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On horizontal cooperation in linear production processes with a supplier that controls a limited resource

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Abstract In this paper we consider a two-echelon supply chain with one supplier that controls a limited resource and a finite set of manufacturers who need to purchase this resource. We analyze the effect of the limited resource on the horizontal cooperation of manufacturers. To this end, we use cooperative game theory and the existence of stable distributions of the total profit among the manufacturers as a measure of the possibilities of cooperation. The game theoretical model that describes the horizontal cooperation involves externalities, which arise because of the possible scarcity of the limited resource and the possible coalition structures that can be formed. Furthermore, manufacturers do not know how the supplier will allocate the limited resource, therefore, how much of this resource they will obtain is uncertain for all concerned. Nevertheless, when the limited resource is not scarce for the grand coalition, the existence of stable distributions of the total profit is guaranteed and consequently the collaboration among the manufacturers is profitable for them all. In the event that the limited resource is insufficient for the grand coalition, we introduce a new cooperative game that assesses the expectations of each coalition of manufacturers regarding the amount of the limited resource they can obtain. We analyze two extreme expectations: the optimistic

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and the pessimistic. In the optimistic case, we cannot reach a conclusion regarding the full cooperation of the manufacturers. In the pessimistic case, with one reasonable assumption, the existence of stable distributions of the total profit is guaranteed and as a result the collaboration among manufacturers is a win-win deal.

Keywords linear production processes · limited resource · horizontal cooperation · externalities

1 Introduction

Production planning is an important part of supply chain management, which entails all the activities in industrial organizations from buying resources to the delivery of products to clients. In this paper, we focus on production processes where several manufacturers can obtain resource bundles, from suppliers that have no restrictions on their capacities, which they use to produce various products via fixed proportion technologies that are available to all firms. Manufacturers differ in the amount of resources they can manage. Thus, they are different in size and therefore they are asymmetric. We assume that manufacturers may collaborate sharing resources and their goal is to maximize their profit, which equals the revenue from their products at the given market prices. These are the so-called linear production (LP) situations, because the optimization programs that arise for maximizing the revenues are linear programs. LP situations and corresponding cooperative games were introduced by Owen (1975).

However, there are many real-life situations in which some resources are very limited and then the demands of the producers could not be completely satisfied. For example, natural resources such as water, fish quotas, coltan or carbon dioxide permits are very important to irrigate in agriculture, food companies, technological companies or pollutant companies respectively, but those resources are limited and the amounts requested by the interested companies could be larger than the natural resources available in each moment. The resources mentioned above are (almost) absolutely necessary to produce, so without water it is not possible to produce any agricultural products, without a fish quota it is not allow to fish, without coltan almost no electronic device can be manufactured and without emission permits the polluting companies can not produce. Moreover, on many occasions these resources are under the control of a single authority that is responsible for allocating the amounts of resources to the companies that demand them. To do this, that authority will use different criteria. Examples of these situations are carbon dioxide emission permits after the Kyoto Protocol in 1997 and the Paris Agreement in 2015, because each signatory country has a limit or target for each period to be divided among the sectors involved; and watershed and irrigators communities in which there is very often an agency that is responsible for the distribution of freshwater.

Bearing this in mind, we consider the aforementioned LP situations when there is a resource limited by a certain amount and managed by an authority that needs to be purchased and is absolutely necessary to manufacture any product. In terms of supply chain management, we have a two-echelon system in which there is only one supplier and n manufacturers. We do not consider the suppliers with unbounded capacity, because we assume that they do not introduce any restriction neither in the operations of the production process nor in the profit margins of the goods produced. The supplier who controls the limited resource can be seen as a monopolist that imposes the price, because the companies are unable to buy the resource from other suppliers.

In this paper we analyze the problem of distributing the limited resource for companies, when they do not know how the supplier will behave in the allocation of this limited resource, i.e, they do not have enough information about what procedure or criteria the supplier will use to allocate the sales of the scarce resource to the claimants. In this sense, the procedure followed by the supplier to assign the resource would be a kind of black box for the producers, who would only know the resource demands of each one but would not be able to determine the result that the black box procedure would give. We use the context of production situations in which horizontal cooperation among manufacturers by utility transfers is possible, trading among companies is allowed and there is a fixed price per unit on the limited resource. As far as we know, this approach -when manufacturers do not know how the limited resource will be sold or allocated- has not been analyzed before. Our research aims to fill this gap in the literature by examining the following key questions:

a) How can the existence of only one supplier with a limited resource affect horizontal cooperation among manufacturers?

b) How does the uncertainty on the procedure used by the supplier to sell this limited resource affect the coalition formation of manufacturers?

In this paper we look for answers to the above questions from a cooperative game theoretical approach. The amount available of the limited resource plays a crucial role in the analysis of the associated games, because externalities can arise due to its possible scarcity. We distinguish two cases: when the cooperation of all manufacturers enables the limited resource to be sufficient for them and when it is not sufficient. In the former case, we can find stable allocations of the revenues, i.e. no subset of manufacturers will have an incentive to break away and act on its own. In the latter situation, cooperation does not always guarantee a stable allocation. The analysis of both cases is carried out taking into account that the exact expression of the corresponding game with externalities is unknown, because manufacturers do not know exactly how the limited resource is to be sold or assigned.

The paper is organized as follows. In Section 2 a review of the related literature is presented. Section 3 contains basic concepts on cooperative transferable utility games. In Section 4 the model of two-echelon chains with one supplier with a limited resource and n manufacturers and their corresponding games with externalities are studied. We show that if the limited resource is not a constraint for the production process of the grand coalition, then the existence of stable distributions of the total profit is guaranteed and consequently the formation of the grand coalition is a win-win deal. In Section 5 we assume that the limited resource is not sufficient to satisfy the production process of the grand coalition and the manufacturers do not know how the limited resource will be distributed. Therefore, we introduce resource games to deal with this uncertainty. Different points of view can be used to define these games. We study the two extremes: the optimistic and the pessimistic. In Section 6 we give some insights about coalition configurations in these situations. Section 7 concludes. All the proofs of the results are included in an Appendix.

2 Literature review

Non cooperative game theory to model interaction between stakeholders in supply chains has received much attention in the supply chain literature, but the cooperative approach has been less common. However, we can find a number of papers in the supply chain literature using cooperative games for analyzing different aspects of the interaction between the stakeholders related to collaboration from many different points of view. Cachon and Netessine (2006) review applications of mainly non cooperative game theory to supply chain analysis, and in Nagarajan and Sosic (2008) and Meca and Timmer (2008) the existing literature on applications of cooperative game theory to supply chain management is reviewed, particularly in profit allocation and stability. Nevertheless, our approach on stability differs from related papers reviewed in Cachon and Netessine (2006), Meca and Timmer (2008), and Nagarajan and Sosic (2008) in several ways as we consider a cooperative game model with externalities, a supplier with a limited resource and it is not known in advance how the limited resource will be sold. Netessine and Zhang (2005) analyze the impact of externalities on the supply chain performance regarding the stocking decisions of retailers in decentralized and centralized management, but not from a cooperative game theoretical point of view. In our analysis, we use a cooperative game model which incorporates externalities and additionally, we carry out the analysis regarding the production planning. Another difference is that in Netessine and Zhang (2005) the retailers purchase a product to the supplier that has unbounded capacity, but in our model manufacturers have to purchase a resource which may be scarce.

Supply chain collaboration has mainly been studied when the partners are the companies, their suppliers and customers, i.e. vertical collaboration. For example, among many others, Akçay and Tan (2008) consider the cooperation of producers that offer a set of products and propose a model that allows to identify the conditions of firms and products that would facilitate cooperation. Chen and Roma (2011) consider a two-level distribution channel with a producer and two retailers and show that, under linear demand curves, group buying is preferable for symmetric retailers. However, for the asymmetric case collaborative purchasing is beneficial to the less efficient player. Cho and Tang (2014) study different implications of the uniform distribution regarding the gaming effect and the benefit for the firms and the supply chain. Li et al. (2017) consider supplier-facilitated transshipment for achieving supply chain coordination in a single supplier, multi-retailer distribution system with non-cooperative retailers. They assume the supplier is an active participant in the system and implements transshipment through a bi-directional adjustment contract, where each retailer can either buy additional inventory or sell back excess inventory to the supplier. Liu et al. (2018) study how a central authority can stabilize the grand coalition for an unbalanced cooperative game, by means of a method that uses simultaneously penalization and subsidization. Meca and Sosic (2014) introduce the class of cost-coalitional problems that possess two distinguished types of agents: the so-called benefactors, a group of players whose participation in an alliance always contributes to the savings of all members, and the so-called beneficiaries, group of players whose cost decreases in such an alliance. They study the role played by these two types of agents in achieving stability.

