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PERFORMANCE ANALYSIS OF LOAD BASED $M/M/3$ TRANSIENT QUEUEING SYSTEM WITH FINITE CAPACITY

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Abstract. This paper deals with the investigation of M/M/3 queueing model under the provision of service rates of jobs depend upon the load of the jobs arrived in the system. The customers (jobs) arrive in the system in Poisson fashion and they are served in the chronological order of their arrival. Differential equations of transient probability distribution functions by using a transition diagram have been set up. Laplace Transform, probability generating function, and Rouchey's Theorem have been applied to get the probability of n customer in time t. Various performance indices such as the mean number of customers in the system, expected number of customers in a queue, probability that one has not to wait, expected mean time spent in a system, expected mean time spent in a queue, probability that the queue size being greater than or equal to N have been obtained analytically. Finally, the analytical results are validated graphically with the help of computing software.

Keywords: Load-Based Teansient Queue, M/M/3 Model, Performance Analysis, Laplace Transform.

AMS Subject Classification: 60K25, 60K20, 62P30, 44A10.

1. Introduction

The queueing system started from the existence of organisms on earth. Human beings also following the lining system from the ancient period but its study begins from 1909 [5, 7]. Chronological development of queueing theory from 1909 to 1969 can be found in Morse [29], Kleinrock [22], Medhi [27], Lawler[26], Jazwinski [18], Gross [13]. Jain [15] investigated a finite multiserver queueing system with queue-dependent heterogeneous servers and obtained the average number of customers in the system by using a recursive method to solve the steady-state equation. Chakravarthy and Parthasarathy [8] studied queueing network model of two different capacity buffers, where customer arrival rate is constant and servers serve customers in heterogeneous service rates. They described

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blocking conditions of the servers and observed steady-state performance measures by using Markov chain theory. Wang and Tai [41] considered M/M/3 queue dependent server model in finite capacity and obtained performances average number of customers and the average number of waiting-customers in steady-state characteristics and he claimed that the problem considered in his paper is more general than the work Kaczynski et al [19], Kasper [20]. Sharma and Tarabia [38] worked for a simple formula for M/M/1/N queueing model. Worthington et al. [42] claimed that the decision-makers understand the level of congestion with the help of load-based steady-state $M/G/\infty$ queueing model.

A lot of literature can be found in which customers joining the queue can be occupied in the system potentially infinite in number. However, sometimes we encounter some unexpected situation in the infinite queueing system, to address such situations instantly we have to make quick decisions and form a finite queue, such as on the battlefield, in the process of evacuation of people, in the pandemic like COVID-19 wherein temporal type queueing models have to be constructed. Such finite queueing models have not been tractable by steady-state queueing model approaches. It is worthwhile to report some of the works done on the line. Kim et al. [21] studied a tandem queueing system with finite and infinite buffers and they analyzed the ergodicity condition and steady-state distribution of the system states.

In our daily life, steady-state systems are not always practicable. Even when the system under study has frequent inherent changes in arrival or service rates. Numerous situations can be found where arrival rates, service rates, and other system parameters may change continuously. In a realistic approach, transient analysis is essential which we experience in a service business that never reaches equilibrium. To overcome such problems transient solutions are desirable. In this transient concept, some works did have been in queueing system [9]. Gorden [32] introduced the time-dependent arrival of customers in the queueing system. Blomqvist [6] developed a transient queueing system and proved the theorem for the standard deviation of waiting time. Mori [28] followed transient study of mean waiting time and results validate in $M/M/1$ and $M/D/1$. Kumar et al. [24] demonstrated how the transient solution for the state probabilities and busy period in a single server Poisson queue with balking can be obtained simply and directly. Law [25] noted many initial transient failure outputs in his study, which was led by Gafarian et al [12]. They studied the problem to determine the point in simulated time when near-equilibrium has been achieved. Literature of transient queueing systems found first in $1950's$. Ammar [1] studied impatient behavior in two heterogeneous servers within a time-dependent approach. Ayyappan et al. [2] developed a single server transient queue with batch service under catastrophe. Ayyppan and Shyamaly [3], Jain and Kanethia [14], Kumar and Madheswri [23] made some contributions to repairmen and catastrophe queueing models in the transient framework. Jain et al. [17] developed a finite queueing model having single and batch service modes for telecommunication systems, wherein the transient state queue size distribution and expected busy period, expected idle period have been obtained. Jain and Singh [16] studied Markovian queue in transient structure analytically by applying the technique of probability generating function. Curry et al. [11] applied transient foundation of queueing system in healthcare departments junction based on hospital responding to demand increases during epidemics and pandemics such as the recent COVID-19. Some of the transient queueing models have been developed in Murray and Kelton [30], Nagpal [31], Odoni and Roth [33], Parthasarthy and Selvaraju [35], Parthasarthy and Lenin [36], Ruchi [37], Suhasiui et al. [39], Sundari and Srinivasan [40].

