b-simulation function defined by $\zeta_b^*(j,k) = \frac{1}{2}k - j$. A mapping $E : V \to V$ and Geraghty function $\eta : [0,\infty) \to [0,1)$ are defined as follows:

$$Ej = \begin{cases} 2, & j \in \{1,2\}, \\ 1, & j = 3, \end{cases} \quad \text{and} \quad \eta(j) = \frac{4}{5}.$$

Also, we define $\alpha, \beta : V \times V \to [0,\infty]$ by

$$\alpha(j,k) = \beta(j,k) = 1 \text{ for all } (j,k) \in V \times V.$$

Now, from inequality (3.15)

$$\begin{aligned} 0 &\leq \zeta_b^*\big(s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek), \eta(M'(j,k))M'(j,k)\big) \\ &= \frac{1}{2}\eta(M'(j,k))M'(j,k) - s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek), \end{aligned}$$

which implies that

$$s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek) = \frac{4}{3}\omega(Ej,Ek) \leq \frac{1}{2}\eta(M'(j,k))M'(j,k) = \frac{2}{5}M'(j,k).$$

Now, we consider the following cases.

1. *If $j = 1$ and $k = 1$, then*
$$\frac{4}{3}\omega(E1,E1) = 0 \leq \frac{2}{5}M'(1,1) = \frac{2}{5}.$$
2. *If $j = 2$ and $k = 2$, then*
$$\frac{4}{3}\omega(E2,E2) = 0 \leq \frac{2}{5}M'(2,2) = 0.$$
3. *If $j = 3$ and $k = 3$, then*
$$\frac{4}{3}\omega(E3,E3) = \frac{2}{3} \leq \frac{2}{5}M'(3,3) = \frac{8}{5}.$$
4. *If $j = 1$ and $k = 2$ (or $j = 2$ and $k = 1$), then*
$$\frac{4}{3}\omega(E1,E2) = 0 \leq \frac{2}{5}M'(1,2) = \frac{2}{5}.$$
5. *If $j = 1$ and $k = 3$ (or $j = 3$ and $k = 1$), then*
$$\frac{4}{3}\omega(E1,E3) = \frac{4}{3} \leq \frac{2}{5}M'(1,3) = \frac{8}{5}.$$
6. *If $j = 2$ and $k = 3$ (or $j = 3$ and $k = 2$), then*
$$\frac{4}{3}\omega(E2,E3) = \frac{4}{3} \leq \frac{2}{5}M'(2,3) = \frac{8}{5}.$$

Thus, all conditions of Corollary 3.2 are satisfied and therefore E has a unique fixed point (namely, $j = 2$) in V.

Corollary 3.3. *Let (V,ω) be a complete b-metric-like space, $\alpha, \beta : V \times V \to [0,\infty)$ be two functions and assuming the mapping $E : V \to V$ with the following assumptions:*

(i) *E is (α,β)-admissible mapping;*

(ii) *there exist $j, k \in V$ such that*
$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek), \eta(\omega(j,k))\omega(j,k)\big) \geq 0, \tag{3.16}$$
where η is a Geraghty function and $\zeta_b^* \in Z_b$;

(iii) *there exists $j_0 \in V$ such that $\alpha(j_0,Ej_0) \geq 1$ and $\beta(j_0,Ej_0) \geq 1$;*

(iv) *either E is continuous or V is (α,β)-regular space.*

Then E has a unique fixed point in V.

Proof. The proof follows from Theorem 3.1 by taking $E = F$ and $M(j,k) = \omega(j,k)$. □

Remark 3.1. *Note that, at point $j = 2, k = 3$ in Example 3.3, the contraction condition (3.16) is not satisfied, i.e.,*

$$\frac{4}{3}\omega(E2, E3) = \frac{4}{3} \nleq \frac{2}{5}\omega(2,3) = \frac{4}{5}.$$

Thus, Corollary 3.2 is a proper extension of Corollary 3.3.