However, horizontal collaboration has received little attention (Pomponi et al. (2013), Chen et al. (2017)). Although there is a growing number of works that use cooperative game theory to model the behaviour of the members in a supply chain. Krajewska et al. (2008) analyze the horizontal cooperation among freight carriers, combining routing, scheduling and cooperative game theory. They analyze the profit margings from horizontal cooperation and how to distribute them fairly among the partners. More recently, Mohebbi and Li (2015) analyze the problem of suppliers' coalition formation to share their capacities and then the fair distribution of the profit obtained from the cooperation among the collaborating suppliers. For this analysis, they develop a model for suppliers' coalition formation through a dynamical coalitional game approach. Roma and Perrone (2016) consider two identical firms that cooperate by sharing a joint venture and compete in setting prices or quantities. They investigate the consequences of using outcome-based versus ex ante-based sharing mechanisms. However, none of the previous papers studies the effect of externalities that can arise when coalitions are formed on the horizontal cooperation as we do in this paper and therefore they do not use cooperative games with externalities either.

Linear production games model production processes in which a group of players agree to share their resources in order to improve their profits. Owen (1975) show that these games always have a non empty core by constructing a stable allocation via a related dual linear program. Gellekom et al. (2000) named the set of all elements that can be found in this way, the Owen set. More general are the production situations studied in Granot (1986), Curiel et al. (1989), Fragnelli et al. (1999), Tijs et al. (2001) and Molina and Tejada (2006). Granot (1986) considers that the capacity of managing inputs for groups of manufacturers is not additive. The paper by Curiel et al. (1989) introduces a generalization involving committee control. In Fragnelli et al. (1999) and Tijs et al. (2001) an infinite number of production techniques is used to study relations between the core of the corresponding game and the Owen set. Molina and Tejada (2006) considers a refinement of the set of players to deal with the analysis of the core, i.e., the set of stable allocations in the model proposed by Curiel et al. (1989).

In our case, we are interested in addressing LP situations à la Owen, but with a supplier who has a limited resource which is essential to the manufacturers to produce. When a limited resource is introduced in an LP situation this leads to a linear production situation with a limited resource ($LPLR$ situation). Although it may intuitively seem that when introducing a small change in the LP model everything will work in a similar way, in this case it is not true because, for instance, the games that arise in these situations are games with externalities. Thus, our model implies the use of games with externalities (or in partition function form) introduced by Thrall and Lucas (1963), where the value of a coalition is determined by taking into account not only what the members of the coalition can do, but also what outsiders can do. The core of these games can be reduced to the core of a related game in characteristic function form and the existence of stable allocations is not always guaranteed. This is another difference when compared with the classic LP games.

Finally, among the papers that deal with the distribution of a limited resource and use a cooperative approach we should mention Funaki and Yamato (1999). They consider a concave production function with labor as the only input and the benefit as output. In this paper we consider a fixed proportion production function with several inputs including the limited resource for producing several outputs and the profit is obtained by solving a linear program. Moreover, in Funaki and Yamato (1999) all players are symmetric, but in our case they are asymmetric. In their model the resource is not bounded from above and the externalities are always negative. In our case, the externalities can be either positive or negative. In Gutierrez et al. (2017, 2018) the production situations considered are the same, but it is known which allocation rule the manager will use in order to distribute the quota of gas emissions. However, this paper deals with a more general framework by considering that it is not known in advance how the limited resource will be allocated. Therefore expectations of the different coalitions of manufacturers concerning the limited resource must be taken into account in order to analyze the problem of horizontal cooperation.

For all the points mentioned above, this paper is, to the best of our knowledge, novel and interesting since the analysis is performed considering externalities, either positive or negative, a supplier with a limited resource and it is unknown how the supplier will allocate the limited resource. This brings innovation to the analysis of horizontal collaboration in a supply chain regarding the production process planning.

3 Preliminaries

In order to deal with our model, we will need to consider transferable utility games with externalities introduced by Thrall and Lucas (1963). In a transferable utility game (TU-game) it is assumed that the utility can be linearly transferred among agents. Formally, let $N = \{1, 2, ..., n\}$ be a non empty finite set of agents who agree to coordinate their actions. Let $\mathcal{P}(N)$ denote the set of all partitions of N and $P = \{S_1, \dots, S_k\}$ represents one of these partitions or coalition structures, where the coalitions $S_1, \dots, S_k$ are disjoint and their union is N. The pair $(S|P)$ such that $S \in P$ is usually called an embedded coalition. A cooperative game with externalities is defined by $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$, where N is the set of players, $\mathcal{P}(N)$ denotes the set of all partitions of N and $V(S|P)$ with $S \in P$ is a real number that represents the profit that a coalition $S \subseteq N$ can obtain when P is formed, with $V(\varnothing|P) = 0$ for all $P \in \mathcal{P}(N)$. Note that the profit that a coalition can obtain depends on the coalitions formed by the other players in $P \in \mathcal{P}(N)$. If $V(S|P)$ does not depend on partition P, i.e. $V(S|P) = v(S)$ for a function $v : 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$, the game (N, v) has no externalities and is a TU-game in characteristic function form.

Given a partition $P \in \mathcal{P}(N)$, a vector $x \in \mathbb{R}^n$ is said to be feasible under P if it satisfies $\sum_{i\in S} x_i \leq V(S|P), \forall S \in P$. We denote by $\mathcal{F}^P$ the set of all feasible vectors under P and $\mathcal{F} = \cup_{P\in\mathcal{P}(N)}\mathcal{F}^P$ denotes the set of all feasible vectors. Given two vectors x, x' in $\mathbb{R}^n$, we say that x dominates x' through S and denote $x \; dom_S \; x'$ if the following conditions are satified:

1. $\sum_{i\in S} x_i \leq V(S|P), \forall P \in \mathcal{P}(N)$ such that $S \in P$,
2. $x_i > x'_i, \forall i \in S$.

We say that x dominates x' if there exists $S \subseteq N$ such that $x \; dom_S \; x'$, and denote $x \; dom \; x'$. The core of a cooperative game with externalities is defined by $C(V) = \{x \in \mathcal{F} \,|\, \nexists x' \in \mathcal{F} \text{ s.t. } x' \; dom \; x\}$. However, if we consider another definition of dominance, then we will obtain a different core. Thus, if we change condition 1 by

$$\bar{1}. \sum_{i\in S} x_i \leq V(S|P), \text{ for some } P \in \mathcal{P}(N) \text{ with } S \in P,$$

we obtain a more restrictive concept of dominance that we denote by $\overline{dom}$ and the corresponding core is defined as $\overline{C}(V) = \left\{x \in \mathcal{F} \,\middle|\, \nexists x' \in \mathcal{F} \text{ s.t. } x' \; \overline{dom} \; x\right\}$.

