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ON DIMENSION OF SOME FINITE ALGEBRAIC GRAPHS OF FINITE RINGS

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Abstract. Suppose that $\Gamma^{\pi}(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is a graph with the vertex set of nonzero zero-divisors of the finite ring $\mathbb{Z}_{p_1p_2\ldots p_\alpha}$, where $\alpha > 1$, and $x - y$ is an edge if and only if x and y are π-prime, where $\pi = \{p_1, p_2, \ldots, p_\alpha\}$ is a set of odd prime numbers and a and b are π-prime if either $(a, b) = 1$ or $(a, b) = p$, $p \notin \pi$. In this paper we study dimension, edge metric dimension and fraction dimension of the graph.

Keywords: diameter, distance, metric dimension, decomposition.

AMS Subject Classification: 05C12, 05C51, 05C75, 05C90.

1. Introduction

The metric dimension of a general metric space was introduced in 1953 by Blumenthal [1]. About twenty years later, it was applied by Slater [7] who introduced the concept of locating set of a graph. Independently, Harary and Melter [3] introduced the same concept as the resolving sets for calculating the metric dimension of a tree. This notion has been frequently used in graph theory, chemistry, biology, robotics and many other disciplines.

Let $G = (V, E)$ be a simple, finite, undirected graph. For vertices x and y of G, we define the distance $d(x, y)$ to be the length of a shortest path from x to y, ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of the graph G is given as $diam(G) = sup\{d(x, y)|$ x and y are vertices of $G\}$. G is said to be connected if there exists a path between any two distinct vertices, and it said to be complete if it is connected with diameter one. The girth of G denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). A subset $S \subseteq V$ is an independent set in G if no two vertices in S are adjacent. The independence number of G is the maximum size of all

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independent sets of vertices, denoted by $\alpha(G)$. For the relevant graph theoretical terms, see [2].

For a non-zero commutative ring R, let $Z^*(R)$ be the set of nonzero zero-divisors of R. In this paper, we consider $\pi = \{p_1, p_2, \ldots, p_\alpha\}$ for $\alpha > 1$, as a set of odd prime numbers and define the graph $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$, with all elements of $Z^*(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ as vertices, and two distinct vertices x and y are adjacent if and only if x and y are π-prime, where a and b are π-prime if either $(a, b) = 1$ or $(a, b) = p$, $p \notin \pi$. We find diameter of this graph, and investigate the various dimensions.

2. Preliminary

Definition 2.1. *A vertex $u \in V$ distinguishes two vertices $x, y \in V$ if $d(u, x) \neq d(u, y)$.*

Definition 2.2. *A metric generator for G is a set $\mathtt{B} \subseteq V$ with the property that, for each pair of vertices $x, y \in V$ there exists a vertex $u \in \mathtt{B}$ which distinguishes x and y. If for some metric generator $\mathtt{A} \subseteq V$, we have that $|\mathtt{A}| = min\{|\mathtt{B}| : \mathtt{B}$ is a metric generator for $G\}$, we say that $\mathtt{A}$ is a metric basis for G and $dim(G) = |\mathtt{A}|$, is the metric dimension of G.*

Definition 2.3. *The distance between the vertex v and the edge e is defined as $d(e, v) = min\{d(u, v),\ d(w, v)\}$, where $e = uw$.*

Definition 2.4. *A vertex $w \in V$ distinguishes two edges $e_1, e_2 \in E$ if $d(w, e_1) \neq d(w, e_2)$.*

Definition 2.5. *A nonempty set $\mathtt{S} \subseteq V$ is an edge metric generator for G if any two edges of G are distinguished by some vertex of S. An edge metric generator with the smallest possible cardinality is called an edge metric basis for G, and its cardinality is the edge metric dimension, which is denoted by $dim_e(G)$.*

Definition 2.6. *For any two vertices x and y of G, $R\{x, y\}$ denotes the set of vertices z such that $d(x, z) \neq d(y, z)$. In this view, a metric generating of G is a subset W of V such that $W \cap R\{x, y\} \neq \emptyset$ for any two distinct vertices x and y of G.*

Definition 2.7. *Let $f : V(G) \longrightarrow [0, 1]$ be a real valued function. For $W \subseteq V$, denote $f(W) = \sum_{v \in W} f(v)$. We call f a resolving function of G if $f(R\{x, y\}) \geq 1$ for any two distinct vertices x and y of G.*

Definition 2.8. *The fractional metric dimension, denoted by $dim_f(G)$, is given by $dim_f(G) = min\{|g| : g$ is a resolving function of $G\}$, where $|g| = g(V(G))$.*