The scope and purpose of this article is to study the load-based M/M/3 queueing model with finite capacity in every epoch beneath the first-come-first-serve discipline. Various performance measures under study have been determined as the functions of time. The probability generating function technique, Laplace transform and its inverse Laplace methods have been employed to solve the system of ordinary differential equations. The model under investigation has its novelty in the way that it takes jobs-dependent servers into account that generates a more complex system of ordinary differential-difference equations, which leads the problem more rigorous. Finding various performance indices of such a model analytically counts as one of the few innovative works. Our model under study can be applied in health care system of corona patient in the way that health care unit is equipped with all the facilities (Doctors, Nurse, oxygen, personal protection equipment, polymerase chain reaction test kit, medicine, isolation ward, bed) and they move from one health care center to other according to rush of the patients in health care centers set up in cluster wise in particular area. This article is organized into sections. The first section covers an introduction of the topic thoroughly. Section 2 involves model description, section 3 generates governing birth-death equations which are derived by making the balance of in-flow and out-flow of system states, section 4 includes performance indices, section 5 presents numerical results and interpretation, and in section 6 concluding remarks & scope of our work have been given.

2. Model Description

The queueing model under study consists of at most three servers. The first server gives the service to the customers with $\mu_1(t)$ service rate if there are $J-1$ customers in the system. If there are greater or equal to J and less than K customers in the system then two servers are employed for the service with service rate $\mu_2(t)$. If there are K to L customers in the system then three servers are provisioned wherein the third server has $\mu_3(t)$ service rate. The model under study assumes that customers arrive at the system in the Poisson process with the rate λ and the service time of servers is exponentially distributed. Differential difference equations have been obtained with the help of a transition diagram. We set, $\mu_1 = \mu_2 = \mu'$, $\mu' = \mu_3$, $\lambda_1 + \lambda_2 = \lambda$ for $n = J(1), J(2), K(1), K(2)$ state.

3. Mathematical Model and Analysis

Under the description of our model, we have constructed the following transient diagram.

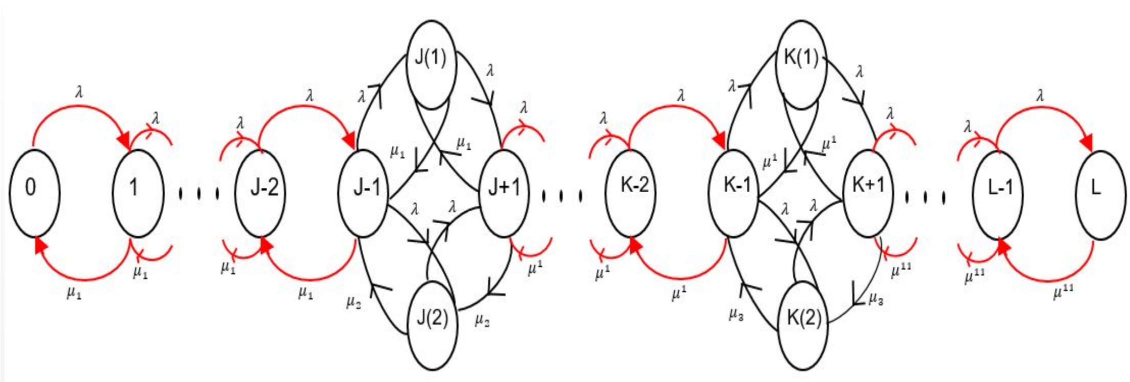

Fig. 1. State-transition diagram

Let X(t) be the number of units in the system at time t. Then X(t) : t > 0 is a continuous-time Markov process with the state space $S = \{0, 1, ..., J-1, J, J+1, ..., K-1, K, K+1, ..., L-1, L\}$ where $J = J(1) + J(2), \quad K = K(1) + K(2)$. Transient equations for the finite M/M/3 queueing system with queue-dependent heterogeneous servers, the system is governed by the following set of differential-difference equations obtained from Fig. 1.