If $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$ is such that, $\forall S \subseteq N, \forall P \in \mathcal{P}(N)$ with $S \in P$, $V(S|P) = v(S)$, then the two definitions of dominance are equivalent, (N, v) is a game in characteristic function form and the above concepts of core reduce to the D-core for games in characteristic function form. However, the definition of core (Gillies, 1953) for games in characteristic function form is given by

$$C(v) = \left\{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N), \text{ and } \sum_{i\in S} x_i \geq v(S), \text{ for all } S \subset N\right\}.$$

Associated with each game with externalities two cooperative games in characteristic function form can be introduced: $\left(N, v^{min}\right)$ and (N, v^{max}), where

$$v^{min}(S) = \min\{V(S|P)\,|P \in \mathcal{P}(N) \text{ such that } S \in P\},$$
$$v^{max}(S) = \max\{V(S|P)\,|P \in \mathcal{P}(N) \text{ such that } S \in P\}.$$

$\left(N, v^{min}\right)$ represents a pessimistic point of view of the gain that a coalition S can achieve working on its own, while (N, v^{max}) can be seen as its optimistic counterpart. It is known (see, for instance, Funaki and Yamato (1999)) that if $V(\{N\}|N) > \sum_{i \in S} V(S|P), \forall P \in \mathcal{P}(N)$, then

a) $\underline{C}(V) = C(v^{min})$, and
b) $\overline{C}(V) = C(v^{max})$.

4 The model

We consider a two-echelon supply chain consisting of a set $N = \{1, \ldots, n\}$ of manufacturers, who produce a set $G = \{1, \ldots, g\}$ of goods from a set $Q = \{1, \ldots, q, q+1\}$ of resources by using the same fixed proportion technology described by the production matrix $A \in \mathcal{M}_{(q+1)\times g}$, where a_{tj} represents the amount of the resource t needed to produce product j, where $a_{(q+1)j} > 0 \quad \forall j \in G$, this means that resource $q+1$ is strictly necessary to produce, and there exists at least another resource $t \in Q \setminus \{q+1\}$ with $a_{tj} > 0 \quad \forall j \in G$. This means that we do not allow for output without input.

Regarding the inputs, there are several suppliers that supply the resources $1, \ldots, q$ which have no restrictions on their capacities, but supplier providing resource $q+1$ has a limited amount of the resource and the manufacturers cannot obtain that resource from any other supplier. This supplier has available an amount of r of the resource $q+1$ at a fixed unitary price c.

With respect to producers, each manufacturer i can manage a different bundle $b^i \in \mathbb{R}^q_+$ of resources in $Q \setminus \{q+1\}$, which s/he obtains from the suppliers without any restriction. These $b^i \in \mathbb{R}^q_+, i \in N$, represent technical limitations associated with manufacturers such as size, production organization or other characteristics of manufacturers. This introduces asymmetry in the problem, i.e., not all manufacturers are identical, they share technology but have their own technical characteristics that make them different from each other, in general. We denote by $B \in \mathcal{M}_{q\times n}$ the matrix of bundles of these resources.

Finally, the produced goods can be sold at given market prices $p \in \mathbb{R}^g_{++}$, i.e. the prices are exogenously given to the problem.

Some additional reasonable assumptions regarding the inputs and the outputs are the following. We assume that for each resource $t \in Q \setminus \{q+1\}$ there is at least one manufacturer i that uses it, i.e. $b^i_t > 0$. This means that all considered resources in the problem are necessary. We also assume that $p_j > a_{(q+1)j}c \quad \forall j \in G$, in order to deal with a profitable production process.

A scheme of the structure of the model is shown in Figure 1.

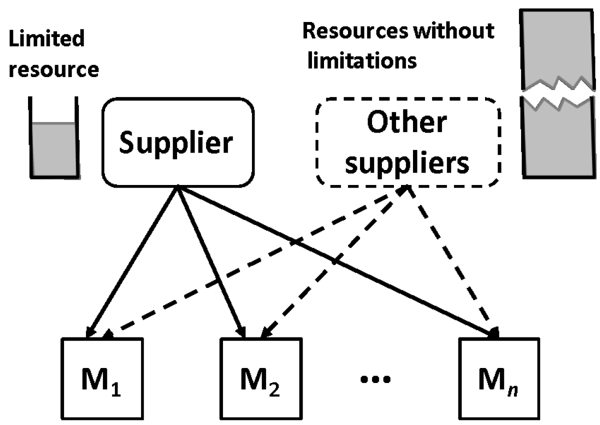

Fig. 1 Structure of the two-echelon supply chain with a limited resource.

Therefore, a linear production with a limited resource situation ($LPLR$) can be described by a 5-tuple (A, B, p, r, c). Note that in this model suppliers with enough resources for all manufacturers are not consider because they do not introduce any restriction in the problem.

Now we consider that manufacturers can collaborate in order to result in a better performance than it would without this cooperation (see, Soosay et al., 2008). We assume that manufacturers can collaborate sharing resources and purchasing jointly the limited resource. This collaboration will be based on shared rewards that yield a win-win deal. Therefore the study of the existence of such reward distributions is an interesting issue in this context. If a coalition S of manufacturers cooperates, then they put all their resources together, $b^S = \sum_{i \in S} b^i$, and so given this amount of resources, the coalition maximizes its profit by solving the following linear program,

$$\begin{array}{ll} \max & \sum_{j=1}^{g} p_j x_j - cz \\ \text{s.t:} & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{array} \tag{1}$$

Thus, an optimal production plan $(x; z)$ for coalition S is an optimal solution of this linear program. With an abuse of notation, we use z to represent the amount of the limited resource that a manufacturer or a group of manufacturers will need. We denote by $value\,(S; z)$ the value of this linear program, for every fixed z and by $d_S = \min\{z \in \mathbb{R}_+ \,|value\,(S; z) \text{ is maximum}\}$, the optimal demand of the limited resource. We should point out that these optimal demands are the desired amount of the limited resource for each coalition S

and can be seen as their greatest aspirations. Note that they are not bounded from above by r.

It is easy to prove that once we know that a positive profit is achieved, all the lower levels of the limited resource also provide positive profits. In the sequel, we will assume that for all S, there is a feasible production plan $(x; z)$ such that $value\,(S; z) > 0$. This implies that $d_S > 0$.

Let us assume that a partition P of manufacturers is formed and the amount of the limited resource finally sold to coalition $S \in P$ by the supplier is $z_S(P)$. Therefore, the profit that a coalition $S \subseteq N$ can obtain is given by solving problem (1) taking $z = z_S(P)$. We should point out that $z_S(P)$ is bounded from above by r, because the supplier cannot sell more than this amount unlike d_S, for all $S \subseteq N$. Furthermore, $z_S(P)$ will be less or equal to its optimal demand d_S, because it does not make sense that they receive more than they demand.

For each partition P we will obtain a different set of demands of the limited resource. Therefore, the amount of the limited resource that a coalition of manufacturers can obtain will depend on the coalitions that make up the rest of the manufacturers. Associated with these situations we can define cooperative games, but these games will be cooperative games with externalities. Formally, this can be expressed as follows.

Definition 1 Let (A, B, p, r, c) be an $LPLR$ situation. The game with externalities associated with this situation is given by $\left(N, \mathcal{P}(N), \{V\,(\bullet|\,P)\}_{P\in\mathcal{P}(N)}\right)$, where N is the set of manufacturers, $\mathcal{P}(N)$ denotes the set of all partitions of N and $V\,(S|\,P)$ with $S \in P$ is obtained from linear program (1), for all $S \subset N$, where z represents the amount of the limited resource allocated to coalition S when partition P is formed.

All elements in Definition 1 are perfectly known for all manufacturers except V, because it will depend on the specific allocation mechanism used by the supplier with the limited resource, but this is not revealed by the supplier or the supplier has not yet made a decision on the procedure to be used when manufacturing coalitions are formed.

However, despite the lack of knowledge of the manufacturers, the next result shows that the total profit that all manufacturers can obtain by collaborating all together is at least as good as the aggregate profit that any partition of manufacturers can obtain. This means that, in this kind of situations, the full collaboration of manufacturers is, in principle, profitable for all of them, i.e. it could be a win-win agreement.

Proposition 1 *Let (A, B, p, r, c) be an $LPLR$ situation with associated game with externalities $\left(N, \mathcal{P}(N), \{V\,(\bullet|\,P)\}_{P\in\mathcal{P}(N)}\right)$. Then,*

$$V\,(N|\,\{N\}) \geq \sum_{S\in P} V\,(S|\,P)\,, \forall P \in \mathcal{P}(N).$$

It is easy to check that $C(V)$ and $\overline{C}(V)$ only include efficient allocations. The following result is given without a proof because it can be derived in a similar manner as in Funaki and Yamato (1999).

