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HYPERGEOMETRIC FUNCTION REPRESENTATION OF THE ROOTS OF A CERTAIN CUBIC EQUATION

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Abstract. The aim in this note is to obtain new hypergeometric forms for the functions

$$(\sqrt{z-1}-\sqrt{z})^b \pm (\sqrt{z-1}-\sqrt{z})^{-b}, \qquad (\sqrt{z-1}+\sqrt{z})^b \pm (\sqrt{z-1}+\sqrt{z})^{-b},$$

where b is an arbitrary parameter, in terms of Gauss hypergeometric functions. An application of these results (when $b=\frac{1}{3}$) is made to obtain the hypergeometric form of the roots of the cubic equation $r^3 - r + \frac{2}{3}\sqrt{\frac{z}{3}} = 0$. This complements the entry in the compendium of Prudnikov *et al.* on page 472, entry (68) of the table, where only the middle root (either real or purely imaginary) is given in hypergeometric form.

Keywords: Gauss hypergeometric function, Roots of a cubic, Pfaff-Kummer transformation.

AMS Subject Classification(2020): 33C05, 33C20, 33C99.

1. Introduction and preliminaries

A remarkable large number of generalized hypergeometric functions and their extensions have been given by many authors (see [3, 12]). These functions play a crucial role in mathematical analysis, engineering and applied sciences. Abdus *et al.* [9] introduced a new confluent hypergeometric gamma function and an associated confluent hypergeometric Pochhammer symbol and studied their many properties like their integral representations, derivative formulas, and generating function relations. The novel expansion of beta function has been given by Musharraf *et al.*[1] by using multi-index Mittag-Leffler function. Srivastava *et al.*[11] introduced an extended version of the Pochhammer symbol and then introduced the corresponding extension of the τ-Gauss hypergeometric function and studied their basic properties. Srivastava *et al.*[10] also gave the extended Pochhammer symbol by using extended gamma function involving Macdonald function. Further

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the authors gave the corresponding extension of the generalized hypergeometric function. The families of generating functions and generating relations for the extended generalized hypergeometric function have also been presented.

In the book by Prudnikov *et al.* [4, p. 472, entry(68)] the middle root (either real or purely imaginary) of the cubic equation

$$r^3 - r + \frac{2}{3}\sqrt{\frac{z}{3}} = 0 \tag{1.1}$$

is given in hypergeometric form by

$$\frac{2}{3}\sqrt{\frac{z}{3}}\,{}_2F_1[\begin{matrix}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{matrix};z]. \tag{1.2}$$

Here, ${}_2F_1(z)$ denotes the Gauss hypergeometric function defined in terms of the Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ by

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\,z^n \qquad (|z|<1)$$

and elsewhere in the complex z-plane cut along $[1,\infty)$ by analytic continuation. It is the aim in this note to give hypergeometric representations for the other two roots of (1.1), following recently published work in [2], [5]–[8]. This is achieved by obtaining similar hypergeometric forms for the functions

$$(\sqrt{z-1}-\sqrt{z})^b \pm (\sqrt{z-1}-\sqrt{z})^{-b}, \qquad (\sqrt{z-1}+\sqrt{z})^b \pm (\sqrt{z-1}+\sqrt{z})^{-b},$$

where b is an arbitrary parameter, thereby complementing those expressions given in [4, p. 486].

From *Mathematica*, the roots of the cubic (1.1) can be expressed in the form

$$r_{1,2}(z) = -\frac{1}{2\sqrt{3}}\left\{(\sqrt{z-1}-\sqrt{z})^{1/3} + (\sqrt{z-1}-\sqrt{z})^{-1/3}\right\}$$
$$\pm\frac{i}{2}\left\{(\sqrt{z-1}-\sqrt{z})^{1/3} - (\sqrt{z-1}-\sqrt{z})^{-1/3}\right\}, \tag{1.3}$$

$$r_3(z) = \frac{1}{\sqrt{3}}\left\{(\sqrt{z-1}-\sqrt{z})^{1/3} + (\sqrt{z-1}-\sqrt{z})^{-1/3}\right\}. \tag{1.4}$$

Elementary considerations show that when $0 \le z \le 1$ the roots are real; when $z > 1$ there is one real root and a complex conjugate pair; when $z < 0$ all three roots are complex with $r_2(z)$ being purely imaginary; see Table 1. When z is complex, it is sufficient to consider only the upper half plane since the roots satisfy $r_j(\overline{z}) = \overline{r_j(z)}$ $(1 \le j \le 3)$, where the bar denotes the complex conjugate.