Definition 2.9. *Let $\pi = \{p_1, p_2, \ldots, p_\alpha\}$ for $\alpha > 1$, be a set of odd prime numbers. We say that a and b are π-prime if either $(a, b) = 1$ or $(a, b) = p$, $p \notin \pi$.*

Definition 2.10. *Let R be a non-zero commutative ring and $Z^*(R)$ be its set of nonzero zero-divisors. Consider $\pi = \{p_1, p_2, \ldots, p_\alpha\}$ for $\alpha > 1$, as a set of odd prime numbers. We define the graph $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ with all elements of $Z^*(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ as vertices, where two distinct vertices x and y are adjacent if and only if x and y are π-prime.*

3. Decomposition

In this section we first decompose the graphs $\Gamma^\pi(\mathbb{Z}_{p_1p_2})$, $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$, $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$ where p_i's are distinct odd prime numbers and then we generalize it to $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$, where $\alpha > 1$.

Remark 3.1. *The number of vertices of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is $|Z^*(\mathbb{Z}_{p_1p_2\cdots p_\alpha})| = |Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha})| - 1$. It is known by Euler's function that $|Z(\mathbb{Z}_n)| = n - \phi(n)$ such that $\phi(n) = n\Pi_{i=1}^{\alpha}(1 - \frac{1}{p_i})$. So, we have $|Z(\mathbb{Z}_{p_1p_2\cdots p_\alpha})| = p_1p_2\cdots p_\alpha - \prod_{i=1}^{\alpha}(p_i - 1)$, and then*

$$|V(\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha}))| = p_1p_2\cdots p_\alpha - \prod_{i=1}^{\alpha}(p_i-1)-1.$$

Definition 3.1. *A decomposition of a graph G is a list of r subgraphs $G_1, G_2, \ldots, G_r$ such that each edge appears in exactly one subgraph in the list. By this terminology, we mean that G is decomposed by $G_1, G_2, \ldots, G_r$, i.e., $G = G_1 + G_2 + \cdots + G_r$.*

Theorem 3.1. $\Gamma^\pi(\mathbb{Z}_{p_1p_2})$ *has the following decomposition;*

$$\Gamma^\pi(\mathbb{Z}_{p_1p_2}) = K_{p_1-1,p_2-1}.$$

Proof. Let $G = \Gamma^\pi(\mathbb{Z}_{p_1p_2})$ and $\pi = \{p_1, p_2\}$. Consider $V_1 = \{kp_1;\quad p_2 \nmid k, k \in \mathbb{Z}\}$, $V_2 = \{kp_2;\quad p_1 \nmid k, k \in \mathbb{Z}\}$. Then $|V_1| = p_2 - 1$, $|V_2| = p_1 - 1$ and $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$. By definition of adjacency in G, it is clear that for any two vertices $u_1 \in V_1$, $u_2 \in V_2$, we have $u_1 \sim u_2$. So, V_1 and V_2 are two independent sets which formed two parts of the complete bipartite graph K_{p_1-1,p_2-1}. □

Example 3.1. *In the following* $\Gamma^\pi(\mathbb{Z}_{15})$ *is shown.*

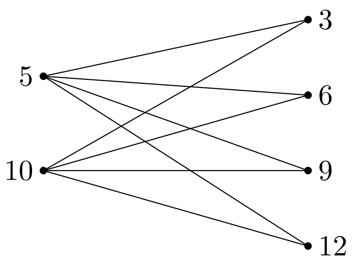


Figure 1. The decomposition of $\Gamma^\pi(\mathbb{Z}_{15})$.

Theorem 3.2. *For the graph* $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$*, the decomposition is as the following.*

$$\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3}) = K_{\theta_1,\theta_2,\theta_3} + \sum_{i=1}^{3} K_{p_i-1,\theta_i}$$

where, $\theta_i = \prod_{\substack{j=1\\ j\neq i}}^{3} p_j - \sum_{\substack{j=1\\ j\neq i}}^{3} p_j + 1.$

Proof. Let $G = \Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$ and $\pi = \{p_1, p_2, p_3\}$. For $1 \le i \le 3$, consider

$$V_i = \{kp_i;\quad p_j \nmid k, 1 \le j \ne i \le 3, k \in \mathbb{Z}\}.$$

Let $x_i \in V_i$ be distinct vertices. We show that $x_1 \sim x_2 \sim x_3 \sim x_1$. Since $(x_1, x_2) = 1$ or $(x_1, x_2) = p$, $p \notin \pi$, x_1 and x_2 are π-prime, i.e., $x_1 \sim x_2$. Similarly, $x_2 \sim x_3$ and $x_3 \sim x_1$. Clearly, V_i contains the nonzero vertices which have only the prime factor p_i. So, by inclusion-exclusion principle, $\theta_i = |V_i| = p_jp_k - (p_j + p_k) + 1$. Therefore, we have the complete 3-partite graph $K_{\theta_1,\theta_2,\theta_3}$ in the decomposition of G. Moreover, for any $x_i = kp_i$ and $u = k'p_jp_k$, $x_i \sim u$ for $1 \le i, j, k \le 3$, $j, k \ne i$. So, the remaining edges take part in three complete bipartite graphs K_{p_i-1,θ_i}. □

In the next theorem we show that $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$ can be decomposed similar to $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$ with more vertices and edges as follows. More details about this decomposition and the proof of the theorem comes after some results which described in the next section.