Remark 3.2. *If we assume usual metric $d(j,k) = |j-k|$ instead of b-metric-like, $M'(j,k) = \omega(j,k)$ and $s = 1$ in Example 3.3, then at point $j = 2, k = 3$,*

$$0 \leq \zeta\big(d(E2,E3), d(2,3)\big) = \frac{1}{2}d(2,3) - d(E2,E3) < 0,$$

which is a contradiction. Thus, it is not a Z-contraction of Khojasteh et al. [20].

Remark 3.3. *If we take b-metric instead of b-metric-like with the values given as $b(0,0) = b(1,1) = b(2,2) = 0,\ b(1,2) = b(2,1) = 1,\ b(1,3) = b(3,1) = 4,\ b(2,3) = b(3,2) = 2$ and $M'(j,k) = \omega(j,k)$ in Example 3.3, then at point $j = 2, k = 3$, the contraction condition (2.1) is not satisfied, i.e.,*

$$0 \leq \zeta_b^*\big(sb(E2,E3), b(2,3)\big) = \frac{1}{2}b(2,3) - \frac{4}{3}b(E2,E3) = 1 - \frac{4}{3} < 0.$$

Thus, it is not a b-simulation function of Demma et al. [15].

Corollary 3.4. *If in Theorem 3.1 we replace condition (ii) by the following condition:*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk) \leq \tau\eta(M(j,k))M(j,k), \text{ where } \tau \in (0,1), \tag{3.17}$$

where $M(j,k)$ is defined as in (3.2), η is a Geraghty function. Then, E and F have a unique common fixed point in V.

Proof. The proof follows from Theorem 3.1 by taking $\zeta_b^*(j,k) = \tau k - j$ for all $j,k \geq 0$, where $\tau \in (0,1)$. □

Remark 3.4. *Taking $\eta(j) = h,\ h \in [0,1)$ and $E = F$, $M(j,k) = \omega(j,k)$ with $\alpha(j,Ej) = \beta(k,Fk) = 1$ in Corollary 3.4 and assume partial metric p instead of b-metric-like then (3.17) reduces to the following:*

$$p(Ej,Ek) \leq \lambda\, p(j,k), \text{ where } \lambda = \tau h \in [0,1), \tag{3.18}$$

which is a variant of Banach contraction in partial metric space. Thus, Corollary 3.4 generalize the result of Matthews [21]. *To see this, if we take partial metric given by $p(j,k) = \max\{j,k\}$ in Example 3.3, then contraction condition (3.18) is not satisfied at point $j = 1, k = 2$, i.e.,*

$$p(E1,E2) = p(2,2) = 2 \nleq \lambda p(1,2) = 2\lambda.$$

Corollary 3.5. *If in Theorem 3.1 we replace condition (ii) by the following condition:*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk) \leq \eta(M(j,k))M(j,k) - \mu\big(\eta(M(j,k))M(j,k)\big),$$

where $M(j,k)$ is defined as in (3.2), η is a Geraghty function and $\mu : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\mu^{-1}(0) = \{0\}$. Then, E and F have a unique common fixed point in V.

Proof. The result follows from Theorem 3.1, by taking $\zeta_b^*(j,k) = k - \mu(k) - j$ for all $j,k \geq 0$. □

Corollary 3.6. *If in Theorem 3.1 we replace condition (ii) by the following condition:*

$$s\alpha(j,Ej)\beta(k,Fk)\omega(Ej,Fk) \le \rho\big(\eta(M(j,k))M(j,k)\big),$$

where $M(j,k)$ is defined as in (3.2), η is a Geraghty function and $\rho : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function with $\rho(j) < j$ for all $j > 0$ and $\rho(0) = 0$. Then, E and F have a unique common fixed point in V.

Proof. The result follows from Theorem 3.1, by taking $\zeta_b^*(j,k) = \rho(k) - j$ for all $j,k \ge 0$. □

4. Application

Suppose, $V = C([0,1])$ be the set of all continuous function defined on $[0,1]$ and $\omega : V \times V \to \mathbb{R}$ be a mapping. Let V be endowed with a b-metric-like defined by

$$\omega(j,k) = \big(|j(t)| + |k(t)|\big)^q, \ \forall\, j,k \in V, \ q > 1. \tag{4.1}$$

Obviously, (V,ω) is a complete b-metric-like space with constant $s = 2^{q-1}$.