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu_1 P_1(t) \tag{1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu_1)P_n(t) + \lambda P_{n-1}(t) + \mu_1 P_{n+1}(t) \qquad (1 \le n \le J-2) \tag{2}$$

$$\begin{aligned} \frac{dP_{J-1}(t)}{dt} &= -(\lambda_1 + \lambda_2 + \mu_1)P_{J-1}(t) + \lambda P_{J-2}(t) + \mu_1 P_{J(1)}(t) + \mu_2 P_{J(2)}(t) \\ &= -(\lambda + \mu_1)P_{J-1}(t) + \lambda P_{J-2}(t) + \mu_1 P_{J(1)}(t) + \mu_2 P_{J(2)}(t) \end{aligned} \tag{3}$$

$$\frac{dP_{J(1)}(t)}{dt} = -(\lambda + \mu_1)P_{J(1)}(t) + \lambda P_{J-1}(t) + \mu_1 P_{J+1}(t) \tag{4}$$

$$\frac{dP_{J(2)}(t)}{dt} = -(\lambda + \mu_2)P_{J(2)}(t) + \lambda P_{J-1}(t) + \mu_2 P_{J+1}(t) \tag{5}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu')P_n(t) + \lambda P_{n-1}(t) + \mu' P_{n+1}(t) \qquad (J+1 \le n \le K-2) \tag{6}$$

$$\begin{aligned} \frac{dP_{K-1}(t)}{dt} &= -(\lambda_1 + \lambda_2 + \mu')P_{K-1}(t) + \lambda P_{K-2}(t) + \mu' P_{K(1)}(t) + \mu_3 P_{K(2)}(t) \\ &= -(\lambda + \mu')P_{K-1}(t) + \lambda P_{K-2}(t) + \mu' P_{K(1)}(t) + \mu_3 P_{K(2)}(t) \end{aligned} \tag{7}$$

$$\frac{dP_{K(1)}(t)}{dt} = -(\lambda + \mu')P_{K(1)}(t) + \lambda P_{K-1}(t) + \mu' P_{K+1}(t) \tag{8}$$

$$\frac{dP_{K(2)}(t)}{dt} = -(\lambda + \mu_3)P_{K(2)}(t) + \lambda P_{K-1}(t) + \mu_3 P_{K+1}(t) \tag{9}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu'')P_n(t) + \lambda P_{n-1}(t) + \mu'' P_{n+1}(t) \qquad (K+1 \le n \le L-1) \tag{10}$$

$$\frac{dP_L(t)}{dt} = -\mu'' P_L(t) + \lambda P_{L-1}(t) \tag{11}$$

Where $\mu' = \mu_1 + \mu_2, \quad \mu'' = \mu_1 + \mu_2 + \mu_3$

By using Laplace transform from equation (1) to (11).

$$(s + \lambda)P_0(s) - \frac{\mu_1}{s^2}P_1(s) = 1 \tag{12}$$

$$(s + \lambda + \frac{\mu_1}{s^2})P_n(s) - \lambda P_{n-1}(s) - \frac{\mu_1}{s^2}P_{n+1}(s) = 0 \qquad (1 \le n \le J-2) \tag{13}$$

$$(s + \lambda + \frac{\mu_1}{s^2})P_{J-1}(s) - \lambda P_{J-2}(s) - \frac{\mu_1}{s^2}P_{J(1)}(s) - \frac{\mu_1}{s^2}P_{J(2)}(s) = 0 \tag{14}$$

$$(s + \lambda + \frac{\mu_1}{s^2})P_{J(1)}(s) - \lambda_1 P_{J-1}(s) - \frac{\mu_1}{s^2}P_{J+1}(s) = 0 \tag{15}$$

$$(s + \lambda + \frac{\mu_1}{s^2})P_{J(2)}(s) - \lambda_2 P_{J-1}(s) - \frac{\mu_2}{s^2}P_{J+1}(s) = 0 \tag{16}$$

$$(s + \lambda + \frac{\mu'}{s^2})P_n(s) - \lambda P_{n-1}(s) - \frac{\mu'}{s^2}P_{n+1}(s) = 0 \qquad (J+1 \le n \le K-2) \tag{17}$$

$$(s + \lambda + \frac{\mu'}{s^2})P_{K-1}(s) - \lambda P_{K-2}(s) - \frac{\mu'}{s^2}P_{K(1)}(s) - \frac{\mu'}{s^2}P_{K(2)}(s) = 0 \tag{18}$$

$$(s + \lambda + \frac{\mu'}{s^2})P_{K(1)}(s) - \lambda_1 P_{K-1}(s) - \frac{\mu'}{s^2}P_{K+1}(s) = 0 \tag{19}$$

$$(s+\lambda+\frac{\mu'}{s^2})P_{K(2)}(s)-\lambda_2 P_{K-1}(s)-\frac{\mu_3}{s^2}P_{K+1}(s)=0 \tag{20}$$