Corollary 1 *Let (A, B, p, r, c) be an LPLR situation, with associated game with externalities $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$ and $\left(N, v^{min}\right)$, (N, v^{max}) the related games in characteristic function form. Then, $C(V) = C\left(v^{min}\right)$ and $\overline{C}(V) = C(v^{max})$.*

The amount of the limited resource will play a crucial role in the analysis of $LPLR$ situations as it is shown in the rest of the paper. We start with the cases when the limited resource is not a constraint on the production process or it is only a restriction for the grand coalition. Before presenting the results we introduce the following notation. Given a partition $P = \{S_1, \ldots, S_k\}$, its total aggregate demand is $d(P) = \sum_{i=1}^{k} d_{S_i}$.

Proposition 2 *Let (A, B, p, r, c) be an LPLR situation with associated game $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$.*

(i) If $d(P) \leq r$ for all P, then $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$ is a game in characteristic function form.

(ii) If $d_N > r$ and $d(P) \leq r$, for all $P \in \mathcal{P}(N)$, $P \neq \{N\}$, then $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$ is a game in characteristic function form.

Let (A, B, p, r, c) be an $LPLR$ situation. The characteristic function form game associated with one of the two previous situations (N, v), where $d_N \leq r$ or $d_N > r$ and $d(P) \leq r$ for all $P \neq \{N\}$, is given by $v(S) = value(S; z)$, with $z = d_S$ for all $S \neq N$ and $z = \min\{d_N, r\}$ for the grand coalition N. This is due to the fact that the limited resource is sufficient to satisfy the demands for all $S \neq N$, but for N the maximum amount available is r.

The next result shows that the characteristic function form game obtained when the limited resource is not a constraint for the production process has a non empty core.

Theorem 1 *Let (A, B, p, r, c) be an LPLR situation where $d(P) \leq r$ for all $P \in \mathcal{P}(N)$. The characteristic function form game (N, v) associated with this situation has a non empty core.*

Corollary 2 *Let (A, B, p, r, c) be an LPLR situation where $d(P) \leq r$ for all $P \in \mathcal{P}(N)$, $\left(N, \mathcal{P}(N), \{V(\bullet|P)\}_{P\in\mathcal{P}(N)}\right)$ the corresponding game with externalities and (N, v) the related game in characteristic function form. Then, $C(V) = \overline{C}(V) = C(v)$.*

As a result of Theorem 1 and Corollary 2, we have that in an $LPLR$ situation where the limited resource is not a constraint on the production process, the existence of stable distributions of the total profit is guaranteed and therefore it is possible for manufacturers to achieve a win-win agreement and consequently cooperation among all of them is feasible.

Following the idea introduced by Gellekom et al. (2000), we can define the Owen set of an $LPLR$ situation (A, B, p, r, c) , $Owen\,(A, B, p, r, c)$, as the set whose elements can be obtained through an optimal solution of the dual problem

$$\begin{array}{ll} \min & \sum_{t=1}^{q} b_t^N y_t + 0 y_{q+1} \\ \text{s.t:} & A^t y \geq p \\ & y_{q+1} \leq c \\ & y \geq \mathbf{0}_{q+1}, \end{array} \tag{2}$$

associated with the optimal solution of (1) $\left(x^N; d_N\right)$ such that $d_N \leq r$. To sum up, when the limited resource is not a constraint on the production process one way to obtain a stable distribution of the total profit is to use an element of the so-called Owen set of the $LPLR$ situation (A, B, p, r, c) . We should mention that this is similar to the classic results in the LP situations. However, it does not always work in the same way. Although the games such as those in Proposition 2 (ii) are characteristic function form games, they can have an empty core, as the following example shows.

Example 1 Let (A, B, r, p, c) be an $LPLR$ situation, with three manufacturers, $N = \{1, 2, 3\}$, who produce three products from three resources and a limited resource, where

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 6 & 6 & 6 \\ 5 & 10 & 6 \\ 2 & 4 & 4 \end{bmatrix}, B = \begin{bmatrix} 15 & 6 & 9 \\ 4 & 18 & 9 \\ 16 & 19 & 2 \end{bmatrix}, p = \begin{pmatrix} 10 \\ 9 \\ 9 \end{pmatrix}, c = 2, r = 10.$$

The demands are

$$\begin{array}{l} d_{\{1\}} = \frac{4}{3}, d_{\{2\}} = 4, d_{\{3\}} = \frac{4}{5}, \\ d_{\{12\}} = \frac{22}{3}, d_{\{13\}} = \frac{17}{3}, d_{\{23\}} = \frac{42}{5}, d_N = \frac{31}{3}, \end{array}$$

and $\min\left\{\frac{31}{3}, 10\right\} = 10$. Since the limited resource is only a restriction for the grand coalition, the corresponding TU-game (N, v) associated with this situation is given by

$$\begin{array}{l} v\left(\{1\}\right) = 4, v\left(\{2\}\right) = 12, v\left(\{3\}\right) = \frac{12}{5}, \\ v\left(\{12\}\right) = 22, v\left(\{13\}\right) = 13, v\left(\{23\}\right) = \frac{126}{5}, v\left(N\right) = 30, \end{array}$$

and $C\,(v) = \varnothing$. Obviously, $C\,(V) = \overline{C}\,(V) = \varnothing$. $\square$

Now we continue with the case in which the limited resource is enough to satisfy the demand of the grand coalition, i.e. $d_N \leq r$. In this case, manufacturers do not need to know how the supplier will allocate the limited resource, because the cooperation of all manufacturers is a win-win deal for them and therefore they behave as just a single agent, as the following theorem shows.

Theorem 2 [1] *Let (A, B, p, r, c) be an LPLR situation with associated game with externalities $\left(N, \mathcal{P}(N), \{V(\bullet| P)\}_{P\in\mathcal{P}(N)}\right)$. If $d_N \leq r$, then $\overline{C}(V) \neq \varnothing$.*

This result is important for several reasons. First, we have found a case in which both cores of the game with externalities are non empty (note that $\overline{C}(V) \subseteq C(V)$). This is relevant because it means that collaboration can be based on shared rewards that yield a win-win deal. Second, it shows that cooperation among all manufacturers is important when it prevents the limited resource from being scarce. Third, from the proof of Theorem 2 given in the Appendix, we observe that one procedure to obtain allocations of the total profit among the manufacturers is by considering elements in the Owen set. This procedure consists of paying each agent a price (the value that the corresponding dual variable takes in the optimal solution) for each resource $t \in Q\setminus\{q+1\}$ that s/he can obtain or manage. This result is similar that done for LP games (Owen, 1975), but the difference is that the price paid for each of these resources has implicitly discounted the cost of purchasing the amount d_N of the limited resource necessary to produce optimally.

In the more general case, when the limited resource could be a constraint on the production process for some partition and it is not enough to satisfy the demand of the grand coalition. In this situation, each coalition of manufacturers will obtain an amount of this resource from a certain procedure decided by the supplier. Regarding the information that manufacturers have on this procedure, we consider that they do not know the way in which r will be sold or allocated because of any of the reasons already aforementioned. The next section will tackle this situation.

5 The resource game

In this section we assume that $d_N > r$ and manufacturers do not know how the limited resource is to be sold or allocated. Therefore, they cannot evaluate the game with externalities V. Thus, they can examine the problem of the amount they will receive from different points of view. We consider that what a coalition of manufacturers S expects to receive from the limited resource can be described by means of a resource game (N, R)[2]. These resource games are cooperative TU-games in characteristic function form and can be defined following different approaches. There are two extreme cases, depending on which point of view is used to deal with the situation, the optimistic and the pessimistic resource games, that will be addressed in this section. Hence, $R(S)$

[1] Similarly, it can be proved that the result holds when we consider $v^{opt}(S) = min\{d_S, r\}$ instead of v^{max}, i.e. $C(v^{opt}) \neq \varnothing$. Note that $v^{opt} \geq v^{max}$, but not equal in general. Therefore, when $d_N \leq r$ it will not be difficult to achieve the agreement of the grand coalition.

