REAL INTERPOLATION OF COMPACT BILINEAR OPERATORS

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Abstract. We establish an analog for bilinear operators of the compactness interpolation result for bounded linear operators proved by Cwikel and Cobos, Kühn and Schonbek. We work with the assumption that $T : (A_0+A_1)\times(B_0+B_1) \longrightarrow E_0+E_1$ is bounded, but we also study the case when this does not hold. Applications are given to compactness of convolution operators and compactness of commutators of bilinear Calderón-Zygmund operators.

1. Introduction

Compact bilinear operators occur rather naturally in harmonic analysis as it has been shown recently by Bényi and Torres [4], Bényi and Oh [3], Hu [28] and other authors. Since interpolation theory is a very useful tool in harmonic analysis (see the books by Bergh and Löfström [5], Triebel [39, 40] or Bennett and Sharpley [1]), it is natural to enquire into the behaviour under interpolation of compact bilinear operators. In fact, already in 1964, Calderón [7, 10.4] established an interpolation theorem for compact bilinear (or multilinear) operators in his seminal paper on the complex interpolation method. He assumed that the couple in the target satisfies a certain approximation property (see [7, 9.6]). Some complementary results have been established by the present authors in the recent paper [24]. Part of them refer to the degenerated case where the couple in the target reduces to a single Banach space or the source is a product of two fixed Banach spaces [24, Corollaries 5.5 and 5.6].

In [24] we also prove results in the non degenerated case for the general real method, but assuming either than the two restrictions of the bilinear operator are compact, or only one of them is compact and the couple in the target satisfies an approximation property. In other words, we obtain the bilinear results which correspond to the compactness results for linear operators established by Hayakawa [27] (see also [9]) and by

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A. Persson [35]. Previous results on real interpolation of bilinear compact operators are due to Fernandez and da Silva [23] but they work under very restrictive conditions on parameters.

As for real interpolation of compact linear operators, the more general result was obtained by Cwikel [19] and Cobos, Kühn and Schonbek [17]. It works without any condition on the couples and it shows that if any restriction of the operator is compact, then the interpolated operator is compact as well.

In the present paper we establish an analog of this result for bilinear operators interpolated by the general real method. The approach that we follow is based on the methods developed in [18] and [17], splitting the bilinear operator by means of certain families of projections on the vector-valued sequence spaces that come up with the construction of the general real method. This is done in Section 3 after fixing notation in Section 2 and reviewing the construction of the general real method and the results of [24] which are needed here. Then, in Section 4, we discuss compactness interpolation results under weaker assumptions on the bilinear operator T than boundedness of $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E_0 + E_1$. The final Section 5 contains applications of the interpolation results. Among other things, we establish there the compactness of commutators of bilinear Calderón-Zygmund operators acting between certain Lorentz spaces and certain Lorentz-Zygmund spaces.

2. Preliminaries

Let A, B, E be Banach spaces, write $V = \{(a, b) \in A \times B : \|a\|_A \leq 1, \|b\|_B \leq 1\}$ and let $T : A \times B \longrightarrow E$ be a bilinear operator. We say that T is *bounded* if $\|T\|_{A\times B,E} = \sup\big\{\|T(a, b)\|_E : (a, b) \in V\big\} < \infty$. The operator T is said to be *compact* if for any bounded set $W \subseteq A \times B$, we have that $T(W)$ is precompact in E. This condition is equivalent to precompactness of $T(V)$ in E, and also equivalent to the fact that for any bounded sequence $(z_n) \subseteq A \times B$, the sequence (Tz_n) has a convergent subsequence in E (see [4, Proposition 1]).

Clearly any compact bilinear operator is bounded. Furthermore, the set $\mathcal{K}(A\times B, E)$ of all compact bilinear operators from $A \times B$ into E is a closed subspace of the space of all bounded bilinear operators from $A \times B$ into E (see [4, Proposition 3]). Examples of distinguished compact bilinear operators can be found in [4, 3, 28].

If $T \in \mathcal{K}(A \times B, E)$ and $R \in \mathcal{L}(E, E_1)$, where E_1 is another Banach space, then $RT = R \circ T$ belongs to $\mathcal{K}(A \times B, E_1)$. Moreover, if A_1, B_1 are Banach spaces and $S_1 \in \mathcal{L}(A_1, A)$, $S_2 \in \mathcal{L}(B_1, B)$ then the operator $T \circ (S_1, S_2)(x, y) = T(S_1, S_2)(x, y) = T(S_1x, S_2y)$ belongs to $\mathcal{K}(A_1 \times B_1, E)$.

When $E = \mathbb{K}$ (the scalar field) it follows from the definition that any bounded bilinear form is compact . If A and B are Hilbert spaces, a more restrictive concept of compact bilinear form has been studied in [16], and the references given there.

For $1 \leq q \leq \infty$, we designate by ℓ_q the usual space of q-summable scalar sequences with $\mathbb{Z}$ as index set. Given a sequence (λ_m) of positive numbers, we put $\ell_q(\lambda_m) = \Big\{\xi = (\xi_m) : (\lambda_m \xi_m) \in \ell_q\Big\}$ endowed with the natural norm.

A Banach space Γ of real valued sequences with $\mathbb{Z}$ as index set is said to be a *Banach sequence lattice* if Γ satisfies the following conditions:

(i) Γ contains all sequences with only finitely many non-zero co-ordinates.

(ii) Whenever $|\xi_m| \leq |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \leq \|(\eta_m)\|_\Gamma$.

(iii) The *Calderón transform*

$$S(\xi_m) = \left(\sum_{k=-\infty}^{\infty} \min(1, 2^{m-k})\xi_k\right)_{m\in\mathbb{Z}}$$

is bounded in Γ.

(iv) For each $k \in \mathbb{Z}$, the *shift operator* τ_k, $\tau_k(\xi_m) = (\xi_{m+k})_{m\in\mathbb{Z}}$, is bounded in Γ and the norms satisfy that $\lim\limits_{n\to\infty} 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0 = \lim\limits_{n\to\infty} \|\tau_{-n}\|_{\Gamma,\Gamma}$.

Since $e_0 = (\delta_m^0)_{m\in\mathbb{Z}} \in \Gamma$, where δ_m^0 is the Kronecker delta, we have that $(\min(1, 2^m)) = Se_0 \in \Gamma$. Moreover,

$$\sup\left\{\sum_{m=-\infty}^{\infty} \min(1, 2^{-m})|\xi_m| \ : \ \|(\xi_m)\|_\Gamma \leq 1\right\} \leq \|e_0\|_\Gamma^{-1}\|S\|_{\Gamma,\Gamma} < \infty.$$

In other words, any Banach sequence lattice Γ is K-*non-trivial* and J-*non-trivial* in the terminology of Nilsson [32].

We associate to Γ the function $f(t) = \|\tau_{[\log_2 t]}\|_{\Gamma,\Gamma}$, $t > 0$, where the logarithm is taken in base 2 and $[\,\cdot\,]$ is the greatest integer function. This function satisfies that $f(t) = o(\max\{1, t\})$ (see [13, Lemma 4.2]).

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is to say, A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. We endow $\Sigma(\bar{A}) = A_0 + A_1$ and $\Delta(\bar{A}) = A_0 \cap A_1$ with the norms $K(1, \cdot)$ and $J(1, \cdot)$, respectively, where for $t > 0$ we put

$$K(t, a) = K(t, a; A_0, A_1) = \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1 \ , \ a_j \in A_j\right\}$$

and

$$J(t, a) = J(t, a; A_0, A_1) = \max\left\{\|a\|_{A_0} \ , \ t\|a\|_{A_1}\right\}.$$

We write $T : \bar{A} \longrightarrow \bar{A}$ to mean that T is a linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{A})$, whose restriction to A_j defines a bounded operator from A_j into A_j for $j = 0, 1$. We put $\|T\|_{\bar{A},\bar{A}} = \max\left\{\|T\|_{A_j,A_j} \,:\, j = 0, 1\right\}$.

Let A be a Banach space such that $\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A})$, where $\hookrightarrow$ means continuous embedding. We say that A is an *interpolation space* with respect to $\bar{A}$ if given any $T : \bar{A} \longrightarrow \bar{A}$, the restriction of T to A defines a bounded operator from A into A. If that is the case, there is a constant $C = C(A, \bar{A})$ such that

$$\|T\|_{A,A} \leq C\|T\|_{\bar{A},\bar{A}} \tag{2.1}$$

for all operators $T : \bar{A} \longrightarrow \bar{A}$ (see [5, Theorem 2.4.2]).