$$(s+\lambda+\frac{\mu''}{s^2})P_n(s)-\lambda P_{n-1}(s)-\frac{\mu''}{s^2}P_{n+1}(s)=0 \qquad (K+1\le n\le L-1) \tag{21}$$

$$(s+\frac{\mu''}{s^2})P_L(s)-\lambda P_{L-1}(s)=0 \tag{22}$$

Adding equation (15) and (16) we get

$$\left(s+\lambda+\frac{\mu_1}{s^2}\right)\left[P_{J(1)}(s)+P_{J(2)}(s)\right]-(\lambda_1+\lambda_2)P_{J-1}(s)-\left(\frac{\mu_1}{s^2}+\frac{\mu_2}{s^2}\right)P_{J+1}(s)=0$$

$$\left(s+\lambda+\frac{\mu_1}{s^2}\right)P_J(s)-\lambda P_{J-1}(s)-\frac{\mu'}{s^2}P_{J+1}(s)=0 \tag{23}$$

Similarly Adding equations (19) and (20)
We get,

$$\left(s+\lambda+\frac{\mu'}{s^2}\right)\left[P_{K(1)}(s)+P_{K(2)}(s)\right]-(\lambda_1+\lambda_2)P_{K-1}(s)-\left(\frac{\mu'}{s^2}+\frac{\mu_3}{s^2}\right)P_{K+1}(s)=0$$

$$\left(s+\lambda+\frac{\mu'}{s^2}\right)P_K(s)-\lambda P_{K-1}(s)-\frac{\mu''}{s^2}P_{K+1}(s)=0 \tag{24}$$

Then equation (13) and (14) can be express in

$$(s+\lambda+\frac{\mu_1}{s^2})P_n(s)-\lambda P_{n-1}(s)-\frac{\mu_1}{s^2}P_{n+1}(s)=0 \qquad (0\le n\le J-1) \tag{25}$$

Similarly, equations (23), (17) and (18) can be express in

$$(s+\lambda+\frac{\mu'}{s^2})P_n(s)-\lambda P_{n-1}(s)-\frac{\mu'}{s^2}P_{n+1}(s)=0 \qquad (J\le n\le K-1) \tag{26}$$

And equations (24), (21) and (22) can be express in

$$(s+\lambda+\frac{\mu''}{s^2})P_n(s)-\lambda P_{n-1}(s)-\frac{\mu''}{s^2}P_{n+1}(s)=0 \qquad (K\le n\le L) \tag{27}$$

Let us define generating function as

$$P(z)=\sum_{n=0}^{\infty}P_n(s)z^n \tag{28}$$

Taking sum of (12) and (25) $\times z^n$, we have

$$(s+\lambda+\frac{\mu_1}{s^2})\sum_{n=0}^{J-1}P_n(s)z^n-\lambda\sum_{n=0}^{J-1}P_{n-1}(s)z^n-\frac{\mu_1}{s^2}\sum_{n=0}^{J-1}P_{n+1}(s)z^n=1$$

$$(s+\lambda+\frac{\mu_1}{s^2})\sum_{n=0}^{J-1}P_n(s)z^n-\lambda z\sum_{n=0}^{J-1}P_{n-1}(s)z^{n-1}-\frac{\mu_1}{s^2 z}\sum_{n=0}^{J-1}P_{n+1}(s)z^{n+1}=1 \tag{29}$$

From (26)

$$(s+\lambda+\frac{\mu'}{s^2})\sum_{n=J}^{K-1}P_n(s)z^n-\lambda\sum_{n=J}^{K-1}P_{n-1}(s)z^n-\frac{\mu'}{s^2}\sum_{n=J}^{K-1}P_{n+1}(s)z^n=0$$

$$(s+\lambda+\frac{\mu'}{s^2})\sum_{n=J}^{K-1}P_n(s)z^n-\lambda z\sum_{n=J}^{K-1}P_{n-1}(s)z^{n-1}-\frac{\mu'}{s^2 z}\sum_{n=J}^{K-1}P_{n+1}(s)z^{n+1}=0 \tag{30}$$

From (27)

$$(s+\lambda+\frac{\mu''}{s^2})\sum_{n=K}^{L}P_n(s)z^n-\lambda\sum_{n=K}^{L}P_{n-1}(s)z^n-\frac{\mu''}{s^2}\sum_{n=K}^{L}P_{n+1}(s)z^n=0$$

$$(s+\lambda+\frac{\mu''}{s^2})\sum_{n=K}^{L}P_n(s)z^n-\lambda z\sum_{n=K}^{L}P_{n-1}(s)z^{n-1}-\frac{\mu''}{s^2z}\sum_{n=K}^{L}P_{n+1}(s)z^{n+1}=0 \qquad (31)$$