[2] Resource games are also used in Granot (1986), but the idea behind the resource game is different. A resource game in Granot (1986) measures the amount of a resource available to each coalition and these amounts are perfectly determined and known for all agents. In our case, the resource game measures the expectation of the different coalitions regarding the amount of limited resource they can obtain.

is what coalition S thinks it can guarantee from the limited resource working on their own. In principle, it could be any value between those obtained from the optimistic and pessimistic points of view.

Once a coalition of manufacturers S receives its share of the resource, $R(S)$, using this amount as z in (1), for all $S \subseteq N$, the $LPLR$ game $\left(N, v^R\right)$ is obtained, where $v^R(S) = value\,(S; R(S))$. In this way, the game associated with the $LPLR$ situation (A, B, r, p, c) obtained from the resource game (N, R) reduces to a characteristic function form game, $\left(N, v^R\right)$, since it does not depend on what the others may do.

The following theorem states a sufficient condition for the $LPLR$ game $\left(N, v^R\right)$ to have a non empty core, no matter from what point of view the resource game (N, R) is defined.

Theorem 3 *Let (A, B, p, r, c) be an LPLR situation, let (N, R) be a resource game associated with it and $\left(N, v^R\right)$ the corresponding LPLR game. When $d_N > r$, if $C(R) \neq \varnothing$, then $C\left(v^R\right) \neq \varnothing$.*

However, the opposite is not true in general as the next example shows.

Example 2 Let (A, B, r, p, c) be an $LPLR$ situation, with two manufacturers, $N = \{1, 2\}$, who produce three products from two resources and a limited resource, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, p = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}, c = 1, r = 4.$$

Consider the resource game, (N, R), such that $R(\{1\}) = R(\{2\}) = R(\{12\}) = 4$. In this case, $C(R) = \varnothing$, $d_N = 5$ and $v^R(\{1\}) = v^R(\{2\}) = 10, v^R(N) = 28$, thus $C\left(v^R\right) \neq \varnothing$.□

The use of the resource game simplifies the mathematical analysis of the problem because we obtain a game without externalities. However, the resource game is derived based on the expectations of the different coalitions of manufacturers, which implies that the interest in collaborating is based on expectations not on real data. Nevertheless, this approach is very useful when there is uncertainty as is the case. Therefore, Theorem 3 establishes that when the expectations on the allocation of the limited resource provides stable results ($C(R) \neq \varnothing$), then the coalition of all manufacturers is achievable based on shared rewards that yield a win-win deal ($C\left(v^R\right) \neq \varnothing$). Furthermore, this ex ante analysis is also valid ex post, because once the manufacturers achieve the agreement of cooperation, they will not be able to know what would have happened if other coalition structures had been formed, because they do not have information about the procedure the supplier uses to sell or allocate the limited resource. Therefore, if manufacturers are able to define a resource game, which will be common knowledge for all of them, then the analysis will be consistent with their expectations.

In the next two sections we study the two extreme cases for resource games: the case of optimistic expectations and the case of pessimistic expectations. It is reasonable to expect that whatever resource game we can define will be between these two extremes, in the following sense:

$$R^{pes}(S) \leq R(S) \leq R^{opt}(S), \forall S \subseteq N.$$

This means that the expectations about the amount of limited resource each coalition can obtain will be between its pessimistic and optimistic expectations.

5.1 The optimistic approach

From an optimistic point of view, a coalition of manufacturers S will obtain its demand. The related resource game (N, R^{opt}) is such that $R^{opt}(S) = \min\{d_S, r\}$. This game can also be derived if manufacturers think that the supplier will use a first-in-first-out (FIFO) procedure, then the optimistic case for a coalition S is to be the first in purchasing the limited resource. Now it is easy to check that the resource game we obtain by using the former reasoning is precisely $R^{opt}(S) = \min\{d_S, r\}$.

Using the amount $R^{opt}(S)$ as z in (1), for all $S \subseteq N$, the optimistic *LPLR game* (N, v^{opt}) is derived.

The core of this class of games can be non empty, as Example 2 illustrates, but it can be empty as the next example shows.

Example 3 Let (A, B, r, p, c) be an $LPLR$ situation, with three manufacturers, $N = \{1, 2, 3\}$, who produce two products from two resources and a limited resource, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 40 & 60 & 80 \\ 60 & 40 & 50 \end{bmatrix}, p = \begin{pmatrix} 50 \\ 60 \end{pmatrix}, c = 14, r = 50$$

and (N, v^{opt}) the related optimistic *LPP game*. In this case, the core of the optimistic *LPLR game* will be all the points in $\mathbb{R}^3$ such that

$$\begin{aligned} & x_1 \geq 720, x_2 \geq 920, x_3 \geq 1150, \\ & x_1 + x_2 \geq 1640, x_1 + x_3 \geq 1936, x_2 + x_3 \geq 2070, x_1 + x_2 + x_3 = 2300, \end{aligned}$$

but it can be seen that there is no point satisfying all the above inequalities, then $C(v^{opt}) = \varnothing$. Taking into account that

$$\begin{aligned} & d_{\{1\}} = 20, d_{\{2\}} = 20, d_{\{3\}} = 25, \\ & d_{\{12\}} = 40, d_{\{13\}} = 46, d_{\{23\}} = 45, d_N = 66 \text{ and } \min\{66, 50\} = 50, \end{aligned}$$

it is easy to check that $C(R^{opt}) = \varnothing$. □

At a first sight, it seems that an easy condition to assure that the core of the optimistic $LPLR$ game is empty, when $d_N > r$, could be $\exists P \in \mathcal{P}(N)$ such that $\sum_{S \in P} d_S > r$. However, it is not true as the next example shows.

Example 4 Let (A, B, r, p, c) be the LPP situation described in Example 2. In this case, $d_{\{1\}} = d_{\{2\}} = 7, d_N = 5$, $v^{opt}(1) = v^{opt}(2) = 10$ and $v^{opt}(N) = 28$. Thus, there is a partition $P = \{\{1\}, \{2\}\}$ where $d_{\{1\}} + d_{\{2\}} > 4$ and the core is non empty. Therefore, the aforementioned condition does not guarantee that the core is empty.

When $d_N > r$ and, $\forall P \in \mathcal{P}(N)$, $\sum_{S \in P} d_S < r$ the core of the optimistic game can be empty as Example 1 shows. But it does not hold in general as the following example illustrates.

Example 5 Let (A, B, r, p, c) be an LPP situation, with three manufacturers, $N = \{1, 2, 3\}$, who produce three products from three resources and a limited resource, where

$$A = \begin{bmatrix} 10 & 8 & 7 \\ 7 & 10 & 5 \\ 3 & 6 & 7 \\ 5 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 9 & 6 & 8 \\ 5 & 18 & 6 \\ 17 & 13 & 3 \end{bmatrix}, p = \begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}, c = 1, r = 5.$$

The corresponding optimistic game is

$$v^{opt}(1) = 1.5, \quad v^{opt}(2) = 5.25, \quad v^{opt}(3) = 3.5,$$
$$v^{opt}(12) = 13.125, v^{opt}(13) = 7.7, v^{opt}(23) = 12.25,$$

and $v^{opt}(N) = 17.5$. The demands are

$$d_{\{1\}} = 1, \quad d_{\{2\}} = 1.5, \quad d_{\{3\}} = 1,$$
$$d_{\{12\}} = 3.75, d_{\{13\}} = 2.2, d_{\{23\}} = 3.5,$$

with $d_N > 5$ and the core is non empty.

Hence, when $d_N > r$ we have from Theorem 3 a condition which is sufficient for the non emptiness of the core. Nevertheless, in general, it is not clear whether the core of the optimistic game is empty or not.

5.2 The pessimistic approach

From a pessimistic point of view, a coalition of manufacturers S will receive what agents outside S leave using the partition that minimizes the remainder for S. This situation can be described as a resource game (N, R^{pes}), where

$$R^{pes}(S) = \min \left\{ \min_{P:S \in P} \left\{ \left(r - \sum\nolimits_{\substack{T \in P \\ T \neq S}} d_T \right)_+ \right\}, d_S \right\}.$$

As in the optimistic case, this game can also be derived if manufacturers think that the supplier will use a FIFO procedure; then the pessimistic case for a coalition S is to be the last in purchasing the limited resource when the most demanding partition P such that $S \in P$ is formed. Now it is easy to check that the resource game we obtain by using the former reasoning is precisely R^{pes}.