Let us consider the two-point boundary value problem of the second order differential equation:

$$\frac{d^2 j}{dt^2} = f(t, j(t)), \ t \in [0,1]; \tag{4.2}$$
$$j(0) = j(1) = 1,$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

The inequality (4.2) is equivalent to the following integral equation:

$$j(t) = \int_0^1 G(t,u) f(u, j(u))\, du, \ t \in [0,1],$$

where Green function associated to (4.2) is defined by

$$G(t,u) = \begin{cases} t(1-u), & \text{if } 0 \le t \le u \le 1, \\ u(1-t), & \text{if } 0 \le u \le t \le 1. \end{cases}$$

Assume that the following conditions hold:

(i) there exist functions $\xi, \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that
$$|f(t,a)| + |f(t,b)| \le |a| + |b|,$$
for all $t \in [0,1]$ and $a,b \in \mathbb{R}$ with $\xi(a,b) > 0$ and $\phi(a,b) > 0$;

(ii) there exists $j_0 \in C[0,1]$ such that $\xi(j_0(t), Ej_0(t) \ge 0$ and $\phi(j_0(t), Ej_0(t)) \ge 0$, $\forall\, t \in [0,1]$ where $E : C[0,1] \to C[0,1]$ is defined by
$$Ej(t) = \int_0^1 G(t,u) f(j, j(u))\, du;$$

(iii) for each $t \in [0,1]$ and $j,k \in C[0,1]$, $\xi(j(t),k(t)) > 0$ implies $\xi(Ej(t), Ek(t)) > 0$ and $\phi(j(t),k(t)) > 0$ implies $\phi(Ej(t), Ek(t)) > 0$;

(iv) for each $t \in [0,1]$, if $\{j_n\}$ is a sequence in $C[0,1]$ such that $j_n \to j^*$ in $C[0,1]$ and $\xi(j_n(t), j_{n+1}(t)) > 0$ and $\phi(j_n(t), j_{n+1}(t)) > 0$, $\forall\, n \in \mathbb{N}$, then $\xi(j_n(t), j^*(t)) > 0$ and $\phi(j_n(t), j^*(t)) > 0$, $\forall\, n \in \mathbb{N}$.

Now, we prove the existence of solution of second order differential equation.

Theorem 4.1. *Under the assumption (i)-(iv), equation* (4.2) *has a solution in* $C^2([0,1])$.

Proof. Define a mapping $E : V \to V$ by

$$Ej(t) = \int_0^1 G(t,u)f(u,j(u))\ du,\ t \in [0,1].$$

Now, for all $j,k \in C[0,1]$ such that $\xi(j(t),k(t)) \geq 0$ and $\phi(j(t),k(t)) \geq 0,\ \forall\ t \in [0,1]$. We have

$$\begin{aligned}
\big(|E(j(t))| + |E(k(t))|\big)^q &= \Big(\Big|\int_0^1 G(t,u)f(u,j(u))\ du\Big| + \Big|\int_0^1 G(t,u)f(u,k(u))\ du\Big| \Big)^q \\
&\leq \Big(\int_0^1 \big|G(t,u)f(u,j(u))\big|\ du + \int_0^1 \big|G(t,u)f(u,k(u))\big|\ du \Big)^q \\
&= \Big(\int_0^1 G(t,u)\big(\big|f(u,j(u))\big| + \big|f(u,k(u))\big|\big)\ du \Big)^q \\
&= \Big(\int_0^1 G(t,u)\big(\big|j(u)\big| + \big|k(u)\big|\big)\ du \Big)^q \\
&\leq \sup_{t\in[0,1]} \Big(\int_0^1 G(t,u)\Big[\big(\big|j(u)\big| + \big|k(u)\big|\big)^q\Big]^{\frac{1}{q}}\ du \Big)^q \\
&\leq \omega(j(t),k(t))\Big[\sup_{t\in[0,1]} \Big(\int_0^1 G(t,u)\ du\Big)^q \Big].
\end{aligned}$$