In what follows we shall make use of the binomial theorem

$$(1-z)^{-b} = {}_1F_0\left[\begin{matrix}b\\ -\end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(b)_n}{n!}\,z^n, \tag{1.5}$$

where $|z| < 1$ and $b \in \mathbf{C}$, and $|z| = 1$ when $\Re(b) < 0$. We shall also require the Pfaff-Kummer transformation

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right] = (1-z)^{-a}{}_2F_1\left[\begin{matrix}a,c-b\\ c\end{matrix};\frac{z}{z-1}\right] \tag{1.6}$$

and Euler's linear transformation

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right] = (1-z)^{c-a-b}{}_2F_1\left[\begin{matrix}c-a,c-b\\ c\end{matrix};z\right], \tag{1.7}$$

TABLE 1. Values of the roots $r_j(z)$ (accurate to 5dp) of the cubic (1.1) for $z \in [-1, 1]$.

-0.2	$-1.01046 - 0.08372i$	$0.16744i$	$1.01046 - 0.08372i$
-0.4	$-1.01983 - 0.11555i$	$0.23109i$	$1.01983 - 0.11555i$
-0.6	$-1.02835 - 0.13846i$	$0.27691i$	$1.02835 - 0.13846i$
-0.8	$-1.03619 - 0.15673i$	$0.31346i$	$1.03619 - 0.15673i$
-1.0	$-1.04347 - 0.17207i$	$0.34414i$	$1.04347 - 0.17207i$
0	-1.00000	0.00000	1.00000
0.2	-1.07696	0.17777	0.89921
0.4	-1.10470	0.26127	0.84343
0.6	-1.12475	0.33611	0.78864
0.8	-1.14094	0.41653	0.72440
1.0	-1.15470	0.57735	0.57735

both holding provided $c \neq 0, -1, -2, \ldots$ and $|\arg(1-z)| < \pi$.

2. Hypergeometric representation of certain functional forms

We have the following theorem:

Theorem 2.1. *Let b be an arbitrary parameter and define the two functions*

$$F_1(b;z) := {}_2F_1\left[\begin{array}{c} -\frac{1}{2}b, \frac{1}{2}b \\ \\ \frac{1}{2} \end{array}; z\right], \qquad F_2(b;z) := {}_2F_1\left[\begin{array}{c} \frac{1}{2}-\frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b \\ \\ \frac{3}{2} \end{array}; z\right].$$

Then the following identities hold:

$$(\sqrt{z-1}-\sqrt{z})^b+(\sqrt{z-1}-\sqrt{z})^{-b} = 2\left\{\cos(\frac{1}{2}\pi b)F_1(b;z) - b\sqrt{z}\sin(\frac{1}{2}\pi b)F_2(b;z)\right\}, \quad (2.1)$$

$$(\sqrt{z-1}-\sqrt{z})^b-(\sqrt{z-1}-\sqrt{z})^{-b} = \pm 2i\left\{\sin(\frac{1}{2}\pi b)F_1(b;z) + b\sqrt{z}\cos(\frac{1}{2}\pi b)F_2(b;z)\right\}, \quad (2.2)$$

$$(\sqrt{z-1}+\sqrt{z})^b+(\sqrt{z-1}+\sqrt{z})^{-b} = 2\left\{\cos(\frac{1}{2}\pi b)F_1(b;z) + b\sqrt{z}\sin(\frac{1}{2}\pi b)F_2(b;z)\right\}, \quad (2.3)$$

$$(\sqrt{z-1}+\sqrt{z})^b-(\sqrt{z-1}+\sqrt{z})^{-b} = \pm 2i\left\{\sin(\frac{1}{2}\pi b)F_1(b;z) - b\sqrt{z}\cos(\frac{1}{2}\pi b)F_2(b;z)\right\}, \quad (2.4)$$

for $z \in \mathbf{C}\backslash[1,\infty)$. The upper or lower signs in (2.2) and (2.4) are chosen according as $0 \leq \arg z \leq \pi$ or $\arg z < 0$, respectively.