Let Γ be a Banach sequence lattice. The *general real interpolation space* $\bar{A}_\Gamma = (A_0, A_1)_\Gamma$ in the form of a K-space, is the collection of all $a \in \Sigma(\bar{A})$ such that $(K(2^m, a)) \in \Gamma$. We put

$$\|a\|_{\bar{A}_{\Gamma;K}} = \|(K(2^m, a))\|_\Gamma.$$

The space $\bar{A}_\Gamma$ is an interpolation space with respect to $\bar{A}$. See [34], [6] and [32] for properties of these spaces.

Since the Calderón transform S is bounded in Γ (condition (iii)), the space $\bar{A}_\Gamma$ can be equivalently defined by means of the J-functional as the set of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $\Sigma(\bar{A})$), where $(u_m) \subseteq A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. Furthermore,

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\left\{\|(J(2^m, u_m))\|_\Gamma \,:\, a = \sum_{m=-\infty}^{\infty} u_m\right\}$$

defines an equivalent norm to $\|\cdot\|_{\bar{A}_{\Gamma;K}}$. By $\|\cdot\|_{\bar{A}_\Gamma}$ we mean any of the norms $\|\cdot\|_{\bar{A}_{\Gamma;K}}$ or $\|\cdot\|_{\bar{A}_{\Gamma;J}}$. This however will not cause any confusion.

Let A_j° be the closure of $\Delta(\bar{A})$ in the norm of A_j. We write $\overline{A^\circ} = (A_0^\circ, A_1^\circ)$. For later use, we point out that

$$(A_0, A_1)_\Gamma = (A_0^\circ, A_1^\circ)_\Gamma. \tag{2.2}$$

This follows from the equality $A_0 \cap A_1 = A_0^\circ \cap A_1^\circ$ and the J-representation of the general real space.

If $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ are other Banach couples, we write $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ to mean that T is a bounded bilinear operator $T : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow \Sigma(\bar{E})$ whose restriction to each $A_j \times B_j$ defines a bounded bilinear operator from $A_j \times B_j$ into E_j for $j = 0, 1$. We put $\|T\|_j$ for the norm of $T : A_j \times B_j \longrightarrow E_j$.

The following norm estimate for interpolated operators was established in [24, Theorem 3.1]. We put $\Gamma_0 * \Gamma_1$ for the collection of all sequences obtained by convolution of sequences of Γ_0 and Γ_1.

Theorem 2.1. *Let Γ_0, Γ_1, Γ_2 be Banach sequence lattices with $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$. Then the restriction of T to $\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}$ defines a bounded bilinear operator $T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}$ whose norm $\|T\|$ satisfies that $\|T\| = 0$ if $\|T\|_0 = 0$ and*

$$\|T\| \leq c\, \|T\|_0 f_1\big(\|T\|_1/\|T\|_0\big)$$

if $\|T\|_0 \neq 0$, where f_1 is the function associated to Γ_1 and c is a constant independent of T.

Now we recall two interpolation results for compact bilinear operators which have been established in [24, Theorems 5.1 and 5.3].

Theorem 2.2. *Let Γ_0, Γ_1 be Banach sequence lattices. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let E be a Banach space. Assume that $T : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow E$ is a bounded bilinear operator such that the restriction $T : A_j \times B_j \longrightarrow E$ is compact for $j = 0$ or 1. Then $T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow E$ is also compact.*

Theorem 2.3. *Let Γ be a Banach sequence lattice. Assume that A, B are Banach spaces and let $\bar{E} = (E_0, E_1)$ be a Banach couple. If $T : A \times B \longrightarrow \Delta(\bar{E})$ is a bounded bilinear operator such that any of the restrictions $T : A \times B \longrightarrow E_j$ is compact for $j = 0, 1$, then $T : A \times B \longrightarrow \bar{E}_\Gamma$ is compact as well.*

Next we give some important examples.

Example 2.4. For $\Gamma = \ell_q(2^{-\theta m})$ with $1 \leq q \leq \infty$ and $0 < \theta < 1$. It is easy to check that $\ell_q(2^{-\theta m})$ is a Banach sequence lattice. Since $\|\tau_k\|_{\ell_q(2^{-\theta m}),\ell_q(2^{-\theta m})} \leq 2^{\theta k}$ we can replace $f(t)$ by t^θ. The space $(A_0, A_1)_{\ell_q(2^{-\theta m})}$ is equal to the real interpolation space $(A_0, A_1)_{\theta,q}$ (see [30, 5, 39, 1]). Theorem 2.1 corresponds to the bilinear interpolation theorem of Lions and Peetre [30, Théorème I.4.1].

Example 2.5. We say that the function $\rho : (0, \infty) \longrightarrow (0, \infty)$ is a *function parameter* if $\rho(t)$ increases from 0 to ∞, $\rho(t)/t$ decreases from ∞ to 0 and, for every $t > 0$, $s_\rho(t) = \sup\{\rho(ts)/\rho(s) : s > 0\}$ is finite with $s_\rho(t) = o(\max\{1, t\})$ as $t \to 0$ and $t \to \infty$. Then $\Gamma = \ell_q(1/\rho(2^m))$ is a Banach sequence lattice and $(A_0, A_1)_{\ell_q(1/\rho(2^m))}$ is the real interpolation space with a function parameter (see [26], [29], [36]). This time $\|\tau_k\|_{\ell_q(1/\rho(2^m)),\ell_q(1/\rho(2^m))} \leq s_\rho(2^k)$, hence we can work with s_ρ instead of f.

Example 2.6. If $g : (0, \infty) \longrightarrow (0, \infty)$ is equivalent to a function parameter ρ, meaning that there are positive constants c_1, c_2 such that $c_1\, g(t) \leq \rho(t) \leq c_2\, g(t)$ for all $t > 0$, then $\Gamma = \ell_q(1/g(2^m))$ is also a Banach sequence lattice. This is the case of

$$g_{(\theta,\gamma)}(t) = t^\theta (1 + |\log t|)^\gamma$$

and

$$g_{(\theta;\alpha_0,\alpha_\infty)}(t) = \begin{cases} t^\theta(1-\log t)^{\alpha_0} & \text{if} \quad 0 < t \le 1, \\ t^\theta(1+\log t)^{\alpha_\infty} & \text{if} \quad 1 \le t < \infty, \end{cases}$$

where $0 < \theta < 1$ and $\gamma, \alpha_0, \alpha_\infty \in \mathbb{R}$. These interpolation spaces have been studied in [31, 21, 22] among other papers. For $\Gamma = \ell_q(1/g_{(\theta,\gamma)}(2^m))$, instead of f we can use the function $t^\theta(1+|\log t|)^{|\gamma|}$ and for $\Gamma = \ell_q(1/g_{(\theta;\alpha_0,\alpha_\infty)}(2^m))$ we can work with $t^\theta(1+|\log t|)^{|\alpha_0|+|\alpha_\infty|}$ (see [15, Lemma 2.1]).

Note that in order to fulfill (iv), we do not allow that θ takes the limit values 0 and 1. Therefore, the results of this paper do not apply to the interpolation methods considered in [10, 14, 15, 8].

Given any Banach sequence lattice Γ, any sequence (W_m) of Banach spaces and any sequence (λ_m) of positive numbers, we put

$$\Gamma(\lambda_m W_m) = \Big\{ w = (w_m) : w_m \in W_m \quad \text{and} \quad (\lambda_m \|w_m\|_{W_m}) \in \Gamma \Big\}.$$

The norm in $\Gamma(\lambda_m W_m)$ is given by $\|w\|_{\Gamma(\lambda_m W_m)} = \|\,(\lambda_m\|w_m\|_{W_m})\,\|_\Gamma$.

The following result is a consequence of ([11, Lemmata 3.4 and 4.5]) and the fact that the Calderón transform is bounded in Γ.

Lemma 2.7. *Let Γ be a Banach sequence lattice and let (W_m) be a sequence of Banach spaces. Then*

$$\big(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m)\big)_\Gamma = \Gamma(W_m) = \big(\ell_1(W_m), \ell_1(2^{-m}W_m)\big)_\Gamma .$$

3. Interpolation of compact bilinear operators

In this section we establish a bilinear analog of the compactness result for linear operators proved by Cwikel [19] and Cobos, Kühn and Schonbek [17]. Subsequently, we write U_A for the closed unit ball of the Banach space A.

Theorem 3.1. *Let Γ_0, Γ_1, Γ_2 be Banach sequence lattices with $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ such that any of the restrictions $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0, 1$. Then $T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}$ is compact as well.*

Proof. We know by (2.2) that $\bar{A}_{\Gamma_0} = \overline{A^\circ}_{\Gamma_0}$ and $\bar{B}_{\Gamma_1} = \overline{B^\circ}_{\Gamma_1}$. Moreover, we have that $T : \overline{A^\circ} \times \overline{B^\circ} \longrightarrow \bar{E}$, and compactness of $T : A_j \times B_j \longrightarrow E_j$ yields that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j$ is compact as well. Hence we can work with the couples $\overline{A^\circ}$, $\overline{B^\circ}$ instead of $\bar{A}$, $\bar{B}$.