Theorem 3.3. *For the graph* $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$*, the decomposition is* $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4}) = H_1 + H_2$*, where*

$$H_1 = K_{\theta_1,\theta_2,\theta_3,\theta_4} + \sum_{i=1}^{4} K_{p_i-1,\theta_i},$$

$$\theta_i = \prod_{\substack{j=1\\ j\neq i}}^{4} p_j - \sum_{i<j} \prod_{\substack{k=1\\ k\neq i,j}}^{4} p_k + \sum_{\substack{j=1\\ j\neq i}}^{4} p_j - 1,$$

and H_2 is an induced subgraph of $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$ formed by the remaining edges of $A_{\{p_i\}}$ and $A_{\{p_i,p_j\}}$.

Theorem 3.4. *In general, we have $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha}) = H_1 + H_2$, where*

$$H_1 = K_{\theta_1,\theta_2,\theta_3,\ldots,\theta_\alpha} + \sum_{i=1}^{\alpha} K_{p_i-1,\theta_i},$$

$$\theta_i = \prod_{\substack{j=1\\ j\neq i}}^{\alpha} p_j - \sum_{i<j} \prod_{\substack{k=1\\ k\neq i,j}}^{\alpha} p_k + \sum_{i<j<k} \prod_{\substack{l=1\\ l\neq i,j,k}}^{\alpha} p_l - \sum_{i<j<k<l} \prod_{\substack{t=1\\ t\neq i,j,k,l}}^{\alpha} p_t + \cdots + (-1)^\alpha \sum_{\substack{j=1\\ j\neq i}}^{\alpha} p_j + (-1)^{\alpha+1},$$

and H_2 is an induced subgraph of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\ldots p_\alpha})$ formed by the remaining adjacencies.

4. Twin equivalence classes of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$

For a vertex u, the open neighborhood of u in G is $N(u) = \{v \in V \mid uv \in E\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. Two vertices u, v are true twins of G if $N[u] = N[v]$. They are false twins if $N(u) = N(v)$. Define a relation $\equiv$ on $V(G)$ by $u \equiv v$ if and only if $u = v$ or u, v are twins. By Lemma 2.6 in [4], $\equiv$ is an equivalence relation. It is not difficult to see that the equivalence classes of the true-twin relations are cliques and those of the false-twin relations are independent sets. There are three possibilities for each twin equivalence class U:

(a) U is a singleton set, or
(b) $|U| > 1$ and $N(u) = N(v)$ for any $u, v \in U$, or
(c) $|U| > 1$ and $N[u] = N[v]$ for any $u, v \in U$.

We will refer to the type (c) as the true twin equivalence classes.

Consider the equivalence relation $\equiv$. For each vertex $v \in V(G)$, let v^* be the set of vertices of G that are equivalent to v under $\equiv$. Let $\{v_1^*, \ldots, v_k^*\}$ be the partition of $V(G)$ induced by $\equiv$, where each v_i is a representative of the set v_i^*. The twin graph of G, denoted by G^*, is the graph with vertex set $V(G^*) := \{v_1^*, \ldots, v_k^*\}$, where $v_i^* v_j^* \in E(G^*)$ if and only if $v_i v_j \in E(G)$. By Lemma 2.10 in [4], one can see that this definition is independent of the choice of representatives.

Note that in $\Gamma^\pi(\mathbb{Z}_{p_1p_2\ldots p_\alpha})$ the vertices can be classified in multiples and common multiples of p_i's. In the next, we show that this partition forms equivalence classes. Also, we obtain the number of the equivalence classes by counting the ways of selecting common multiples of p_i's.

Notation 4.1. *For any nonempty proper subset $S \subset \pi$, let*

$$A_S = \{x \in Z^*(\mathbb{Z}_{p_1p_2\ldots p_\alpha}); \quad p \mid x \Longleftrightarrow p \in S\}.$$

Set $\mathcal{A} = \{A_S; \quad S \subset \pi\}$ and for all $1 \leq i \leq \alpha$, $\mathcal{A}_i = \{A_S; \quad S \subset \pi, |S| = i\}$.