Using this amount $R^{pes}(S)$ as z in (1), for all $S \subseteq N$, the pessimistic $LPLR$ game (N, v^{pes}) is obtained.

When $d_N > r$ the core of the $LPLR$ pessimistic game can be empty as Example 1 shows and, therefore, $C(v^{max}) = \overline{C}(V) = C(v^{min}) = C(V) = \varnothing$. We should point out that in Example 1 the optimistic and pessimistic games coincide. The following result states a condition for the non emptiness of the core of the pessimistic game.

Theorem 4 *Let (A, B, p, r, c) be an LPLR situation and (N, v^{pes}) the pessimistic LPLR game associated with it. If $d_N > r$ and $\sum_{i \in N} d_i \geq r$, then $C(v^{pes}) \neq \varnothing$.*

This result is strong because the pessimistic resource game is very conservative in the sense that $v^{pes} \leq v^{min}$ but not equal in general. Therefore, it is easy to guarantee the existence of stable outcomes under lower expectations of the manufacturers. However, even in this case we need an extra condition, $\sum_{i \in N} d_i \geq r$, which is not negligible.

When $d_N > r$ and $\sum_{i \in N} d_i < r$, the core of the pessimistic game can be empty as in Example 1 or it can be non empty as Example 5 illustrates, where the core of the optimistic game is non empty and the core of the pessimistic one is also non empty. Thus, in this case we have obtained similar results to those in which we apply the optimistic approach.

Another situation in which the pessimistic core is non empty is when all manufacturers are symmetric, i.e. $b^i = b \in \mathbb{R}^q_+, \forall i \in N$, as the following theorem shows.

Theorem 5 *Let (A, B, p, r, c) be an LPLR situation such that $b^i = b \in \mathbb{R}^q_+, \forall i \in N$. The core of its associated pessimistic game, (N, v^{pes}), is always non empty, i.e. $C(v^{pes}) \neq \varnothing$.*

6 Coalition configurations. A first look

The results obtained in the previous section show that the grand coalition will not be always formed based on the distribution of profits among all manufacturers by means of side payments because the core can be empty. In fact, the non emptiness of the core is only guaranteed in some cases. Therefore, it is reasonable to think that other coalition configurations than the grand coalition will be formed between manufacturers. What configurations of coalitions will occur? The answer is not simple, but we next provide a definition of stability that allows us, in some sense, to extend the concept of the core which we are using in this paper as key for cooperation to take place.

Definition 2 Let (N, v) be a cooperative game. A coalition configuration $P \in \mathcal{P}(N)$ is said to be *cc-stable* if the following two conditions hold $\forall S \in P$

(1) $C(v_S) \neq \varnothing$, *and*

(2) $\nexists \{T_k\}_{k=1}^{l} \in P$ *such that* $C\left(v^{S \cup \{\bigcup_{k=1}^{l} T_k\}}\right) \neq \varnothing$,

where (S, v_S) *is the game reduced to coalition* S when coalition configuration P is formed.

Note the following. First, a coalition configuration is nothing other than a partition of all the agents. Second, this definition of stability holds for games in partition function form (with whatever concept of dominance) and in characteristic function form. Finally, when the core of a game is non empty, then the grand coalition is the only one which is cc-stable.

If we now focus on the problem at hand, and use the resource game, which is the one that collects the expectations of the different coalitions with respect to the limited resource, the configurations of coalitions that can be formed will depend on these expectations.

One possible approach would be that the coalitions that were formed tended to be as large as possible. In this sense, we introduce the concept of maximally stable coalition.

Definition 3 Let (N, v) be a cooperative game. A coalition S is said to be *maximally stable* if the following two conditions holds:

1. $C(v_S) \neq \varnothing$.
2. $C(v_{S \cup T}) = \varnothing, \forall T \subseteq N \backslash S$.

However, the previous definitions are not enough to analyze the situation under a limited resource, because when a coalition configuration forms, the sum of the expectations of each coalition must be compatible with the availability of the scarce resource. In this sense, we say that a coalition configuration, $\{S_1, ..., S_k\}$, is *compatible* with the available limited resource, r, if $\sum_{j=1}^{k} R(S_j) \leq r$.

Theorem 6 *Let* (A, B, p, r, c) *be an LPLR situation and* (N, v^{pes}) *the pessimistic game associated with it. There always exists at least one cc-stable compatible coalition configuration.*

The proof of this result is constructive and shows how to obtain a cc-stable compatible coalition configuration based on the concept of maximally stable. It also looks like the way to build the Dutta-Ray solution (Dutta and Ray, 1989).

7 Conclusions

First of all, the relation between the total amount of limited resource, r, and the demand of the limited resource for the grand coalition, d_N, plays an important role in providing the answers to the key questions posed in the Introduction. First, the presence of a limited resource leads to the emergence of externalities. These externalities hinder the possibility that the grand coalition is formed, because it is not possible to achieve stable distributions of the total profit obtained from the collaboration of all manufacturers, i.e. the cores of these games can be empty. This result contrasts with those obtained, for example, in Owen (1975) and Funaki and Yamato (1999). However, when horizontal cooperation among all manufacturers prevents the limited resource from being scarce, $d_N \leq r$, then the cores of the corresponding game are always non empty. Therefore, it is possible for all manufacturers to achieve a win-win agreement. When $d_N > r$, the uncertainty about how the supplier will sell or allocate the limited resource comes to play a relevant role in the analysis. This uncertainty compels us to analyze the problem from the point of view of expectations. In this sense we obtain that if a resource game based on the expectations of the different coalitions of manufacturers has nonempty core, then the formation of the grand coalition is possible based on those expectations. The converse of this result does not hold in general. The answers to the posed questions open a door to analyze the coalition structures that could be formed using concepts of stability related to coalition structures (see, for example, Aumann and Dreze, 1974) or new ones, as the used in this paper, which can better reflect the situations analyzed.

Therefore, the existence of a limited resource has significant implications on the configuration of alliances among the manufacturers in order to enhance their profits. This fosters the no collaboration among all manufacturers, because this makes it difficult to achieve a final distribution of the profit after compensations and side payments that satisfy the interests of all the parties. A sharp decline in the supply of limited resources can lead to a reconfiguration of alliances between manufacturers. For example, a cooperative that groups several manufacturers can be broken as a result of the scarcity of a resource, in the sense that it may be more difficult to continue collaborating based on a distribution of profits depending on the expectations of the stakeholders especially when they are asymmetric. And conversely, a sharp rise in the supply of limited resources may favor the formation of the grand coalition, and that this may result in a cooperative or other form of stable business collaboration among the stakeholders, regardless of the asymmetries between them.

Finally, in view of the aforementioned comments, if the supplier of the limited resource were also a kind of regulator, it could influence the number of alliances or mergers that could be established between manufacturers by modifying the scarce resource supply. In addition, it could influence the mix of products manufactured by manufacturers through variations in the cost of limited resources.

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Appendix

In this appendix we include all proofs of the results given throughout the paper. This is done in order to improve the readability of the paper for those readers not interested in the proofs, but only in the results and insights provided on horizontal cooperation of manufacturers when there is a limited resource.