For $m \in \mathbb{Z}$, put

$$F_m = (A_0^\circ \cap A_1^\circ\,,\, J(2^m, \cdot\,; A_0^\circ, A_1^\circ))\ , \quad G_m = (B_0^\circ \cap B_1^\circ\,,\, J(2^m, \cdot\,; B_0^\circ, B_1^\circ))$$

$$\text{and} \qquad W_m = (E_0 + E_1\,,\ K(2^m, \cdot\,; E_0, E_1))\,.$$

The space $\overline{A^\circ}_{\Gamma_0}$ realized as a J-space is the quotient space of $\Gamma_0(F_m)$ given by the surjective map $\pi : \Gamma_0(F_m) \longrightarrow \overline{A^\circ}_{\Gamma_0}$ defined by $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0^\circ + A_1^\circ$). This map also satisfies that

$$\pi : \ell_1(F_m) \longrightarrow A_0^\circ \quad \text{and} \quad \pi : \ell_1(2^{-m}F_m) \longrightarrow A_1^\circ$$

are bounded with norm less than or equal to 1. The same happens for $\overline{B^\circ}_{\Gamma_1}$ with respect to $\Gamma_1(G_m)$. As for $\bar{E}_{\Gamma_2}$, if we realize this space as a K-space and we put $jz = (\dots, z, z, z, \dots)$ then the map $j : \bar{E}_{\Gamma_2} \longrightarrow \Gamma_2(W_m)$ is a metric injection. Note also that the maps

$$j : E_0 \longrightarrow \ell_\infty(W_m) \quad \text{and} \quad j : E_1 \longrightarrow \ell_\infty(2^{-m}W_m)$$

are bounded with norm less than or equal to 1.

Let $\widehat{T} = jT(\pi, \pi)$. By the properties of π and j, to show that $T : \overline{A^\circ}_{\Gamma_0} \times \overline{B^\circ}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}$ is compact, it suffices to prove compactness of $\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \longrightarrow \Gamma_2(W_m)$. This fact and Lemma 2.7 lead us to work with the couples

$$\overline{\ell_1(F)} = \big(\ell_1(F_m), \ell_1(2^{-m}F_m)\big)\ , \quad \overline{\ell_1(G)} = \big(\ell_1(G_m), \ell_1(2^{-m}G_m)\big)$$

and

$$\overline{\ell_\infty(W)} = \big(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m)\big)\,.$$

On these couples of vector-valued sequences we can consider certain families of projections which will help in splitting the operator $\widehat{T}$ in suitable pieces. For $n \in \mathbb{N}$, put

$$\begin{aligned} R_n(z_m) &= (\dots, 0, 0, z_{-n}, z_{-n+1}, \dots, z_{n-1}, z_n, 0, 0, \dots), \\ R_n^+(z_m) &= (\dots, 0, 0, z_{n+1}, z_{n+2}, z_{n+3}, \dots), \\ R_n^-(z_m) &= (\dots, z_{-n-3}, z_{-n-2}, z_{-n-1}, 0, 0, \dots). \end{aligned}$$

Then $(R_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}\left(\Sigma(\overline{\ell_1(F)}), \Delta(\overline{\ell_1(F)})\right)$, $(R_n^+)_{n\in\mathbb{N}}, (R_n^-)_{n\in\mathbb{N}} \subseteq \mathcal{L}\left(\Sigma(\overline{\ell_1(F)}), \Sigma(\overline{\ell_1(F)})\right)$ and these operators satisfy the following properties:

(3.1) They have norm 1 acting from $\ell_1(F_m)$ into $\ell_1(F_m)$, from $\ell_1(2^{-m}F_m)$ into $\ell_1(2^{-m}F_m)$ and from $\Gamma_0(F_m)$ into $\Gamma_0(F_m)$.

(3.2) The identity operator I on $\Sigma(\overline{\ell_1(F)})$ can be written as $I = R_n + R_n^+ + R_n^-$, $n \in \mathbb{N}$.

(3.3) For each $n \in \mathbb{N}$, we have that $R_n^+ : \ell_1(F_m) \longrightarrow \ell_1(2^{-m}F_m)$ and $R_n^- : \ell_1(2^{-m}F_m) \longrightarrow \ell_1(F_m)$ are bounded with

$$\|R_n^+\|_{\ell_1(F_m),\ell_1(2^{-m}F_m)} = 2^{-(n+1)} = \|R_n^-\|_{\ell_1(2^{-m}F_m),\ell_1(F_m)}.$$

Moreover, $\|R_n\|_{\ell_1(F_m),\ell_1(2^{-m}F_m)} = 2^n = \|R_n\|_{\ell_1(2^{-m}F_m),\ell_1(F_m)}$.

We denote by S_n, S_n^+, S_n^- and P_n, P_n^+, P_n^- the corresponding projections acting on the couples $\overline{\ell_1(G)}$ and $\overline{\ell_\infty(W)}$, respectively. They have similar properties to R_n, R_n^+, R_n^-.

Assume that $T : A_1 \times B_1 \longrightarrow E_1$ is compact. The case where $T : A_0 \times B_0 \longrightarrow E_0$ is compact can be treated analogously. Given any $n \in \mathbb{N}$, we can decompose

$$\begin{aligned}
\widehat{T} &= (P_n + P_n^+ + P_n^-)\widehat{T} = P_n\widehat{T} + P_n^+\widehat{T} + P_n^-\widehat{T}(R_n + R_n^+ + R_n^-\,,\, S_n + S_n^+ + S_n^-)\\
&= P_n\widehat{T} + P_n^-\widehat{T}(R_n, S_n) + P_n^+\widehat{T} + P_n^-\widehat{T}(R_n, S_n^+) + P_n^-\widehat{T}(R_n^+, S_n) + P_n^-\widehat{T}(R_n^+, S_n^+)\\
&+ P_n^-\widehat{T}(R_n, S_n^-) + P_n^-\widehat{T}(R_n^+, S_n^-) + P_n^-\widehat{T}(R_n^-, S_n) + P_n^-\widehat{T}(R_n^-, S_n^+) + P_n^-\widehat{T}(R_n^-, S_n^-).
\end{aligned}$$

Our plan is to show that $P_n\widehat{T}$ and $P_n^-\widehat{T}(R_n, S_n)$ are compact from $\Gamma_0(F_m) \times \Gamma_1(G_m)$ into $\Gamma_2(W_m)$, and that the norms of the remaining operators tend to 0 as $n \to \infty$. This will show that $\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \longrightarrow \Gamma_2(W_m)$ is the limit of a sequence of compact operators and therefore that it is compact.

For $P_n\widehat{T}$ we have the diagram

$$\begin{tikzcd}
\ell_1(F_m) \times \ell_1(G_m) \arrow[r, "\widehat{T}"] & \ell_\infty(W_m) \arrow[dr, "P_n"] & & \\
& & \Delta(\overline{\ell_\infty(W)}) \arrow[r, hook] & \Gamma_2(W_m) \\
\ell_1(2^{-m}F_m) \times \ell_1(2^{-m}G_m) \arrow[r, "\widehat{T}"] & \ell_\infty(2^{-m}W_m) \arrow[ur, "P_n"] & &
\end{tikzcd}$$

where the last embedding is a consequence of Lemma 2.7. Compactness of $T : A_1 \times B_1 \longrightarrow E_1$ yields that

$$P_n\widehat{T} : \ell_1(2^{-m}F_m) \times \ell_1(2^{-m}G_m) \longrightarrow \Gamma_2(W_m)$$

is compact. Then Theorem 2.2 and Lemma 2.7 give that $P_n\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \longrightarrow \Gamma_2(W_m)$ is compact.

As for $P_n^-\widehat{T}(R_n, S_n)$ we use the factorization

$$\begin{tikzcd}
\Gamma_0(F_m) \times \Gamma_1(G_m) \arrow[dd, hook] & & \\
& \ell_1(F_m) \times \ell_1(G_m) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(W_m) \\
\Sigma(\overline{\ell_1(F)}) \times \Sigma(\overline{\ell_1(G)}) \arrow[ur, "{(R_n,S_n)}"] \arrow[dr, "{(R_n,S_n)}"] & & \\
& \ell_1(2^{-m}F_m) \times \ell_1(2^{-m}G_m) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(2^{-m}W_m).
\end{tikzcd}$$

Since $P_n^-\widehat{T}(R_n,S_n):\Gamma_0(F_m)\times\Gamma_1(G_m)\longrightarrow\ell_\infty(2^{-m}W_m)$ is compact, applying Theorem 2.3 and Lemma 2.7 we conclude that $P_n^-\widehat{T}(R_n,S_n):\Gamma_0(F_m)\times\Gamma_1(G_m)\longrightarrow\Gamma_2(W_m)$ is compact.