Add equations (29), (30) and (31) we get

$$\begin{aligned}&(s+\lambda+\frac{\mu''}{s^2})\left[\sum_{n=0}^{J-1}P_n(s)z^n+\sum_{n=J}^{K-1}P_n(s)z^n+\sum_{n=K}^{L}P_n(s)z^n\right]\\&-\lambda z\left[\sum_{n=0}^{J-1}P_{n-1}(s)z^{n-1}+\sum_{n=J}^{K-1}P_{n-1}(s)z^{n-1}+\sum_{n=K}^{L}P_{n-1}(s)z^{n-1}\right]\\&-\frac{\mu''}{s^2z}\left[\sum_{n=0}^{J-1}P_{n+1}(s)z^{n+1}+\sum_{n=J}^{K-1}P_{n+1}(s)z^{n+1}+\sum_{n=K}^{L}P_{n+1}(s)z^{n+1}\right]=1\end{aligned}$$

or $\quad (s+\lambda+\frac{\mu''}{s^2})\sum_{n=0}^{L}P_n(s)z^n-\lambda z\sum_{n=0}^{L}P_{n-1}(s)z^{n-1}-\frac{\mu''}{s^2z}\sum_{n=0}^{L}P_{n+1}(s)z^{n+1}=1$

or $\quad (s+\lambda+\frac{\mu''}{s^2})\sum_{n=0}^{L}P_n(s)z^n-\lambda z\sum_{n=1}^{L}P_{n-1}(s)z^{n-1}-\frac{\mu''}{s^2z}\left[\sum_{n=0}^{L}P_n(s)z^n-P_0(s)\right]=1$

or $\quad (s+\lambda+\frac{\mu''}{s^2})P(z)-\lambda zP(z)-\frac{\mu''}{s^2z}\left[P(z)-P_0(s)\right]=1 \qquad (32)$

$$\text{where}\qquad P(z)=\sum_{n=0}^{L}P_n(s)z^n$$

$$P(z)\left(s+\lambda+\frac{\mu''}{s^2}-\lambda z-\frac{\mu''}{s^2z}\right)=1-\frac{\mu''\times P_0(s)}{s^2z}$$

$$P(z)\frac{(s^3z+s^2\lambda z+\mu''z-s^2\lambda z^2-\mu'')}{s^2z}=\frac{s^2z-\mu''\times P_0(s)}{s^2z}$$

$$P(z)=\frac{s^2z-\mu''\times P_0(s)}{-\{s^2\lambda z^2-(s^3+s^2\lambda+\mu'')z+\mu''\}} \qquad (33)$$

We may write

$$P(z)=\frac{N(z,s)}{D(z,s)}=\frac{N}{D}$$

Where $D=-\{s^2\lambda z^2-(s^3+s^2\lambda+\mu'')z+\mu''\}=-s^2\lambda\left[z-\alpha_1(s)\right]\left[z-\alpha_2(s)\right]$
Two zeros of D are,

$$\alpha_1(s)=\frac{(s^3+s^2\lambda+\mu'')+\sqrt{(s^3+s^2\lambda+\mu'')^2-4s^2\lambda\mu''}}{2s^2\lambda} \qquad (34)$$

$$\&\quad \alpha_2(s)=\frac{(s^3+s^2\lambda+\mu'')-\sqrt{(s^3+s^2\lambda+\mu'')^2-4s^2\lambda\mu''}}{2s^2\lambda} \qquad (35)$$

We wish to show that $\mid \alpha_2(s) \mid < \mid \alpha_1(s) \mid$ for $Re(s) > 0$, or that $\mid \alpha_2(s) \mid^2 - \mid \alpha_1(s) \mid^2 > 0$. Defining $m \triangleq \sqrt{(s^3 + s^2\lambda + \mu'')^2 - 4s^2\lambda\mu''}$ and substituting for $\alpha_1(s)$ and $\alpha_2(s)$ in the latter inequality yields the equivalent condition.

$$Re(s^3 + s^2\lambda + \mu'')Re(m) + Im(s^3 + s^2\lambda + \mu'')Im(m) > 0 \tag{36}$$

Since $\mid \alpha_2(s) \mid \leq \mid \alpha_1(s) \mid$, the one zero of D of interest must be $\alpha_2(s)$. By the analyticity of P(z) for $\mid z \mid < 1 \quad (Re(s) > 0)$, $\alpha_2(s)$ is a zero of the numerator [by Rouchey Theorem]. Hence,

$$s^2\alpha_2(s) - \mu'' P_0(s) = 0 \Rightarrow P_0(s) = \frac{s^2\alpha_2(s)}{\mu''} \tag{37}$$