Proof. For convenience in presentation let us define $Z := z/(z-1)$. Consider the expression

$$H(b;z) := (\sqrt{z-1}-\sqrt{z})^b = (z-1)^{b/2}(1-Z^{1/2})^b = (z-1)^{b/2}{}_1F_0\left[\begin{array}{c} -b; \\ \\ -; \end{array} Z\right]$$

$$= (z-1)^{b/2}\left\{\sum_{n=0}^{\infty}\frac{(-b)_{2n}}{(2n)!}Z^n + \sum_{n=0}^{\infty}\frac{(-b)_{2n+1}}{(2n+1)!}Z^{n+1/2}\right\}$$

by (1.5) and in the series decomposition we suppose that $|Z| < 1$. Employing the identities

$$(a)_{2n} = 2^{2n}(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n, \qquad (a)_{2n+1} = 2^{2n}a(\frac{1}{2}a+\frac{1}{2})n(\frac{1}{2}a+1)_n,$$

we find

$$H(b;z) = (z-1)^{b/2}\left\{ {}_2F_1\left[\begin{matrix} -\frac{1}{2}b, \frac{1}{2}-\frac{1}{2}b \\ \frac{1}{2} \end{matrix}; Z\right] - bZ^{1/2}{}_2F_1\left[\begin{matrix} \frac{1}{2}-\frac{1}{2}b, 1-\frac{1}{2}b \\ \frac{3}{2} \end{matrix}; Z\right]\right\}.$$

Application of the Pfaff-Kummer transformation (1.6) then yields

$$H(b;z) = (z-1)^{b/2}\left\{(1-z)^{-b/2}F_1(b;z) - bZ^{1/2}(1-z)^{-b/2+1/2}F_2(b;z)\right\}.$$

Using the fact that

$$\sqrt{z-1} = \pm i\sqrt{1-z}, \qquad \begin{cases} 0 \le \arg\, z \le \pi \\ \arg\, z < 0, \end{cases}$$

we obtain

$$H(b;z) = e^{\pm\pi i b/2}\left(F_1(b;z) \pm ib\sqrt{z}F_2(b;z)\right), \tag{2.5}$$

where the upper or lower sign is taken according as $0 \le \arg\, z \le \pi$ or $\arg\, z < 0$, respectively. The result (2.5) has been obtained under the assumption that $|Z| < 1$, but may be extended by analytic continuation to $z \in \mathbf{C}\backslash[1,\infty)$.

Upon noting that $F_j(-b;z) = F_j(b;z)$ $(j = 1, 2)$, it follows immediately upon reversing the sign of b that

$$H(-b;z) = e^{\mp i\pi b/2}\left(F_1(b;z) \mp ib\sqrt{z}F_2(b;z)\right). \tag{2.6}$$

Hence some routine algebra produces

$$H(b;z) + H(-b;z) = 2\{\cos(\frac{1}{2}\pi b)F_1(b;z) - b\sqrt{z}\sin(\frac{1}{2}\pi b)F_2(b;z) \tag{2.7}$$

and

$$H(b;z) - H(-b;z) = \pm 2i\left\{\sin(\frac{1}{2}\pi b)F_1(b;z) + b\sqrt{z}\cos(\frac{1}{2}\pi b)F_2(b;z)\right\} \tag{2.8}$$

which are the results stated in (2.1) and (2.3). □

The proofs of (2.2) and (2.4) follow the same steps and so will be omitted.