Next we proceed to estimate the norm of the other operators. For this aim we rely on the estimate given in Theorem 2.1 and the properties of the function f_1. Namely,

$$\begin{aligned}(3.4)\qquad & f_1(t)=o(\max\{1,t\})\ ,\ f_1(st)\le cf_1(s)f_1(t)\ \text{ for all }\ t,s>0\\ & \text{and if }\ s<t\ \text{ then }\ f_1(s)\le cf_1(t)\ \text{ and }\ \frac{f_1(t)}{t}\le c\frac{f_1(s)}{s}\end{aligned}$$

(see [13, Lemma 4.2]). For any of the remaining sequences of operators, say (V_n), we show that either $(\|V_n\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_n)})$ is bounded and $(\|V_n\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_n)})$ converges to 0 , or $(\|V_n\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_n)})$ converges to 0 and the sequence $(\|V_n\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_n)})$ is bounded. These facts together with Theorem 2.1 and (3.4) imply that the sequence of norms of the interpolated operators go to 0 as $n\to\infty$.

We start with $P_n^+\widehat{T}$. We have $\|P_n^+jT\|_{A_0^\circ\times B_0^\circ,\ell_\infty(W_m)}\le\|T\|_{A_0^\circ\times B_0^\circ,E_0}$. Hence, Theorem 2.1, Lemma 2.7 and (3.4) yields

$$\begin{aligned}&\|P_n^+\widehat{T}\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)}\le\|P_n^+jT\|_{\overline{A^\circ}_{\Gamma_0}\times\overline{B^\circ}_{\Gamma_1},\Gamma_2(W_m)}\\ &\le c_1\,\|P_n^+jT\|_{A_0^\circ\times B_0^\circ,\ell_\infty(W_m)}\,f_1\left(\frac{1}{\|P_n^+jT\|_{A_0^\circ\times B_0^\circ,\ell_\infty(W_m)}}\right)\,f_1\left(\|P_n^+jT\|_{A_1^\circ\times B_1^\circ,\ell_\infty(2^{-m}W_m)}\right)\\ &\le c_2\,f_1\left(\|P_n^+jT\|_{A_1^\circ\times B_1^\circ,\ell_\infty(2^{-m}W_m)}\right).\end{aligned}$$

So, to check that the norm of $P_n^+\widehat{T}$ between the interpolated spaces goes to 0 as $n\to\infty$, it is enough to show that $\|P_n^+jT\|_{A_1^\circ\times B_1^\circ,\ell_\infty(2^{-m}W_m)}\longrightarrow 0$ as $n\to\infty$. Take any $\varepsilon>0$. Using compactness of $jT:A_1^\circ\times B_1^\circ\longrightarrow\ell_\infty(2^{-m}W_m)$ we can find a finite set $\{z_1,\cdots,z_r\}\subseteq U_{A_1^\circ\times B_1^\circ}$ such that

$$jT\left(U_{A_1^\circ\times B_1^\circ}\right)\subseteq\bigcup_{k=1}^{r}\{jTz_k+\frac{\varepsilon}{2}U_{\ell_\infty(2^{-m}W_m)}\}.$$

Since $A_0\cap A_1$ is dense in A_1° and $B_0\cap B_1$ is dense in B_1°, without loss of generality we may assume that $\{z_1,\cdots,z_r\}\subseteq\Delta(\overline{A^\circ})\times\Delta(\overline{B^\circ})$. So $jTz_k\in\Delta(\overline{\ell_\infty(W)})$ for $1\le k\le r$. By the corresponding property to (3.3) for P_n^+, there exits $N\in\mathbb{N}$ such that for any $n\ge N$ and any $1\le k\le r$, we have $\|P_n^+jTz_k\|_{\ell_\infty(2^{-m}W_m)}\le\varepsilon/2$. Whence, given any $z\in U_{A_1^\circ\times B_1^\circ}$, if we choose z_k such that $\|jTz-jTz_k\|_{\ell_\infty(2^{-m}W_m)}\le\varepsilon/2$, we get

$$\|P_n^+jTz\|_{\ell_\infty(2^{-m}W_m)}\le\|P_n^+(jTz-jTz_k)\|_{\ell_\infty(2^{-m}W_m)}+\|P_n^+jTz_k\|_{\ell_\infty(2^{-m}W_m)}\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Consider now the operators $P_n^-\widehat{T}(R_n, S_n^+)$, $P_n^-\widehat{T}(R_n^+, S_n)$ and $P_n^-\widehat{T}(R_n^+, S_n^+)$. Their norms from $\ell_1(2^{-m}F_m) \times \ell_1(2^{-m}G_m)$ into $\ell_\infty(2^{-m}W_m)$ are bounded by $\|T\|_{A_1^\circ \times B_1^\circ, E_1}$. By Theorem 2.1, Lemma 2.7 and (3.4), it is enough to show that their norms acting from $\ell_1(F_m) \times \ell_1(G_m)$ into $\ell_\infty(W_m)$ go to 0 as $n \to \infty$. The factorization

$$\ell_1(F_m)\times\ell_1(G_m) \xrightarrow{(R_n,S_n^+)} \ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m) \xrightarrow{\widehat{T}} \ell_\infty(2^{-m}W_m) \xrightarrow{P_n^-} \ell_\infty(W_m)$$

and (3.3) yields

$$\begin{aligned}
&\|P_n^-\widehat{T}(R_n, S_n^+)\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_m)} \le \|R_n\|_{\ell_1(F_m),\ell_1(2^{-m}F_m)} \\
&\quad\times \|S_n^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} \|\widehat{T}\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \\
&\quad\le 2^n\, 2^{-(n+1)}\, \|T\|_{A_1^\circ\times B_1^\circ,E_1}\, 2^{-(n+1)} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

In the same way

$$\begin{aligned}
&\|P_n^-\widehat{T}(R_n^+, S_n)\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_m)} \le \|R_n^+\|_{\ell_1(F_m),\ell_1(2^{-m}F_m)} \\
&\quad\times \|S_n\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} \|\widehat{T}\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \\
&\quad\le 2^{-(n+1)}\, 2^n\, \|T\|_{A_1^\circ\times B_1^\circ,E_1}\, 2^{-(n+1)} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|P_n^-\widehat{T}(R_n^+, S_n^+)\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_m)} \le \|R_n^+\|_{\ell_1(F_m),\ell_1(2^{-m}F_m)} \\
&\quad\times \|S_n^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} \|\widehat{T}\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \\
&\quad\le 2^{-(n+1)}\, 2^{-(n+1)}\, \|T\|_{A_1^\circ\times B_1^\circ,E_1}\, 2^{-(n+1)} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Consider now the operator $P_n^-\widehat{T}(R_n, S_n^-)$. We have

$$\|P_n^-\widehat{T}(R_n, S_n^-)\|_{\ell_1(F_m)\times\ell_1(G_m),\ell_\infty(W_m)} \le \|T\|_{A_0^\circ\times B_0^\circ,E_0}. \tag{3.5}$$

The other restriction satisfies that

$$\begin{aligned}
\|P_n^-\widehat{T}(R_n, S_n^-)\|&_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \\
&\le \|T(\pi R_n, \pi S_n^-)\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),E_1}.
\end{aligned}$$

We claim that

$$\lim_{n\to\infty} \|T(\pi R_n, \pi S_n^-)\|_{\ell_1(2^{-m}F_m)\times\ell_1(2^{-m}G_m),E_1} = 0. \tag{3.6}$$

Indeed, if this would not be the case, then there would exist $\lambda > 0$, a subsequence (n_1) and vectors $a_{n_1} \in U_{\ell_1(2^{-m}F_m)}$, $b_{n_1} \in U_{\ell_1(2^{-m}G_m)}$ such that

$$\lim_{n_1\to\infty} \|T(\pi R_{n_1} a_{n_1}, \pi S_{n_1}^- b_{n_1})\|_{E_1} = \lambda.$$

By (3.1), the sequence $(\pi R_{n_1} a_{n_1})$ is bounded in A_1° and $(\pi S_{n_1}^- b_{n_1})$ is bounded in B_1°. Using the compactness of $T : A_1^\circ \times B_1^\circ \longrightarrow E_1$, we may assume, passing to another subsequence if necessary, that $\big(T(\pi R_{n_2}a_{n_2}, \pi S^-_{n_2}b_{n_2})\big)$ converges to some w in E_1. So $\|w\|_{E_1} = \lambda > 0$. But

$$\begin{aligned}\|T(\pi R_{n_2}a_{n_2}, \pi S^-_{n_2}b_{n_2})\|_{E_0+E_1} &\le \|T\|_{\Sigma(\overline{A^\circ})\times\Sigma(\overline{B^\circ}),\Sigma(\bar{E})}\|R_{n_2}a_{n_2}\|_{\Sigma(\overline{\ell_1(F)})}\|S^-_{n_2}b_{n_2}\|_{\Sigma(\overline{\ell_1(G)})}\\ &\le \|T\|_{\Sigma(\overline{A^\circ})\times\Sigma(\overline{B^\circ}),\Sigma(\bar{E})}\|R_{n_2}a_{n_2}\|_{\ell_1(2^{-m}F_m)}\|S^-_{n_2}b_{n_2}\|_{\ell_1(G_m)}\\ &\le c\,2^{-(n_2+1)} \longrightarrow 0 \ \text{ as } \ n\to\infty\end{aligned}$$

where we have used (3.3) in the last inequality. By compatibility we conclude that $w = 0$ contradicting $w \ne 0$.