Then from equation 33, we have

$$\begin{aligned} P(z) &= \frac{s^2 z - \mu'' \times \frac{s^2\alpha_2}{\mu''}}{-s^2\lambda(z-\alpha_1)(z-\alpha_2)} \\ &= \frac{1}{\lambda\alpha_1}\left(1 + \frac{z}{\alpha_1} + \frac{z^2}{\alpha_1^2} + ... + \frac{z^n}{\alpha_1^n} + ...\right) \end{aligned}$$

Equating the like coefficient of z^n, we have

$$P_n(s) = \frac{1}{\lambda\alpha_1} \times \frac{1}{\alpha_1^n} = P_0(s)\frac{1}{\alpha_1^n} \tag{38}$$

By using the normalize condition $$\sum_{n=0}^{L} P_n(s) = \frac{1}{s} \tag{39}$$

From equations (38) and (39)

$$\sum_{n=0}^{L} \frac{1}{\alpha_1^n} P_0(s) = \frac{1}{s} \Rightarrow P_0(s) = \frac{1}{s}\left[\sum_{n=0}^{L} \frac{1}{\alpha_1^n}\right]^{-1} \tag{40}$$

Then equation (38)

$$P_n(s) = \frac{1}{s\alpha_1^n}\left[\sum_{n=0}^{L} \frac{1}{\alpha_1^n}\right]^{-1} \tag{41}$$

The inverse transform $P_n(t)$ is given by the well−known inversion formula

$$P_n(t) = \left(\frac{1}{2\pi i}\right)\int_{a-i\infty}^{a+i\infty} e^{st} P_n(s)ds$$

or, alternately,

$$P_n(t) = \left(\frac{e^{at}}{\pi}\right)\int_0^\infty [\text{Re}\{P_n(s)\}\cos wt - \text{Im}\{P_n(s)\}\sin wt]dw, \tag{42}$$

$s = a + iw$, where a can be any real number greater than α. Equation (42) can also be represented by cosine transform pair

$$P_n(t) = \left(\frac{2e^{at}}{\pi}\right)\int_0^\infty \text{Re}\{P_n(s)\}\cos wt\, dw, \tag{43}$$

or by the sine transform pair,

$$P_n(t) = \left(-\frac{2e^{at}}{\pi}\right)\int_0^\infty \text{Im}\{P_n(s)\}\sin wt\, dw. \tag{44}$$

The technique for numerically inverting $P_n(s)$ is essentially a trapezoidal rule approximation to (43) which involves only $\text{Re}(P_n(s))$. An essential feature of the method is that an expression for the error in the computed inverse transform is available which allows us to control the maximum error in the inversion technique $P_n(t) = P_c(t) - E_c$ where $P_c(t)$ is the approximation to $P_n(t)$ defined by

$$P_c(t) = \left(\frac{2e^{at}}{T}\right)\left[\frac{1}{2}P(a) + \sum_{k=1}^{\infty} Re\{P(a + \frac{k\pi i}{T})\}cos(\frac{k\pi t}{T})\right] \tag{45}$$

with the error

$$E_c = \sum_{m=1}^{\infty} exp(-2mat)\{P(2mT + t) + exp(2at)P(2mT - t)\} \tag{46}$$

where the parameter a, T satisfy the condition $T \geq t$, and $a \geq \alpha$. Using the bound $|P(t)| \leq Me^{at}$(shifting theorem) in (46) and summing the resulting geometric series, we find that

$$E_c \leq Me^{at}[exp\{2(a - \alpha)t\} + 1]/[exp\{2(a - \alpha)T\} - 1] \tag{47}$$

By choosing $a - \alpha$ sufficiently large with $T > t$, the error E_c can be made as small as desired.

One shortcoming of the method is that usually the series (45) converges slowly. One must often sum hundreds of terms before convergence to three significant figures is attained. If we take $T = 2t$ to make above series converge fast

$$P_c(t) = \left(\frac{e^{at}}{t}\right)\left[\frac{1}{2}P(a) + \sum_{k=1}^{\infty} Re\{P(a + \frac{k\pi i}{t})\}(-1)^k\right] \tag{48}$$

For numerical results we have used the equation (41) in equation (42) which is the numerical inversion of (41) that gives completely $P_n(t)$ in explicit form.

4. Performance Indices

We have been observed the following six performance of the model.