One sees that $|\mathcal{A}_i| = \binom{\alpha}{i}$. In the next theorem we show that for all $S \subset \pi$, A_S is an equivalence class.

Remark 4.1. *Every equivalence class A_S of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\ldots p_\alpha})$ is an independent set; by adjacency definition, $x \nsim y$ for all $x, y \in A_S$.*

Theorem 4.2. *The number of twin equivalence classes of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is $2^\alpha - 2$.*

Proof. Let $S = \{p_{i_1}, \ldots, p_{i_s}\} \subset \pi$. We show that $N(x) = N(y)$ for every $x, y \in A_S \in \mathcal{A}_s$. Suppose that $z \in N(x)$. So, for any $p \in S$, $p \nmid z$. Since all divisors of y belong to S, $d(y, z) = 1$. So, $z \in N(y)$. Thus, A_S is an equivalence class.

By assumption, we have $\binom{\alpha}{s}$ sets of A_S's. Therefore, the number of the equivalence classes is equal to $\sum\limits_{i=1}^{\alpha} \binom{\alpha}{i} - 1 = 2^\alpha - 2$. □

Example 4.1. *In the following we describe the twin graphs of* $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$.

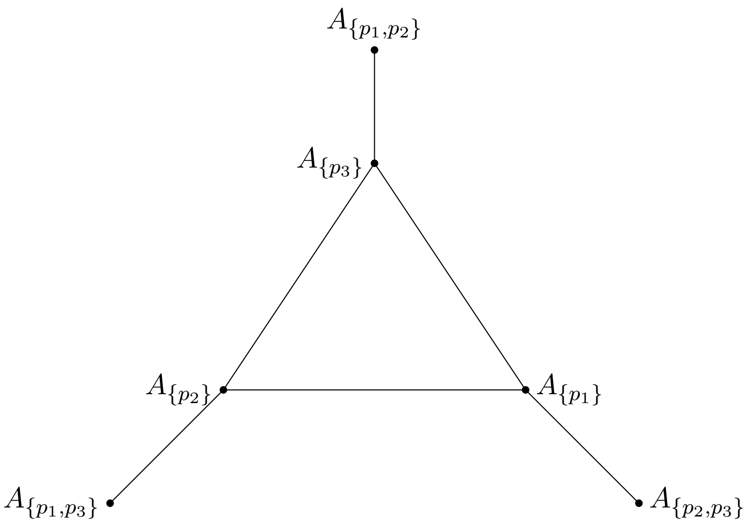


Figure 2. The twin graph of $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3})$).

Example 4.2. *Twin graphs of* $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$ *is described as the following.*

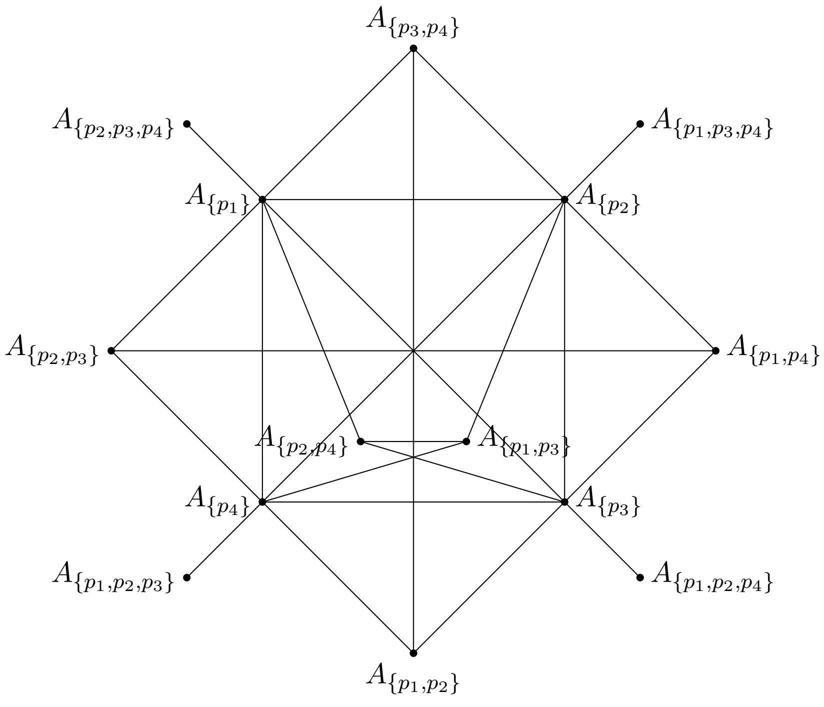


Figure 3. The twin graph of $\Gamma^\pi(\mathbb{Z}_{p_1p_2p_3p_4})$.