Now, using (3.5) and (3.6), it follows from Theorem 2.1, Lemma 2.7 and (3.4) that

$$\|P^-_n\widehat{T}(R_n, S^-_n)\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)} \longrightarrow 0 \ \text{ as } \ n\to\infty.$$

Operators $P^-_n\widehat{T}(R^+_n, S^-_n)$, $P^-_n\widehat{T}(R^-_n, S_n)$, $P^-_n\widehat{T}(R^-_n, S^+_n)$ and $P^-_n\widehat{T}(R^-_n, S^-_n)$ can be treated similarly to $P^-_n\widehat{T}(R_n, S^-_n)$.

The proof is complete. □

For the case of the real method with a function parameter (Example 2.5), the result reads as follows.

Theorem 3.2. *Assume that ρ_0, ρ_1, ρ_2 are function parameters such that for some constant $C > 0$ we have*

$$\rho_0(t)\,\rho_1(s) \le C\,\rho_2(ts)\,,\ t,s>0. \tag{3.7}$$

Let $1 \le p,q,r \le \infty$ with $1/p + 1/q = 1 + 1/r$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be Banach couples. If $T : \bar{A}\times\bar{B} \longrightarrow \bar{E}$ and $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or $j = 1$, then

$$T : (A_0, A_1)_{\rho_0,p} \times (B_0, B_1)_{\rho_1,q} \longrightarrow (E_0, E_1)_{\rho_2,r}$$

is compact as well.

Proof. We have $\Gamma_0 = \ell_p(1/\rho_0(2^m)), \Gamma_1 = \ell_q(1/\rho_1(2^m))$ and $\Gamma_2 = \ell_r(1/\rho_2(2^m))$. Since $1/p + 1/q = 1 + 1/r$, Young's inequality for convolution yields that $\ell_p * \ell_q \hookrightarrow \ell_r$. Take any $\xi = (\xi_m) \in \ell_p(1/\rho_0(2^m))$ and $\eta = (\eta_m) \in \ell_q(1/\rho_1(2^m))$. For $\xi*\eta = \big(\sum_{k=-\infty}^{\infty}\xi_k\eta_{m-k}\big)_{m\in\mathbb{Z}}$ using (3.7) and Young's inequality, we obtain

$$\begin{aligned}\|\xi*\eta\|_{\ell_r(1/\rho_2(2^m))} &\le C\Big(\sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty}\big(|\xi_k|/\rho_0(2^k)\big)\big(|\eta_{m-k}|/\rho_1(2^{m-k})\big)\Big)^r\Big)^{1/r}\\ &\le C\|\xi\|_{\ell_p(1/\rho_0(2^m))}\|\eta\|_{\ell_q(1/\rho_1(2^m))}.\end{aligned}$$

Hence, $\ell_p(1/\rho_0(2^m)) * \ell_q(1/\rho_1(2^m)) \hookrightarrow \ell_r(1/\rho_2(2^m))$ and the result is a consequence of Theorem 3.1. □

For the special case of the real method (Example 2.4) we get the following

Theorem 3.3. *Let $0 < \theta < 1$ and $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be Banach couples. If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or $j = 1$, then*

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r}$$

is compact as well.

Proof. The result follows from Theorem 3.2 by taking $\rho_0(t) = \rho_1(t) = \rho_2(t) = t^\theta$. □

4. Interpolation under weaker assumptions

So far we have been working with bilinear operators $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$. Hence, they satisfy

$$T : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow \Sigma(\bar{E}) \quad \text{boundedly.} \tag{4.1}$$

However, sometimes in applications we do not have (4.1) but only that T is defined on $\Delta(\bar{A}) \times \Delta(\bar{B})$ with values in $\Delta(\bar{E})$ and that there are constants $M_j > 0$ such that

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}),\ j = 0, 1. \tag{4.2}$$

Subsequently, we write $\mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ for the collection of all those bilinear operators T satisfying (4.2).

Assumption (4.2) is the one used by Calderón for the bilinear (and multilinear) interpolation theorem for the complex method [7, 10.1] (see also [5, 4.4]).

Since (4.2) is weaker than the assumption $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$, it is natural to enquire into the validity of the compactness results when we only assume that T belongs to $\mathcal{B}(\bar{A}, \bar{B}; \bar{E})$. This is the aim of this section.

Given any $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$, the operator T may be uniquely extended to a bilinear operator $T_j : A_j^\circ \times B_j^\circ \longrightarrow E_j$, $j = 0, 1$. We put

$$\|T\|_j = \|T_j\|_{A_j^\circ \times B_j^\circ, E_j}, \quad j = 0, 1.$$

We say that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j$ is compact if T_j does it.

Recall that we have pointed out in (2.2) that $(A_0, A_1)_\Gamma = (A_0^\circ, A_1^\circ)_\Gamma$.

Let e_k be the sequence of scalars which is 0 at all co-ordinates but the kth co-ordinate where it is 1. We are going to assume in the later results that some Banach sequence lattices Γ satisfy

$$\xi = \lim_{n\to\infty} \sum_{k=-n}^{n} \xi_k e_k \quad \text{(convergence in } \Gamma\text{) for all} \quad \xi = \{\xi_m\} \in \Gamma. \tag{4.3}$$

In this case it is easy to check that for any Banach couple $\bar{A}$ it holds that $\Delta(\bar{A})$ is dense in $\bar{A}_\Gamma$.

We also recall that given any Banach sequence lattice Γ and any Banach couple $\bar{A}$, for any $a \in \Delta(\bar{A})$ there is a J-representation $a = \sum_{m=-\infty}^{\infty} u_m$ with only a finite number of terms u_m different from 0 such that

$$\|(J(2^m, u_m))\|_\Gamma \le 8\, \|a\|_{\bar{A}_{\Gamma;K}} \quad \text{(see [12, Lemma 2.2]).} \tag{4.4}$$

Using (4.4) we can adapt the arguments in the proof of [24, Theorems 3.1] to cover operators in $\mathcal{B}(\bar{A}, \bar{B}; \bar{E})$.

Theorem 4.1. *Let Γ_0, Γ_1 be Banach sequence lattices satisfying (4.3) and let Γ_2 be another Banach sequence lattice with $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$. Suppose that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ are Banach couples and let $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$. Then T may be uniquely extended to a bilinear operator from $\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}$ to $\bar{E}_{\Gamma_2}$ and its norm $\|T\|$ satisfies that $\|T\| = 0$ if $\|T\|_0 = 0$ and*

$$\|T\| \le c\, \|T\|_0 f_1\big(\|T\|_1/\|T\|_0\big)$$

if $\|T\|_0 \neq 0$, where f_1 is the function associated to Γ_1 and c is a constant independent of T.