(i) The mean number of customers in the system is

$$Ls(t) = \sum_{n=1}^{L} nP_n(t) = \sum_{n=1}^{J-1} nP_n(t) + \sum_{n=J}^{K-1} nP_n(t) + \sum_{n=K}^{L} nP_n(t)$$

(ii) Expected number of customers in the queue is

$$Lq(t) = \sum_{n=1}^{L}(n-1)P_n(t) = \sum_{n=1}^{J-1}(n-1)P_n(t) + \sum_{n=J}^{K-1}(n-1)P_n(t) + \sum_{n=K}^{L}(n-1)P_n(t)$$

(iii) Probability that the queue size being greater than equal to N

$$P(\text{Queue size} \geq N) = \sum_{n=N}^{L} P_n(t) = \sum_{n=0}^{L} P_n(t) - \sum_{n=0}^{N-1} P_n(t)$$

(iv) Probability that one has not to wait $= P_0(t)$

(v) Expected mean time spent in the system $W_s(t) = \frac{L_s(t)}{\lambda}$

(vi) Expected mean time spent in the queue $W_q(t) = \frac{L_q(t)}{\lambda}$

5. Numerical Results and Interpretation

The matlab software has been used to obtain various measures of performance such as the transient probability distribution $(P_n(t))$, mean number of customers in the system $(L_s(t))$, mean number of customers in the queue $(L_q(t))$, mean-time that a customer spent in the system $(W_s(t))$, mean of waiting time that a customer has to wait in the queue $(W_q(t))$.