Proof Of Theorem 3.3 Similar to the proof of Theorem 3.2, some edges of G induce the graph $H_1 = K_{\theta_1,\theta_2,\theta_3,\theta_4} + \sum_{i=1}^{4} K_{p_i-1,\theta_i}$. Let H_2 be the graph formed by the remaining edges of $A_{\{p_i\}}$ and $A_{\{p_i,p_j\}}$. It is clear that for all $u \in A_{\{p_i\}}$, $v \in A_{\{p_j,p_k\}}$, $u \sim v$. Also, for all $x \in A_{\{p_i,p_j\}}$, $y \in A_{\{p_k,p_l\}}$, $x \sim y$. Since we have four equivalence classes $A_{\{p_i\}}$, $1 \leq i \leq 4$, and six equivalence classes $A_{\{p_i,p_j\}}$, $1 \leq i, j \leq 4$, $i \neq j$. So, the twin graph of H_2 is a 3-regular graph on ten vertices, which is isomorphic to Petersen graph as the Figure 3.

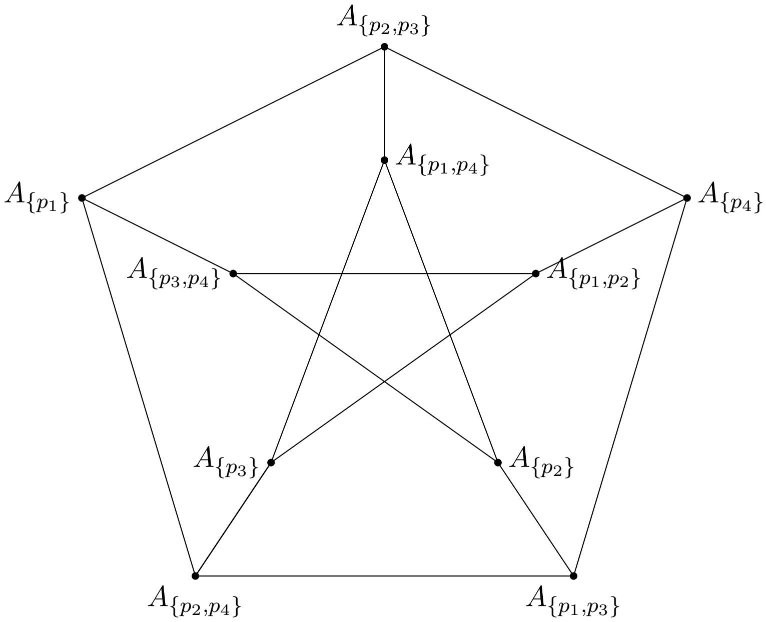


Figure 4. The twin graph of H_2.

Corollary 4.1. *Let $G = \Gamma^{\pi}(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$, then*

(i) *The clique number of G is α.*

(ii) *The independence number of G is $Max\{\theta_i : 1 \leq i \leq \alpha\}$.*

Proof. One can see both of items by presence of the induced subgraph $K_{\theta_1,\theta_2,\theta_3,\ldots,\theta_\alpha}$ in decomposition of G, by Theorem 3.4, and being independent sets for equivalence classes, by Remark 4.1. □

Corollary 4.2. *The girth of $\Gamma^{\pi}(\mathbb{Z}_{p_1p_2\ldots p_\alpha})$ is as the following.*

(i) $gr(\Gamma^{\pi}(\mathbb{Z}_{p_1p_2})) = 4$;

(ii) $gr(\Gamma^{\pi}(\mathbb{Z}_{p_1p_2\ldots p_\alpha})) = 3$, *where* $\alpha \geq 3$.

Proof. It is easy to see in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4. □

Theorem 4.3. *Let $X, Y \subset \pi$ be two nonempty proper subsets of $\pi = \{p_1, \ldots, p_\alpha\}$. For any $x \in A_X$ and $y \in A_Y$;*

(i) $d(x, y) = 1$ *if and only if* $X \cap Y = \emptyset$;

(ii) $d(x, y) = 2$ *if and only if* $X \cap Y \neq \emptyset$, $X \cup Y \neq \pi$;

(iii) $d(x, y) = 3$ *if and only if* $X \cap Y \neq \emptyset$, $X \cup Y = \pi$.

Proof. (i) This statement is an equivalent definition for the adjacency of graph G.