Proof. Take any $a \in \Delta(\bar{A})$, $b \in \Delta(\bar{B})$. If $\|T\|_0 = 0$ then $T(a,b) = 0$ and therefore $\|T\| = 0$. Assume that $\|T\|_0 \neq 0$. Take any representation $a = a_0 + a_1$ with $a_j \in A_j$. Then $a_j \in \Delta(\bar{A})$ for $j = 0, 1$. Choose also a J-representation $b = \sum_{m=-\infty}^{\infty} u_m$ with only a finite number of terms u_m different from 0 and satisfying the corresponding inequality to (4.4). Let $s \in \mathbb{Z}$ such that $2^s \le \|T\|_1/\|T\|_0 < 2^{s+1}$. Since $b = \sum_{k=-\infty}^{\infty} u_{k+s}$ we obtain

$$\begin{aligned}
K(2^m, T(a,b); \bar{E}) &\le \sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+s}); \bar{E}) \\
&\le \sum_{k=-\infty}^{\infty} \|T(a_0, u_{k+s})\|_{E_0} + 2^m \|T(a_1, u_{k+s})\|_{E_1} \\
&\le \sum_{k=-\infty}^{\infty} \Big(\|T\|_0 \|a_0\|_{A_0} + 2^{m-k-s} \|T\|_1 \|a_1\|_{A_1}\Big)\, J(2^{k+s}, u_{k+s}; \bar{B}).
\end{aligned}$$

Therefore

$$K(2^m, T(a,b); \bar{E}) \le \max\{\|T\|_0, 2^{-s}\|T\|_1\} \sum_{k=-\infty}^{\infty} K(2^{m-k}, a; \bar{A}) J(2^{k+s}, u_{k+s}; \bar{B})$$
$$\le 2\,\|T\|_0 \sum_{n=-\infty}^{\infty} K(2^n, a; \bar{A}) J(2^{k+s-n}, u_{k+s-n}; \bar{B}).$$

Having in mind the embedding $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$, that $\|\tau_s\|_{\Gamma_1,\Gamma_1} \le f_1(2^s) \le c_1 f_1\big(\|T\|_1/\|T\|_0\big)$ and the choice of (u_m), we get

$$\begin{aligned}\|T(a,b)\|_{\bar{E}_{\Gamma_2}} &\le c_2\,\|T\|_0 \|a\|_{\bar{A}_{\Gamma_0}} \|(J(2^{k+s}, u_{k+s}; \bar{B}))\|_{\Gamma_1}\\ &\le c_2\,\|T\|_0 \|\tau_s\|_{\Gamma_1,\Gamma_1} \|a\|_{\bar{A}_{\Gamma_0}} \|(J(2^k, u_k; \bar{B}))\|_{\Gamma_1}\\ &\le c\,\|T\|_0 f_1\big(\|T\|_1/\|T\|_0\big) \|a\|_{\bar{A}_{\Gamma_0}} \|b\|_{\bar{B}_{\Gamma_1}}.\end{aligned}$$

Since $\Delta(\bar{A})$ is dense in $\bar{A}_{\Gamma_0}$, $\Delta(\bar{B})$ is dense in $\bar{B}_{\Gamma_1}$ and $\bar{E}_{\Gamma_2}$ is complete, we derive the desired conclusion. □

Regarding interpolation of compact bilinear operators, in the case of Theorem 2.3 where the source space reduces to a product of two fixed Banach spaces, i.e. $A_0 = A_1 = A$ and $B_0 = B_1 = B$, note that assumption $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ is the same as $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$.

The situation is different for Theorem 2.2 even when Γ_0, Γ_1 satisfy (4.3). The reason is that there is still a lot of freedom with Γ_0 and Γ_1. Next we show it by means of examples.

Counterexample 4.2. Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $0 < \tau < \theta < 1$. The Banach sequence lattices $\Gamma_0 = \ell_p(2^{-\theta m})$, $\Gamma_1 = \ell_{p'}(2^{-\tau m})$ satisfy (4.3). Consider the Banach couples $(\ell_p, \ell_p(2^{-m}))$, $\big(\ell_{p'}(2^m), \ell_{p'}\big)$, and let T by the operator given by

$$T(\xi, \eta) = \sum_{m=-\infty}^{\infty} \xi_m 2^{-m} \eta_{-m}\,, \quad \xi = (\xi_m)\,,\ \eta = (\eta_m).$$

Then $T \in \mathcal{B}\left((\ell_p, \ell_p(2^{-m}))\,, \big(\ell_{p'}(2^m), \ell_{p'}\big)\,; (\mathbb{K}, \mathbb{K})\right)$ because

$$\begin{aligned}|T(\xi,\eta)| = \Big| \sum_{m=-\infty}^{\infty} \xi_m 2^{-m} \eta_{-m} \Big| &\le \left(\sum_{m=-\infty}^{\infty} |\xi_m|^p \right)^{1/p} \left(\sum_{m=-\infty}^{\infty} |2^{-m}\eta_{-m}|^{p'} \right)^{1/p'}\\ &= \|\xi\|_{\ell_p} \|\eta\|_{\ell_{p'}(2^m)}\end{aligned}$$

and similarly

$$|T(\xi,\eta)| \le \left(\sum_{m=-\infty}^{\infty} |2^{-m}\xi_m|^p \right)^{1/p} \left(\sum_{m=-\infty}^{\infty} |\eta_{-m}|^{p'} \right)^{1/p'} = \|\xi\|_{\ell_p(2^{-m})} \|\eta\|_{\ell_{p'}}.$$

Both restrictions

$$T : \ell_p \times \ell_{p'}(2^m) \longrightarrow \mathbb{K} \quad , \quad T : \ell_p(2^{-m}) \times \ell_{p'} \longrightarrow \mathbb{K}$$

are compact because the target space is $\mathbb{K}$, but the interpolated operator

$$T : \big(\ell_p, \ell_p(2^{-m})\big)_{\Gamma_0} \times \big(\ell_{p'}(2^m), \ell_{p'}\big)_{\Gamma_1} \longrightarrow \mathbb{K}$$

is not even bounded. Indeed, by [5, Theorem 5.6.1] we have

$$\big(\ell_p, \ell_p(2^{-m})\big)_{\Gamma_0} = \ell_p(2^{-\theta m}) \quad , \quad \big(\ell_{p'}(2^m), \ell_{p'}\big)_{\Gamma_1} = \ell_{p'}(2^{(1-\tau)m}).$$

Vectors $2^{\theta n} e_n$, $2^{(1-\tau)n} e_{-n}$, satisfy that

$$\|2^{\theta n} e_n\|_{\ell_p(2^{-\theta m})} = 1 = \|2^{(1-\tau)n} e_{-n}\|_{\ell_{p'}(2^{(1-\tau)m})}$$

but

$$T(2^{\theta n} e_n, 2^{(1-\tau)n} e_{-n}) = 2^{\theta n} 2^{-n} 2^{(1-\tau)n} = 2^{(\theta-\tau)n}$$

which is not bounded.

Counterexample 4.3. Consider the couples $\big(\ell_2(n), \ell_2(n^{-1})\big)$, $\big(\ell_2(n^{-3/4}), \ell_2(n)\big)$ of sequence spaces with $\mathbb{N}$ as index set, and the Banach sequence lattices $\Gamma_0 = \ell_2(2^{-m/4})$, $\Gamma_1 = \ell_2(2^{-m/7})$ which satisfy (4.3). According to [5, Theorem 5.5.1]) , we have

$$\big(\ell_2(n), \ell_2(n^{-1})\big)_{\Gamma_0} = \ell_2(n^{1/2}) \quad , \quad \Big(\ell_2(n^{-3/4}), \ell_2(n)\Big)_{\Gamma_1} = \ell_2(n^{-1/2}).$$

Let $T(\xi, \eta) = (\xi_n \eta_n)$ for $\xi = (\xi_n)$, $\eta = (\eta_n)$. Since

$$\|T(\xi, \eta)\|_{\ell_1} = \sum_{n=1}^{\infty} |n^{-1}\xi_n||n\eta_n| \leq \|\xi\|_{\ell_2(n^{-1})} \|\eta\|_{\ell_2(n)},$$

we have that $T : \ell_2(n^{-1}) \times \ell_2(n) \longrightarrow \ell_1$ is bounded. Moreover,
$T : \ell_2(n) \times \ell_2(n^{-3/4}) \longrightarrow \ell_1$ is the limit of the sequence of finite rank operators

$$T_k(\xi, \eta) = (\xi_1\eta_1, \cdots, \xi_k\eta_k, 0, 0, \cdots), \ k \in \mathbb{N},$$

because

$$\|(T - T_k)(\xi, \eta)\|_{\ell_1} = \sum_{n=k+1}^{\infty} n^{-1/4} |n\xi_n||n^{-3/4}\eta_n| \leq (k+1)^{-1/4} \|\xi\|_{\ell_2(n)} \|\eta\|_{\ell_2(n^{-3/4})}.$$

Hence,

$$T \in \mathcal{B}\Big(\big(\ell_2(n^{-1}), \ell_2(n)\big), \big(\ell_2(n), \ell_2(n^{-3/4})\big); (\ell_1, \ell_1)\Big)$$

and

$$T : \ell_2(n) \times \ell_2(n^{-3/4}) \longrightarrow \ell_1 \quad \text{compactly.}$$

However, the interpolated operator $T : \ell_2(n^{1/2}) \times \ell_2(n^{-1/2}) \longrightarrow \ell_1$ is bounded but it is not compact. Indeed, the sequences $(n^{-1/2}e_n)$ and $(n^{1/2}e_n)$ are formed by unit vectors in $\ell_2(n^{1/2})$ and $\ell_2(n^{-1/2})$, respectively, but $\big(T(n^{-1/2}e_n, n^{1/2}e_n)\big) = (e_n)$ does not have any convergent subsequence.