3. Hypergeometric form of the roots of the cubic (1.1)

When $b = \frac{1}{3}$, we obtain from (2.1)-(2.4) the representations:

$$\left.\begin{aligned}
(\sqrt{z-1}-\sqrt{z})^{1/3} + (\sqrt{z-1}-\sqrt{z})^{-1/3} &= \sqrt{3}F_1(\tfrac{1}{3};z) - \tfrac{\sqrt{z}}{3}F_2(\tfrac{1}{3};z),\\
(\sqrt{z-1}-\sqrt{z})^{1/3} - (\sqrt{z-1}-\sqrt{z})^{-1/3} &= \pm i\left\{F_1(\tfrac{1}{3};z) + \sqrt{\tfrac{z}{3}}F_2(\tfrac{1}{3};z)\right\},\\
(\sqrt{z-1}+\sqrt{z})^{1/3} + (\sqrt{z-1}+\sqrt{z})^{-1/3} &= \sqrt{3}F_1(\tfrac{1}{3};z) + \tfrac{\sqrt{z}}{3}F_2(\tfrac{1}{3};z),\\
(\sqrt{z-1}+\sqrt{z})^{1/3} - (\sqrt{z-1}+\sqrt{z})^{-1/3} &= \pm i\left\{F_1(\tfrac{1}{3};z) - \sqrt{\tfrac{z}{3}}F_2(\tfrac{1}{3};z)\right\},
\end{aligned}\right\} \tag{3.1}$$

where

$$F_1(\frac{1}{3};z) = {}_2F_1\left[\begin{array}{c} -\frac{1}{6},\frac{1}{6} \\ \\ \frac{1}{2} \end{array};z\right] = \sqrt{1-z}\,{}_2F_1\left[\begin{array}{c} \frac{1}{3},\frac{2}{3} \\ \\ \frac{1}{2} \end{array};z\right],\ F_2(\frac{1}{3};z) = {}_2F_1\left[\begin{array}{c} \frac{1}{3},\frac{2}{3} \\ \\ \frac{3}{2} \end{array};z\right]. \tag{3.2}$$

The alternative form of $F_1(\frac{1}{3};z)$ follows from Euler's transformation (1.7).

We then obtain from (2.1) and (2.2) the desired hypergeometric forms of the roots given in the following theorem:

Theorem 3.1.. *The roots* r_j $(1 \le j \le 3)$ *in (2.1) and (2.2) of the cubic equation*

$$r^3 - r + \frac{2}{3}\sqrt{\frac{z}{3}} = 0,$$

where $0 \le \arg z \le \pi$, *have the hypergeometric representations*

$$\left.\begin{array}{l} r_1(z) = -F_1(\frac{1}{3};z) - \frac{1}{3}\sqrt{\frac{z}{3}}F_2(\frac{1}{3};z), \\ r_2(z) = \frac{2}{3}\sqrt{\frac{z}{3}}F_2(\frac{1}{3};z), \\ r_3(z) = F_1(\frac{1}{3};z) - \frac{1}{3}\sqrt{\frac{z}{3}}F_2(\frac{1}{3};z), \end{array}\right\} \tag{3.3}$$

where the hypergeometric functions $F_1(\frac{1}{3};z)$ *and* $F_2(\frac{1}{3};z)$ *are defined in (3.2). The roots satisfy* $r_j(\overline{z}) = \overline{r_j(z)}$ $(1 \le j \le 3)$ *for conjugate values of* z.

Proof. The results in (3.3) follow immediately by insertion of the first two formulas in (3.1) (with the upper sign when $0 \le \arg z \le \pi$) into the expressions for the roots $r_j(z)$ in (2.1) and (2.2). □

4. Conclusions

In this paper, we have obtained the hypergeometric forms of some new functions (not recorded earlier) in terms of combinations of Gauss hypergeometric functions by using series manipulation and the Pfaff-Kummer linear transformation. In addition, we have also obtained the hypergeometric form of the two missing roots (not recorded in [4]) of the cubic equation (1.1). The various functional forms have all been verified numerically with the help of *Mathematica*.

We can obtain the hypergeometric forms of the real and complex roots of higher degree polynomial equations in an analogous manner. Moreover, these results could have potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

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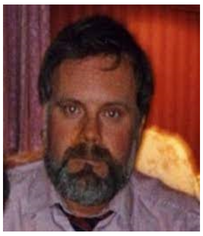
M. I. Qureshi for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.10, N.4.



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