(ii) Since $X \cap Y \neq \emptyset$, $d(x, y) \neq 1$ by part (i), and for all $z \in A_Z$; $Z \subseteq \pi \setminus X \cup Y$, $x \sim z \sim y$. So, $d(x, y) = 2$. Conversely, let $d(x, y) = 2$. By part (i), $X \cap Y \neq \emptyset$. Also,

$x \sim z \sim y$ for some $z \in A_Z$ such that $Z \neq X, Y$, $Z \cap X = \emptyset$, $Z \cap Y = \emptyset$. Thus, $X \cup Y \neq \pi$.

(iii) Let $d(x, y) = 3$, then $X \cap Y \neq \emptyset$ and $X \cup Y = \pi$, by parts (i), (ii). Conversely, let $X \cap Y \neq \emptyset$, $X \cup Y = \pi$. We show that $d(x, y) = 3$. Suppose that there is a path of length greater than 3 between x and y. We may assume that $x - u - v - w - y$ is a path of length four such that $u \in A_U$, $v \in A_V$, $w \in A_W$, where $U, V, W \subset \pi$. Since $X \cap Y \neq \emptyset$, $d(x, y) \neq 1$. Also, If $d(x, y) = 2$, then there exists $z \in A_Z$ such that $x \sim z \sim y$. So, $Z \cap X = \emptyset$, $Z \cap Y = \emptyset$, which is a contradiction. If $V \cap X = \emptyset$ or $V \cap Y = \emptyset$, then $x \sim v$ or $v \sim y$. So, $d(x, y) = 3$.
Assume that $V \cap X \neq \emptyset$, $V \cap Y \neq \emptyset$. Since $w \sim y$, $W \cap Y = \emptyset$ by part (i). Also, $X \cup Y = \pi$, so we have $W \subset X \setminus Y$. Thus, $W \cap X \neq \emptyset$ and $d(x, w) = 2$ by part (ii). Therefore, $d(x, y) = 3$.

□

Corollary 4.3. *$diam(\Gamma^\pi(\mathbb{Z}_{p_1p_2\ldots p_\alpha})) = 3$.*

Proof. According to Theorem 4.3, (iii), we can shorten any path of length greater than four between x and y to a path of length four. □

Corollary 4.4.
(i) *$\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is a connected graph.*
(ii) *The edge connectivity of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is $min\{p_i - 1; 1 \leq i \leq \alpha\}$.*

Proof. (i) Consider two distinct vertices x, y of G and the equivalence classes A_X, A_Y where $X, Y \subset \pi$, $x \in A_X$ and $y \in A_Y$. Then by Theorem 4.3, since $d(x, y) \leq 3$, there exists a path of length 1, 2 or 3 between x and y. So, $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ is connected.

(ii) By the structure of G, we see that the twin graph of $\Gamma^\pi(\mathbb{Z}_{p_1p_2\cdots p_\alpha})$ has α vertices of degree one, which are the representatives of the equivalence classes A_S such that $|S| = \alpha - 1$. So, by removing the edges between A_S and A_T such that $|T| = 1$, G becomes disconnected. The minimum size of such A_S is the edge connectivity of G. □

5. DIMENSION

In this section we obtain some types of dimension for the graph $\Gamma^\pi(Z(\mathbb{Z}_{p_1p_2\ldots p_\alpha}))$. First, we note that determining whether a given set $\mathtt{B}$ of vertices of G is a metric generating set of G, one needs to investigate only the pairs of vertices in $V(G) - \mathtt{B}$, since $u \in \mathtt{B}$ is the only vertex of G whose distance from u is 0.

Theorem 5.1. *[5] If G^* is the twin graph of G, then $dim(G) \geq n(G) - n(G^*)$.*

Theorem 5.2. *Let $G = \Gamma^\pi(\mathbb{Z}_{p_1p_2\ldots p_\alpha})$. Then $dim(G) = n(G) - 2^\alpha + 2$.*

Proof. By Theorems 5.1 and 4.2, $dim(G) \geq n(G) - 2^\alpha + 2$. Set R as a complete set of representative vertices of equivalence classes. By Theorem 4.2, $|R| = 2^\alpha - 2$. We show that $\mathtt{M} = V(G) - R$ is a metric basis for G.
Let $u, v \in R$ and suppose S and T be two subsets of π such that $u \in A_S$ and $v \in A_T$. There exists $L \subset \pi$ such that $L \cap S \neq \emptyset$ and $L \cap T = \emptyset$. So, for every vertex $x \in A_L$, $x \notin R$; $d(x, v) = 1$ and $d(x, u) \in \{2, 3\}$, by Lemma 4.3.
Hence, $\mathtt{M}$ is a metric basis and $dim(G) \leq n(G) - 2^\alpha + 2$. □

Lemma 5.1. *Let y and z be twins. If $e, f \in E(G)$ such that $e = xy$ and $f = xz$, then every edge metric basis $\mathtt{E}$ contains at least one of y and z.*

Proof. Let S, T be two nonempty proper subsets of π such that $S \cap T = \emptyset$. Let $x \in A_S$ and $y, z \in A_T$. Consider two edges $e = xy$ and $f = xz$. It is clear that $d(e, x) = d(f, x) = 0$. So, x doesn't distinguish e and f. Let $v \neq x, y, z$.