Remark 4.4. As for the complex interpolation method, since $[\ell_p, \ell_p(2^{-m})]_\theta = \ell_p(2^{-\theta m})$ and $\big[\ell_{p'}(2^m), \ell_{p'}\big]_\tau = \ell_{p'}(2^{(1-\tau)m})$ (see [5, Theorem 5.5.3]), Counterexample 4.2 (and also Counterexample 4.3) shows that [24, Corollary 5.5] fails if we replace assumption

$$T : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow E \quad \text{boundedly}$$

by

$$\|T(a,b)\|_E \le M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}),\ j = 0,1.$$

Next we show that it is possible to establish a positive result if the couple in the target of the operator satisfies the following approximation property.

We say that a Banach couple $\bar{E} = (E_0, E_1)$ satisfies *condition* $(\mathfrak{H})$ if for any compact subset $K \subseteq E_0$, there is a family of operators $\{P_\lambda\}_{\lambda\in\Lambda} \subseteq \mathfrak{L}\left(\Sigma(\bar{E}), \Delta(\bar{E})\right)$ and a constant $C > 0$ such that

$$P_\lambda : \Sigma(\bar{E}) \longrightarrow \Delta(\bar{E}) \quad \text{is compact, } \lambda \in \Lambda. \tag{4.5}$$

$$\|P_\lambda\|_{E_j,E_j} \le C\,, \quad j = 0,1\,, \quad \lambda \in \Lambda. \tag{4.6}$$

(4.7) For every $\varepsilon > 0$, there is $\lambda_0 \in \Lambda$ such that $\|x - P_{\lambda_0}x\|_{E_0} \le \varepsilon$ for every $x \in K$.

Similar conditions have been used by Calderón [7, 10.4], A. Persson [35, p. 216] and Edmunds and Teixeira [38, p. 133].

Next, we show examples of couples satisfying $(\mathfrak{H})$.

Proposition 4.5. *Let (Ω,μ) be any measure space and let E_0, E_1 be both interpolation spaces with respect to the couple $(L_1(\Omega), L_\infty(\Omega))$. If the simple functions are dense in E_0, then (E_0, E_1) satisfies $(\mathfrak{H})$.*

Proof. We are going to associate the same family of operators to any compact subset $K \subseteq E_0$, namely the set of operators

$$Pf = \sum_{k=1}^{M} \Big(\mu(O_k)^{-1} \int f\chi_{O_k}\, d\mu\Big)\chi_{O_k} \tag{4.8}$$

where $O_1, \dots, O_M$ is any finite collection of pairwise disjoint measurable sets of finite measure. As it was shown in [24, Proposition 4.1], any of these operators P satisfies that $P \in \mathfrak{L}\big(L_1(\Omega)+L_\infty(\Omega), L_1(\Omega)\cap L_\infty(\Omega)\big)$ with $\|P\|_{L_1(\Omega),L_1(\Omega)} = 1 = \|P\|_{L_\infty(\Omega),L_\infty(\Omega)}$. Since E_j, $j = 0,1$, are interpolation spaces, we have $P \in \mathfrak{L}(E_j, E_j)$. Moreover, by (2.1), there is a constant $C > 0$ such that $\|P\|_{E_j,E_j} \le C$, $j = 0,1$. Clearly, $P : \Sigma(\bar{E}) \longrightarrow \Delta(\bar{E})$ compactly because it has finite rank. Finally, given any $\varepsilon > 0$, since K is compact and the simple functions are dense in E_0, we can find a finite collection of simple functions $f_1 \cdots, f_N$ such that

$$K \subseteq \bigcup_{j=1}^{N} \left\{ f_j + \frac{\varepsilon}{2(1+C)} U_{E_0} \right\}.$$

We can also choose pairwise disjoint measurable sets $O_1 \cdots, O_M$ such that each of the functions f_j assumes a constant value on each of the sets O_k. Then the operator P defined by (4.8) satisfies (4.7) because given $f \in K$ there is $1 \leq j \leq N$ such that $\|f - f_j\|_{E_0} \leq \frac{\varepsilon}{2(1+C)}$ and so

$$\|f - Pf\|_{E_0} \leq \|f - f_j\|_{E_0} + \|f_j - Pf_j\|_{E_0} + \|P(f_j - f)\|_{E_0} \leq \varepsilon.$$

□

Theorem 4.6. *Let Γ_0, Γ_1 be Banach sequence lattices satisfying (4.3) and let Γ_2 be another Banach sequence lattices with $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be Banach couples with $\bar{E}$ satisfying condition $(\mathfrak{H})$. If $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ and $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ is compact, then T may be uniquely extended to a compact bilinear operator from $\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}$ to $\bar{E}_{\Gamma_2}$.*

Proof. By Theorem 4.1, T may be uniquely extended to a bounded bilinear operator from $\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}$ to $\bar{E}_{\Gamma_2}$. Since $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ is compact, the closure in E_0 of the image by T_0 of the unit ball of $A_0^\circ \times B_0^\circ$ is a compact set K. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the family of operators associated to K by condition $(\mathfrak{H})$. Factorization

$$\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \xrightarrow{\quad T \quad} \bar{E}_{\Gamma_2} \hookrightarrow \Sigma(\bar{E}) \xrightarrow{\quad P_\lambda \quad} \Delta(\bar{E}) \hookrightarrow \bar{E}_{\Gamma_2}$$

and (4.5) show that for any $\lambda \in \Lambda$ the operator

$$P_\lambda T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2} \quad \text{is compact.}$$

Hence, it suffices to show that T can be uniformly approximated by operators $P_\lambda T$.

Using that $\|T - P_\lambda T\|_1 \leq (1+C)\|T\|_1$, Theorem 4.1 and (3.4), we get

$$\begin{aligned} \|T - P_\lambda T\|_{\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}, \bar{E}_{\Gamma_2}} &\leq c\, \|T - P_\lambda T\|_0 f_1 \left(\frac{\|T - P_\lambda T\|_1}{\|T - P_\lambda T\|_0} \right) \\ &\leq c_1 \|T - P_\lambda T\|_0 f_1 \left(\frac{1}{\|T - P_\lambda T\|_0} \right). \end{aligned}$$

Now, given any $\varepsilon > 0$, using (3.4) we can find t_0 such that $tf(1/t) < \varepsilon / c_1$ for any $t \leq t_0$. By (4.7), there exits $\lambda_0 \in \Lambda$ such that $\|T - P_{\lambda_0} T\|_0 \leq t_0$. Consequently,

$$\|T - P_{\lambda_0} T\|_{\bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1}, \bar{E}_{\Gamma_2}} < \varepsilon.$$

□

We close this section by writing down the corresponding versions of Theorem 4.6 for the case of the real method with a function parameter and for the case of the classical real method.

Theorem 4.7. *Assume that* ρ_0, ρ_1, ρ_2 *are function parameters such that for some constant* $C > 0$ *we have* $\rho_0(t)\,\rho_1(s) \le C\,\rho_2(ts)$, $t, s > 0$. *Let* $1 \le p, q < \infty$ *and* $1/r = 1/p + 1/q - 1$. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and* $\bar{E} = (E_0, E_1)$ *be Banach couples with* $\bar{E}$ *satisfying condition* ($\mathfrak{H}$). *If* $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ *and* $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ *is compact, then* T *may be uniquely extended to a compact bilinear operator from* $(A_0, A_1)_{\rho_0,p} \times (B_0, B_1)_{\rho_1,q}$ *to* $(E_0, E_1)_{\rho_2,r}$.

Theorem 4.8. *Assume that* $0 < \theta < 1$, $1 \le p, q < \infty$ *and* $1/r = 1/p + 1/q - 1$. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and* $\bar{E} = (E_0, E_1)$ *be Banach couples with* $\bar{E}$ *satisfying condition* ($\mathfrak{H}$). *If* $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ *and* $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ *is compact, then* T *may be uniquely extended to a compact bilinear operator from* $(A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}$ *to* $(E_0, E_1)_{\theta,r}$.

5. Applications

In this section we apply the previous results to show compactness of certain bilinear operators acting between function spaces.