Proof of Proposition 1

Given $P \in \mathcal{P}(N), V\left(S|\, P\right) = value\left(S; z_S(P)\right), \forall S \in P$ such that $\sum_{S\in P} z_S(P) \leq r$. Let be $\left(x^S; z_S(P)\right)$ an optimal plan for each coalition $S \in P$. Thus, $Ax^S \leq \begin{pmatrix} b^S \\ z_S(P) \end{pmatrix}$ and

$$A\left(\sum_{S\in P} x^S\right) \leq \begin{pmatrix} \sum_{S\in P} b^S \\ \sum_{S\in P} z_S(P) \end{pmatrix} \leq \begin{pmatrix} b^N \\ r \end{pmatrix}.$$

Then, $\left(\sum_{S\in P} x^S; \sum_{S\in P} z_S(P)\right)$ is a feasible production plan for N and

$$\sum_{S\in P} value\left(S; z_S(P)\right) \leq value\left(N; \sum_{S\in P} z_S(P)\right) \leq V\left(N|\, \{N\}\right).$$

■

Proof of Proposition 2

(i) We know that $z_S(P) = d_S$ for all $S \subseteq N$, due to $d(P) \leq r$ for all P. Therefore, for each $S \subseteq N$, $V(S|P) = V(S|P')$ for all $P, P' \in \mathcal{P}(N)$ such that $S \in P$, i.e. for each coalition the value does not depend on the coalitions formed by other players.

(ii) Since $d_N > r$ and $d(P) \leq r$, for all $P \in \mathcal{P}(N)$, $P \neq \{N\}$, similarly to (i), for each $S \subseteq N, V(S|P) = V(S|P')$ for all $P, P' \in \mathcal{P}(N)$, i.e. the value of coalition S does not depend on the coalitions formed by other players. On the other hand, the value of the grand coalition

$$\begin{array}{ll} \max \sum_{j=1}^{g} p_j x_j - cr & \\ \text{s.t:} \;\; Ax \leq \begin{pmatrix} b^N \\ r \end{pmatrix} & \\ \qquad x \geq \mathbf{0}_g . & \end{array} \tag{3}$$

only depends on its own, since there is no partition including N as a proper subset. Therefore, the related game has no externalities. ■

Proof of Theorem 1

The dual problem of (1)[3] for the grand coalition, N, is

$$\begin{array}{ll} \min \sum_{t=1}^{q} b_t^N y_t + 0 y_{q+1} & \\ \text{s.t:} \;\; A^t y \geq p & \\ \qquad y_{q+1} \leq c & \\ \qquad y \geq \mathbf{0}_{q+1} . & \end{array} \tag{4}$$

An optimal solution of (1) for the grand coalition N is given by $\left(x^N; d_N\right)$ with $d_N \leq r$, and the related dual optimal solution is $\left(y_q^N; y_{q+1}^N\right)$, where with an abuse of notation from now on, we represent by y_q^N the vector $\left(y_1^N, ..., y_q^N\right)$. From duality, it is known that

$$\sum_{j=1}^{g} p_j x_j^N - c d_N = \sum_{t=1}^{q} b_t^N y_t^N + 0 y_{q+1}^N = v(N) .$$

Therefore, somehow, the cost of the common resource is charged to (discounted from) the value of the resources. It is easy to check that $\left(y_q^N; y_{q+1}^N\right)$ is feasible in the dual problem of (1) for every coalition $S \subset N$. Moreover, we have that

[3] We use this problem because it is known by hypotesis that $\exists z \leq r$ for all problems, i.e. the limited resource is not scarce in any case.

for a dual optimal solution $\left(y_q^S; y_{q+1}^S\right)$ associated with the optimal solution $\left(x^S; d_S\right)$, it holds that

$$\sum_{t=1}^{q} b_t^S y_t^N + 0y_{q+1}^N \geq \sum_{t=1}^{q} b_t^S y_t^S + 0y_{q+1}^S = v\left(S\right).$$

Thus, $\sum_{i\in S}\left(\sum_{t=1}^{q} b_t^i y_t^N + 0y_{q+1}^N\right) \geq v\left(S\right), \forall S \subset N$, and this implies that $\left(b^i y^N\right)_{i\in N} \in C\left(v\right)$. ■

Proof of Theorem 2

Before giving the proof of Theorem 2 we need the following technical lemma, which is given without a proof because it is easy to derive. It give us two linear programs that, although they have different optimal solution sets, also have the same optimal values, i.e. they are optimally equivalents. Note that an optimal solution of the second one is the optimal demand of the limited resource for each coalition S, d_S. We should highlight that they only differ in a redundant constraint, $z \leq d_S$, however, this is the key with which to prove the next theorem.

Lemma 1 *Let (A, B, p, r, c) be an LPLR situation. For all S, the following linear programs are optimally equivalents ,*

$$\begin{array}{ll} \max & \sum_{j=1}^{g} p_j x_j - cz \\ s.t.\!: & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & z \leq d_S \\ & x \geq \mathbf{0}_g, z \geq 0. \end{array} \tag{5}$$

$$\begin{array}{ll} \max & \sum_{j=1}^{g} p_j x_j - cz \\ s.t.\!: & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0. \end{array} \tag{6}$$

The previous result provides us with two different, but equivalent, ways in which to tackle the linear programs. In the proof of the following theorem, we use one or the other depending on which will be more helpful.

Proof of Theorem 2. Since $d_N \leq r$, consider the linear program (5) for the grand coalition,

$$\begin{array}{ll} \max & \sum_{j=1}^{g} p_j x_j - cz \\ \text{s.t:} & Ax \leq \begin{pmatrix} b^N \\ z \end{pmatrix} \\ & z \leq d_N \\ & x \geq \mathbf{0}_g, z \geq 0. \end{array} \tag{7}$$

its dual is given by

$$\begin{array}{ll} \min \sum_{t=1}^{q} b_t^N y_t + 0y_{q+1} + d_N y_{q+2} & \\ \text{s.t:} \;\; A^t y \geq p & \\ \quad\;\; y_{q+1} - y_{q+2} \leq c & \\ \quad\;\; y \geq \mathbf{0}_{q+2}. & \end{array} \tag{8}$$

Let $\left(x^N; d_N\right)$ and $\left(y_q^N, y_{q+1}^N, 0\right)$ be primal and dual optimal solutions for (7) and (8), respectively with $d_N \leq r$ and $y_{q+2}^N = 0$. Effectively, by the lemma we know that linear problems (1) and (5) are optimally equivalent, and therefore their dual problems will also be optimally equivalent. Now if $\left(y_q^N, y_{q+1}^N\right)$ is an optimal solution of the dual problem (4) associated with the linear problem (1), then $\left(y_q^N, y_{q+1}^N, 0\right)$ is an optimal solution of the dual problem (5). Therefore, we can take a dual optimal solution with $y_{q+2}^N = 0$ associated with the optimal solution $\left(x^N; d_N\right)$.

Now, it is easy to check that $\left(y_q^N, y_{q+1}^N, 0\right)$ is a feasible solution for the dual problem of (5) for every coalition S. If $\left(y_q^S, y_{q+1}^S, y_{q+2}^S\right)$ is an optimal dual solution associated with $\left(x^S; d_S\right)$, it holds that

$$\sum_{t=1}^{q} b_t^S y_t^N + 0y_{q+1}^N + d_S y_{q+2}^N \geq \sum_{t=1}^{q} b_t^S y_t^S + 0y_{q+1}^S + d_S y_{q+2}^S = value(S; d_S) \geq V(S \mid P),$$

for all $P \in \mathcal{P}(N)$. Therefore, $\sum_{i \in S} \left(\sum_{t=1}^{q} b_t^i y_t^N\right) \geq V(S \mid P)$, $\forall S \subseteq N$, and for all $P \in \mathcal{P}(N)$ such that $(S \mid P)$ is an embedded coalition, and this implies that $\left(b^i y^N\right)_{i \in N} \in \overline{C}\,(V)$. ■

Proof of Theorem 3

Since $C\,(R) \neq \varnothing$, there is $u \in \mathbb{R}^N$ such that $u(S) = \sum_{i \in S} u_i \geq R\,(S)$, for all S, and $u(N) = r$. Let y^* be an optimal solution of the dual problem of (3). From duality theory, we know that $\sum_{t=1}^{q} b_t^N y_t^* + r y_{q+1}^* - cr = v^R(N)$. On the other hand, $\forall S \subseteq N$

$$\sum_{t=1}^{q} b_t^S y_t^* + u\,(S)\, y_{q+1}^* - cu\,(S) \geq \sum_{t=1}^{q} b_t^S y_t^* + R\,(S)\,(y_{q+1}^* - c) \geq v^R(S),$$

where the last inequality holds because y^* is feasible for the dual problem of coalition S and $y_{q+1}^* > c$ since $d_N > r$. In effect, assume that $y_{q+1}^* \leq c$. Consider a variation Δr such that the basic feasible solution does not change and $r + \Delta r < d_N$. Then,

$$value\,(N; r + \Delta r) = value\,(N; r) + \left(y_{q+1}^* - c\right) \Delta r.$$