If $v \in A_S$, then $d(e, v) = d(f, v) = 1$. If $v \notin A_S$, then $v \in A_T$ for some $T \subset \pi$ and by Lemma 4.3, $d(e, v) = d(f, v) \in \{1, 2, 3\}$. So, any vertex $v \neq x, y, z$ doesn't distinguish e and f. Thus, at least one of y and z must be in an edge metric basis $\mathtt{E}$. □

Remark 5.1. *The argument in the proof of Lemma 5.1 can be repeated for any pair of incident edges. Then every edge metric basis* $\mathtt{E}$ *of* G *contains at least* $|A_S| - 1$ *vertices of the equivalence class* A_S.

Theorem 5.3. *Let* $G = \Gamma^{\pi}(\mathbb{Z}_{p_1 p_2 \ldots p_\alpha})$. *Then* $dim_e(G) = n(G) - 2^\alpha + 2$.

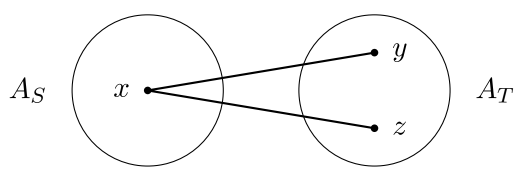
Proof. For any edge metric basis $\mathtt{E}$ of G, by Theorem 4.2, $|\mathtt{E}| \geq n(G) - 2^\alpha + 2$. Let R be a set of representative vertices of equivalence classes. By Theorem 4.2, $|R| = 2^\alpha - 2$. We claim that $\mathtt{E} = V(G) - R$ is an edge metric basis for G. According to the structure of the graph and by the view of Theorem 4.3, for any pair of edges e and f we have the following cases. In each case, we show that there is $x \in \mathtt{E}$ which distinguishes e and f. First consider that e and f have a common endpoint. We have two cases.

Case A1. Let $e = xy$, $f = xz$ such that $x \in A_S$, $y, z \in A_T$ for $S, T \subset \pi$ such that $S \cap T = \emptyset$. It is clear that $d(e, x) = d(f, x) = 0$. Also, we know that at most one of y and z belongs to R. Let $y \in R$, then $d(e, z) = 1$ and $d(f, z) = 0$. Otherwise, $y, z \in \mathtt{E}$ and both of y and z distinguish e and f.

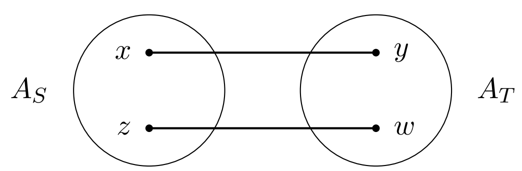
Case A2. Let $e = xz$, $f = zy$ such that $x \in A_S$, $y \in A_T$, $z \in A_K$ for $S, T, K \subset \pi$ where $S \cap T \neq \emptyset$, $S \cup T \neq \pi$ and $S \cap K = T \cap K = \emptyset$. If x or y doesn't belong to R, say $x \notin R$, then $d(e, x) = o$, $d(f, x) = 1$. So, x distinguishes e and f. If $x, y \in R$, then there exists $v \in A_L$ such that $L \cap S = \emptyset$, $L \cap T \neq \emptyset$, $L \cap K \neq \emptyset$, and $d(e, v) = 1$, $d(f, v) = 2$.

Now, assume that e and f are two distinct edges. There are three cases.

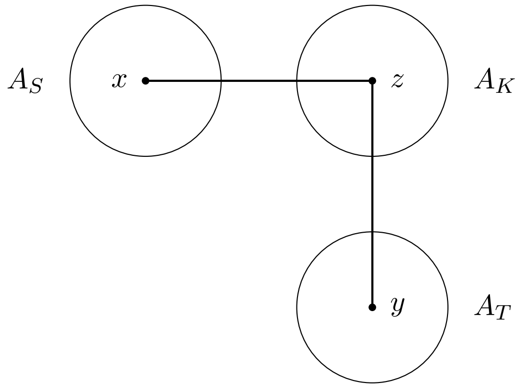
Case B1. If $e = xy$, $f = zw$ such that $x, z \in A_S$, $y, w \in A_T$ for $S, T \subset \pi$ with $S \cap T = \emptyset$, then there exists a vertex of $\{x, y, z, w\}$ which does not belong to R, say $x \notin R$. So, $d(e, x) = o$, $d(f, x) = 1$.