Since, $\int_0^1 G(t,u)\ du = -\frac{t^2}{2} + \frac{t}{2},\ \forall\ t \in [0,1]$, we have $\sup_{t\in[0,1]} \Big(\int_0^1 G(t,u)\ du\Big)^q = \Big(\frac{1}{8}\Big)^q$, then it follows that

$$\omega(Ej,Ek) = \big(|E(j(t))| + |E(k(t))|\big)^q \leq \Big(\frac{1}{8}\Big)^q \omega(j(t),k(t)). \tag{4.3}$$

Let $\zeta_b^*(a,b) = \frac{3}{4}b - a, \forall\ a,b \in [0,\infty)$ and Geraghty function is defined by $\eta(t) = \frac{1}{2},\ \forall\ t \geq 0$. For $t \in [0,1]$ the following is defined:

$$\alpha(j,k) = \begin{cases} 1, & \text{if } \xi(j(t),k(t)) > 0, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta(j,k) = \begin{cases} 1, & \text{if } \phi(j(t),k(t)) > 0, \\ 0, & \text{otherwise}. \end{cases}$$

From (4.3), we have

$$s\omega(Ej,Ek) = 2^{q-1}\omega(Ej,Ek) \leq 2^{q-1}\frac{1}{8^q}\omega(j,k) \leq \frac{3}{8}\omega(j,k). \tag{4.4}$$

Now, using (4.4), we get

$$\frac{3}{4}\eta(\omega(j,k))\omega(j,k) - s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek) = \frac{3}{8}\omega(j,k) - s\omega(Ej,Ek) \geq 0.$$

Hence,

$$\zeta_b^*\big(s\alpha(j,Ej)\beta(k,Ek)\omega(Ej,Ek), \eta(\omega(j,k))\omega(j,k)\big) \geq 0.$$

Therefore the mapping E is (α,β)-Z_b-Geraghty type contraction. From (ii), there exists $j_0 \in C[0,1]$ such that $\alpha(j_0,Ej_0) \geq 1$ and $\beta(j_0,Ej_0) \geq 1$.
Now using (iii), we get

$$\begin{aligned}
\alpha(j,k) \geq 1 \quad &\Longrightarrow \quad \xi(j(t),k(t)) > 0 \\
&\Longrightarrow \quad \xi(Ej(t),Ek(t)) > 0 \\
&\Longrightarrow \quad \alpha(Ej,Ek) \geq 1.
\end{aligned}$$

Similarly,

$$\begin{aligned} \beta(j,k) \geq 1 &\implies \phi(j(t),k(t)) > 0 \\ &\implies \phi(Ej(t),Ek(t)) > 0 \\ &\implies \beta(Ej,Ek) \geq 1. \end{aligned}$$

So, the mapping E is (α,β)-admissible.
Therefore, all the hypotheses of Corollary 3.3 are satisfied. Hence, E must have a fixed point in $C[0,1]$ (say j), which is a solution of (4.2). □

5. Conclusion

Our results deals with a new class of (α,β)-Z_b-Geraghty type contraction in a wider structure such as a b-metric-like space and extends the result of Matthews [21], Khojasteh et al. [20], Demma et al. [15], Chandok [12] and others. Indeed, the two-point boundary value problem of the second order differential equation is solved using this new class of contraction condition.

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Smita Negi is currently a Ph.D student in Mathematics at BGR Campus of H. N. B. Garhwal University, Pauri Garhwal, Uttarakhand, India. Her research interests are Nonlinear analysis and fixed point theory and its applications.

Swati Antal is a research scholar in the department of Mathematics at BGR Campus of H. N. B. Garhwal University, Pauri Garhwal, Uttarakhand, India. Her research interests includes fixed point theory and its applications, and fractal geometry.

U. C. Gairola is a Professor in the Department of Mathematics, BGR Campus of H. N. B. Garhwal University, Pauri Garhwal, Uttarakhand, India.