We distinguish two cases:

1. If $\Delta r > 0$, then

$$\begin{aligned} &value\,(N; r+\Delta r) \leq value\,(N; r) \\ &r < r+\Delta r < d_N \Rightarrow \exists \alpha \in (0,1) \text{ such that } r+\Delta r = \alpha r + (1-\alpha)\, d_N. \end{aligned}$$

Since $\alpha x^r + (1-\alpha)\, x^{d_N}$ is a feasible solution for the problem with $r+\Delta r$, $\left(x^r, x^{d_N}\right)$ is an optimal solution for r and d_N, we have

$$value\,(N; r+\Delta r) \geq \alpha value\,(N; r) + (1-\alpha)\, value\,(N; d_N)\,.$$

However, $value\,(N; d_N) > value\,(N; r+\Delta r)$ by definition of d_N and $value\,(N; r) \geq value\,(N; r+\Delta r)$. Therefore, we have a contradiction.

2. If $\Delta r < 0$, then

$$\begin{aligned} &value\,(N; r+\Delta r) = value\,(N; r) + \left(y^*_{q+1} - c\right) \Delta r \Rightarrow \\ &\Rightarrow value\,(N; r+\Delta r) \geq value\,(N; r)\,. \\ &r+\Delta r < r < d_N \Rightarrow \exists \alpha \in (0,1) \text{ such that } \alpha\,(r+\Delta r) + (1-\alpha)\, d_N = r \end{aligned}$$

Being that $\alpha x^{r+\Delta r} + (1-\alpha)\, x^{d_N}$ is a feasible solution for the problem with r, $\left(x^{r+\Delta r}, x^{d_N}\right)$ is an optimal solution with $r+\Delta r$ and d_N, respectively. We obtain

$$value\,(N; r) \geq \alpha value\,(N; r+\Delta r) + (1-\alpha)\, value\,(N; d_N)\,.$$

But $value\,(N; d_N) > value\,(N; r)$ by definition of d_N and $value\,(N; r+\Delta r) \geq value\,(N; r)$. Thus, there is a contradiction.

Thereby we can conclude that $y^*_{q+1} > c$.

Therefore, $\left(b^i y^* + u_i \left(y^*_{q+1} - c\right)\right)_{i\in N} \in C\left(v^R\right) \neq \varnothing$. ■

Proof of Theorem 4

Let $(d_i)_{i\in N}$ be the individual demands of agents in N. Consider the resource game (N, w), where $w\,(S) = \left(r - \sum_{i\notin S} d_i\right)_+, \forall S \subsetneq N$, and $w\,(N) = r$.

We will distinguish two cases:

a) If $\sum\limits_{i\in N} d_i > r$, (N, w) is a standard bankruptcy game (O'Neill, 1982) and, therefore, it has a non empty core. Then, an $u \in \mathbb{R}^N$ such that $u\,(N) = r$ exists and

$$\begin{aligned} u\,(S) &\geq \left(r - \textstyle\sum_{i\notin S} d_i\right)_+ \geq \min_{P: S\in P} \left\{ \left(r - \textstyle\sum_{\substack{T\in P\\ T\neq S}} d_T \right)_+ \right\} \\ &\geq \min \left\{ \min_{P: S\in P} \left\{ \left(r - \textstyle\sum_{\substack{T\in P\\ T\neq S}} d_T \right)_+ \right\}, d_S \right\} = R^{pes}\,(S)\,. \end{aligned}$$

Therefore, $C\,(R^{pes}) \neq \varnothing$ and using the same arguments as in Theorem 3, the result holds.

b) If $\sum_{i \in N} d_i = r$,

$$d(S) = \sum_{i \in S} d_i \geq \left(r - \sum\nolimits_{i \notin S} d_i\right)_+ \geq \min_{P: S \in P} \left\{ \left(r - \sum\nolimits_{\substack{T \in P \\ T \neq S}} d_T\right)_+ \right\} \geq$$

$$\geq \min \left\{ \min_{P: S \in P} \left\{ \left(r - \sum\nolimits_{\substack{T \in P \\ T \neq S}} d_T\right)_+ \right\}, d_S \right\} = R^{pes}(S).$$

Thus, $C(R^{pes}) \neq \varnothing$ and, then, from Theorem 3 $C(v^{pes}) \neq \varnothing$. ■

Proof of Theorem 5

If $d_N \leq r$, the result immediately follows by using a similar argument as in Theorem 2.

If $d_N > r$, we first note the following:

i) Since the manufacturers are symmetric $d_i = d \in \mathbb{R}_+, \forall i \in N$
ii) If $(x; z)$ is an optimal plan for just one manufacturer, then by the linearity of the production system, $(sx; sz)$ is an optimal plan for s manufacturers and, consequently, $d_S = |S|d$. Therefore, $d > \frac{r}{n}$.
iii) $R^{pes}(S) = (r - (n - |S|)d)_+$
iv) $r - (n - s)d \leq r - (n - s)\frac{r}{n} = \frac{sr}{n}$.

Taking into account $(i)-(iv)$, it is not difficult to prove that $\left(\frac{r}{n}\right)_{i \in N}$ belongs to $C(R^{pes})$, therefore $C(R^{pes}) \neq \varnothing$. Now by Theorem 3, we have that $C(v^{pes}) \neq \varnothing$. ■

Proof of Theorem 6

Before giving the proof of Theorem 6 we need the following technical lemma.

Lemma 2 *Let (A, B, p, r, c) be an LPLR situation and (N, R^{pes}) the associated pessimistic resource game. Each coalition configuration $\{S_1, ..., S_k\}$ is compatible.*

Proof . Given $\{S_1, ..., \S_k\}$, we have to prove that

$$\sum_{j=1}^{k} R^{pes}(S_j) \leq r.$$

First we recall that

$$R^{pes}(S) = \min \left\{ \min_{P: S \in P} \left\{ \left(r - \sum\nolimits_{\substack{T \in P \\ T \neq S}} d_T\right)_+ \right\}, d_S \right\}.$$

We now distinguish two cases:

1. $\sum_{j=1}^{k} d_{S_j} \leq r$. In this situation, the result immediately holds.
2. $\sum_{j=1}^{k} d_{S_j} > r$. First, note that this relationship implies that

$$d_{S_h} > r - \sum_{j:j\neq h} d_{S_j} > r.$$

Now we have the following chain of inequalities:

$$\sum_{j=1}^{k} R^{pes}(S_j) \leq \sum_{j=1}^{k} \min\left\{ \min_{P:S_j\in P} \left\{ \left(r - \sum\nolimits_{\substack{T\in P\\ T\neq S_j}} d_T \right)_+ \right\}, d_{S_j} \right\}$$
$$\leq \sum_{j=1}^{k} \left(r - \sum_{h:h\neq j} d_{S_h} \right)_+ \leq r,$$

where the last inequality holds because the $\sum_{j=1}^{k} \left(r - \sum_{h:h\neq j} d_{S_h} \right)_+$ is a non decreasing function in r and when $r = \sum_{j=1}^{k} d_{S_j}$ the left hand side equals r.

■

Proof of Theorem 6. First, we take a maximally stable coalition $S_1 \subseteq N$. This is always possible to do it because in the worst case a singleton is maximally stable. Next we take a maximally stable coalition $S_2 \subseteq N \backslash S_1$ and so on so forth. This procedure ends in a finite number of steps k and it is well defined. The coalition configuration obtained $\{S_1, ..., S_k\}$ is by construction cc-stable compatible. Indeed, Conditions (1) and (2) in Definition 2 hold by definition of maximally stable coalition. Finally, by Lemma 2 the constructed coalition configuration is compatible. Therefore, the result holds. ■