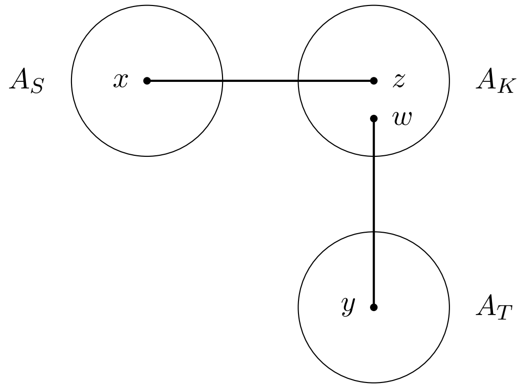
Case $A1$



Case $B1$



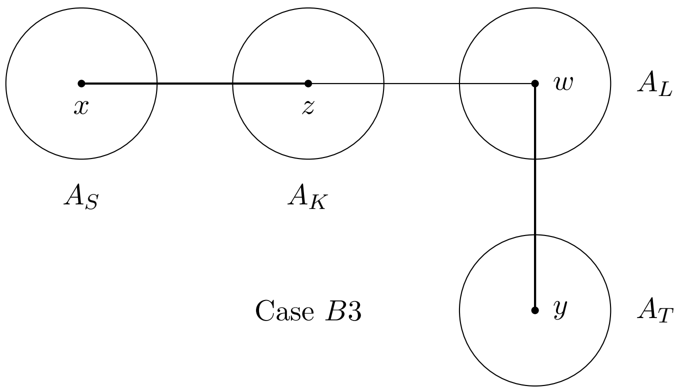
Case $A2$



Case $B2$

Case B2. Let $e = xz$, $f = yw$ such that $x \in A_S$, $y \in A_T$, $z, w \in A_K$ for $S, T, K \subset \pi$ such that $S \cap T \neq \emptyset$, $S \cup T \neq \pi$ and $S \cap K = T \cap K = \emptyset$. If x or y doesn't belong to R, say $x \notin R$, then $d(e, x) = o$, $d(f, x) = 1$. So, x distinguishes e and f. Let $x, y \in R$, since at most one of z and w belongs to R, consider $z \in R$, then $d(e, w) = 1$, $d(f, w) = 0$. It means that w distinguishes e and f.

Case B3. Let $e = xz$, $f = yw$ such that $x \in A_S$, $z \in A_K$, $y \in A_T$, $w \in A_L$ for $S, T, K, L \subset \pi$ such that $S \cap T \neq \emptyset$, $S \cap K = L \cap K = T \cap L = \emptyset$ and $T \cap K \neq \emptyset$. If x or y doesn't belong to R, say $x \notin R$, then $d(e, x) = 0$, $d(f, x) = 2$. So, x distinguishes e and f. Let $x, y \in R$, then there exists $u \in A_S$ such that $d(e, u) = 1$, $d(f, u) = 2$. So, u distinguishes e and f.



Case $B3$

In each case, E is an edge metric basis of G. Thus, $dim_e(G) \leq n(G) - 2^{\alpha} + 2$. □

Lemma 5.2. *For any twin vertices x, y of a connected graph G, $R\{x, y\} = \{x, y\}$.*

Proof. Let $z \in R\{x, y\} - \{x, y\}$, then $d(x, z) \neq d(y, z)$. So, $z \notin N(x) \cap N(y)$. Since G is connected and x, y are twins, $d(x, z) = d(y, z)$, which is a contradiction. □

Theorem 5.4. *(See [6]) Let G be a connected graph of order at least two. Then $dim_f(G) = \frac{|V(G)|}{2}$ if and only if there exists a bijection $\varphi : V(G) \longrightarrow V(G)$ such that $\varphi(v) \neq v$ and $|R\{v, \varphi(v)\}| = 2$ for all $v \in V(G)$.*

Theorem 5.5. *Let $G = \Gamma^{\pi}(\mathbb{Z}_{p_1 p_2 \dots p_\alpha}))$. Then $dim_f(G) = \frac{n(G)}{2}$.*

Proof. Let $\varphi : V(G) \longrightarrow V(G)$ such that $\varphi(x) = x + \prod\limits_{p_i | x} p_i$. Then it is easy to check that φ is a bijection which takes any vertex x to it's twin and $\varphi(x) \neq x$. Moreover, by Lemma 5.2, $R\{x, \varphi(x)\} = \{x, \varphi(x)\}$, and the result follows by Theorem 5.4. □

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