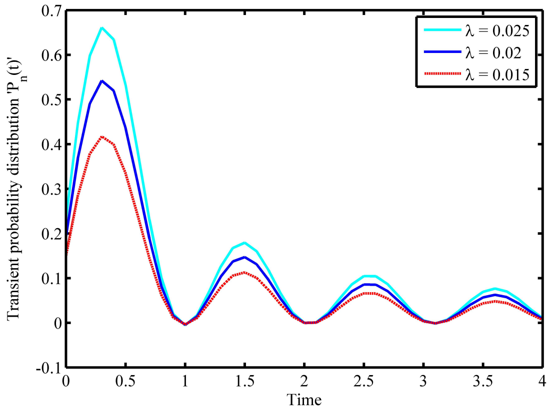

Fig.2. Transient probability distribution versus time

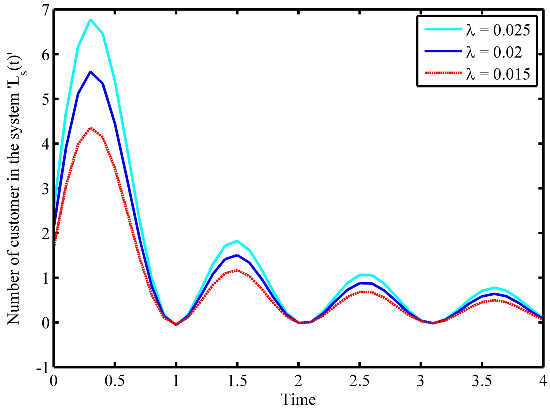

Fig.3. Number of customers in the system versus time

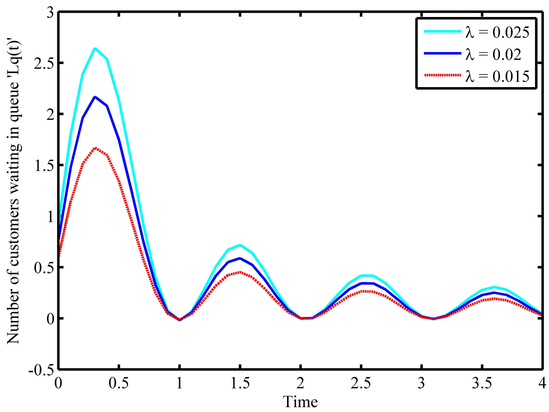

Fig.4. Number of customers waiting in queue versus time

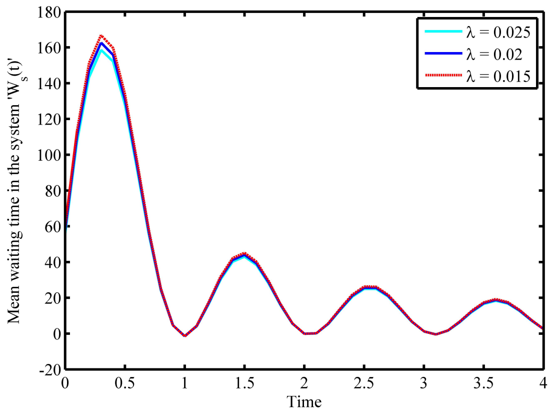

Fig.5. Mean waiting time in the system versus time

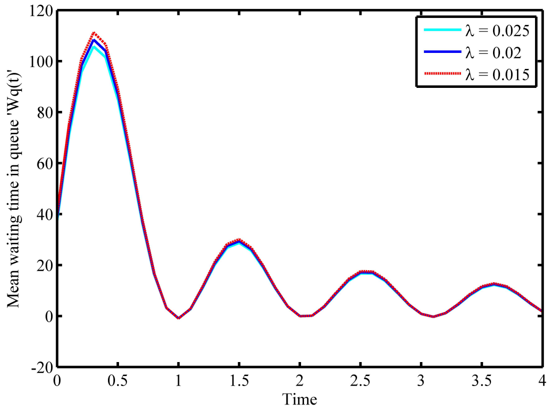

Fig.6. Mean waiting time in queue versus time

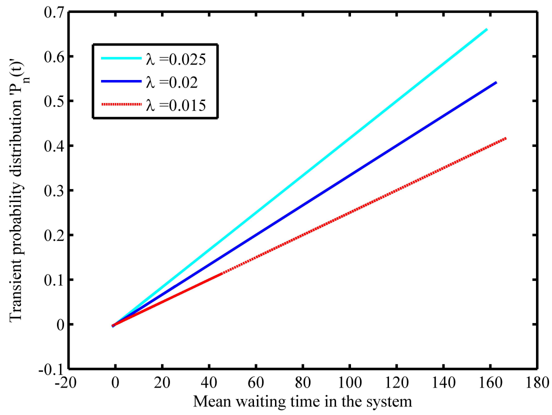

Fig.7. Transient probability distribution vs Mean waiting time

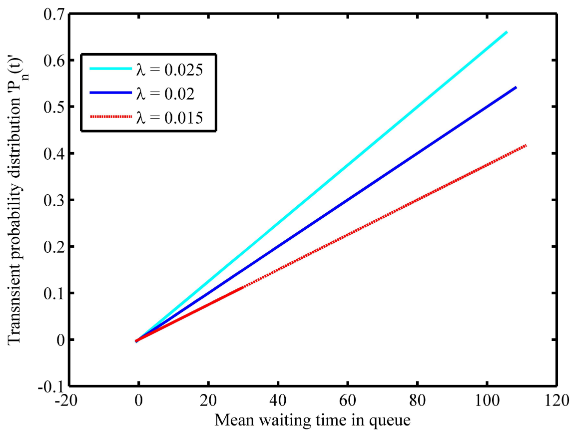

Fig.8. Transient probability distribution vs mean waiting time in queue

For the numerical computations, we have taken values of some parameters as; $J = 2, K = 4, L = 6, a = 0.01, w = 0.12$, and service rates $\mu_1 = 0.023, \mu_2 = 0.024, \mu_3 = 0.025$. Graph of transient probability distribution drawn against time for various arrival rates $\lambda = 0.015, \lambda = 0.02, \lambda = 0.025$ shown in the Fig. 2, The soar of arrival rate showed to increase the transient probability with time increases, which is experienced in real-life situations. Graphs start with $P_n(t)$ axis is meant to have servers busy from the beginning. In all the figures graphs come to at zero levels because servers are idle for a moment, which has been shown by some valley on the curve. Fig. 3 displays that at the beginning as the arrival rate increases the number of customers in the system $L_s(t)$ increases with time. But as time passes on the number of customers decrease gradually. Which is also realistic. Fig. 4 is the graph of the number of customers waiting in queue against time which is of the same nature as that of the number of customers in the system but in the system, the number of customers in the queue is less which is inherently true. Graphs of system time and waiting time against the time have been shown in Fig. 5 and Fig. 6 respectively shows that as the arrival rate increases the system time as well as the waiting time in queue decreases since after fixed number of customers there are provisions of introducing a new server. Graphs also show that increasing arrival rates do not show a significant change in system time and waiting time.

Fig. 7 and Fig. 8 show an increase of arrival rates, increase of system time and waiting time yields the increase in transient probability which proportionate have been illustrated by the graphs.

6. Concluding Remark and Scope

(i) Various methods of solving transient queueing systems explained in detail can be the guideline for researchers in the future.
(ii) Our model can be extended to a more general $M/M/n$ model by taking the $'n'$ number of servers, however, it will become more complex to tackle.
(iii) The model under study can be converted to the steady-state condition when $t \to \infty$. The model can be studied in both time-varying arrival rates and service rates so that the model can be much more realistic.
The model under study has paramount applications in telecommunications in which after a rushing of telephone calls when it exceeds a certain number there can be switching of guard channels to serve the calls. The model can be experienced in peak hours in service sectors such as the bank, government tax offices, etc., where employees are used in additional service counters, after completion of peak hours, they join their regular job. In a flexible manufacturing system, an automatic switch-over system can be equipped to fit our model.

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