Let (Ω, μ) be a σ-finite measure space. Let $1 < p < \infty$, $1 \le q \le \infty$ and $b \in \mathbb{R}$. Recall that the Lorentz-Zygmund space $L_{p,q}(\log L)_b(\Omega)$ is formed by all those measurable functions f on Ω having a finite norm

$$\|f\|_{L_{p,q}(\log L)_b(\Omega)} = \Big(\int_0^{\mu(\Omega)} \big[t^{1/p-1} (1 + |\log t|)^b \int_0^t f^*(s) ds \big]^q \frac{dt}{t} \Big)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$) (see [2, 1, 20]). Here f^* stands for the non-increasing rearrangement of f. If $\Omega = \mathbb{R}^n$ and μ is the Lebesgue measure then we skip the measure space in the notation.

For the couple $(L_1(\Omega), L_\infty(\Omega))$ it turns out that $K(t, f) = \displaystyle\int_0^t f^*(s) ds$ (see [5, Theorem 5.2.1] or [39, 1.18.6, p. 133]). This yields that $(L_1(\Omega), L_\infty(\Omega))_{1-1/p,q} = L_{p,q}(\Omega)$ and $(L_1(\Omega), L_\infty(\Omega))_{\rho,q} = L_{p,q}(\log L)_b(\Omega)$ where $\rho(t) = t^{1-1/p}(1 + |\log t|)^{-b}$. If $p_0 \ne p_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\rho(t) = t^\theta (1 + |\log t|)^{-b}$, then $(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\rho,q} = L_{p,q}(\log L)_b(\Omega)$ (see, for example, [36, Prop. 6.2]).

Let (Ω_j, μ_j) be σ-finite measure spaces for $j = 0, 1, 2$. In what follows, the Banach couple $\bar{A} = (A_0, A_1)$ [respectively, $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$] is formed by certain Lorentz-Zygmund spaces on (Ω_0, μ_0) [respectively, (Ω_1, μ_1) and (Ω_2, μ_2)]. The bilinear operator T is defined for all pairs (f, g), where $f \in \Delta(\bar{A})$ and $g \in \Delta(\bar{B})$, taking values in $\Delta(\bar{E})$.

Next we establish a reinforced version of [1, Theorem IV.7.7].

Theorem 5.1. *Suppose* $1 < p < \infty$ *and* $1 \leq \alpha, \beta, \gamma \leq \infty$ *with* $1/\alpha + 1/\beta = 1 + 1/\gamma$. *Let* $\bar{A} = (L_1(\Omega_0), L_\infty(\Omega_0))$, $\bar{B} = (L_1(\Omega_1), L_\infty(\Omega_1))$ *and* $\bar{E} = (L_1(\Omega_2), L_\infty(\Omega_2))$. *If* $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ *and* $T : L_1(\Omega_0) \times L_1(\Omega_1) \longrightarrow L_1(\Omega_2)$ *is compact, then* T *may be uniquely extended to a compact bilinear operator from* $L_{p,\alpha}(\Omega_0) \times L_{p,\beta}(\Omega_1)$ *to* $L_{p,\gamma}(\Omega_2)$.

Proof. Apply Theorem 4.8 with $\theta = 1 - 1/p$. The couple $\bar{E}$ satisfies $(\mathfrak{H})$ by Proposition 4.5. □

Operators belonging to $T \in \mathcal{B}\left((L_1(\Omega_0), L_\infty(\Omega_0)), (L_1(\Omega_1), L_\infty(\Omega_1)); (L_1(\Omega_2), L_\infty(\Omega_2))\right)$ are referred in [1] and [37] as *tensor-product operators*. An example is

$$T(f, g)(x, y) = (f \otimes g)(x, y) = f(x)g(y).$$

The following result deals with *integral operators* in the terminology of [37]. That is, operators T belonging to $\mathcal{B}\left((L_1(\Omega_0), L_\infty(\Omega_0)), (L_\infty(\Omega_1), L_1(\Omega_1)); (L_1(\Omega_2), L_\infty(\Omega_2))\right)$. The prototype is

$$T(f, g)(x) = \int_{\Omega_1} f(x, y)g(y)\, d\mu_1.$$

Theorem 5.2. *Suppose* $1 < p < \infty$, $1/p + 1/p' = 1$ *and* $1 \leq \alpha, \beta, \gamma \leq \infty$ *with* $1/\alpha + 1/\beta = 1 + 1/\gamma$. *Let* $\bar{A} = (L_1(\Omega_0), L_\infty(\Omega_0))$, $\bar{B} = (L_\infty(\Omega_1), L_1(\Omega_1))$ *and* $\bar{E} = (L_1(\Omega_2), L_\infty(\Omega_2))$. *If* $T \in \mathcal{B}(\bar{A}, \bar{B}; \bar{E})$ *and* $T : L_1(\Omega_0) \times L_\infty(\Omega_1) \longrightarrow L_1(\Omega_2)$ *compactly, then* T *may be uniquely extended to a compact bilinear operator from* $L_{p,\alpha}(\Omega_0) \times L_{p',\beta}(\Omega_1)$ *into* $L_{p,\gamma}(\Omega_2)$.

Proof. It follows from Theorem 4.8 with $\theta = 1 - 1/p$. □

Next we consider *convolution operators* in the sense of [33] and [1]. That is, a bilinear operators T satisfying

$$\begin{cases} \|T(f, g)\|_{L_1(\Omega_2)} \leq \|f\|_{L_1(\Omega_0)} \|g\|_{L_1(\Omega_1)}, \\ \|T(f, g)\|_{L_\infty(\Omega_2)} \leq \|f\|_{L_\infty(\Omega_0)} \|g\|_{L_1(\Omega_1)}, \\ \|T(f, g)\|_{L_\infty(\Omega_2)} \leq \|f\|_{L_1(\Omega_0)} \|g\|_{L_\infty(\Omega_1)}. \end{cases} \tag{5.1}$$

The common example being

$$T(f, g)(x) = (f * g)(x) = \int_{\Omega_1} f(x - y)g(y)\, d\mu_1.$$

Theorem 5.3. *Suppose* $1 < p, q, r < \infty$, $1 \leq \alpha, \beta < \infty$, $1 \leq \gamma \leq \infty$ *with* $1/p + 1/q = 1 + 1/r$ *and* $1/\alpha + 1/\beta = 1 + 1/\gamma$. *Let* T *be a bilinear operator satisfying (5.1) and such that* $T : L_1(\Omega_0) \times L_1(\Omega_1) \longrightarrow L_1(\Omega_2)$ *compactly. Then* T *may be uniquely extended to a compact bilinear operator from* $L_{p,\alpha}(\Omega_0) \times L_{q,\beta}(\Omega_1)$ *to* $L_{r,\gamma}(\Omega_2)$.

Proof. Since $1/p + 1/q = 1 + 1/r$ and $1/q < 1$, we have $1/p > 1/r$. Let $\theta = 1 - 1/r$ and $u = \theta(1/p - 1/r)^{-1}$. Then $1/p = 1 - \theta + \theta/u$. Since the operator T belongs to $\mathcal{B}\left((L_\infty(\Omega_0), L_1(\Omega_0)), (L_1(\Omega_1), L_\infty(\Omega_1)); (L_\infty(\Omega_2), L_\infty(\Omega_2))\right)$, using the complex bilinear interpolation theorem [5, Theorem 4.4.1] with $\eta = 1/u$ we derive that $T : L_u(\Omega_0) \times L_{u'}(\Omega_1) \longrightarrow L_\infty(\Omega_2)$ boundedly, where $1/u + 1/u' = 1$. This yields that

$$T \in \mathcal{B}\left((L_1(\Omega_0), L_u(\Omega_0)), (L_1(\Omega_1), L_{u'}(\Omega_1)); (L_1(\Omega_2), L_\infty(\Omega_2))\right).$$

Applying now Theorem 4.8 and using that $(L_1(\Omega_2), L_\infty(\Omega_2))_{\theta,\gamma} = L_{r,\gamma}(\Omega_2)$, and similarly $(L_1(\Omega_0), L_u(\Omega_0))_{\theta,\alpha} = L_{p,\alpha}(\Omega_0)$ and $(L_1(\Omega_1), L_{u'}(\Omega_1))_{\theta,\beta} = L_{q,\beta}(\Omega_1)$ because $1 - \theta + \theta/u' = 1/q$, we conclude the wanted result. $\square$

Finally we consider commutators of bilinear Calderón-Zygmund operators. We take $(\Omega_0, \mu_0) = (\Omega_1, \mu_1) = (\Omega_2, \mu_2) = (\mathbb{R}^n, dx)$ so in what follows we skip the measure space in our notation for function spaces. By a *bilinear Calderón-Zygmund operator* T we mean a bounded bilinear operator $T : L_p \times L_q \longrightarrow L_r$ where $1 < p, q < \infty$, $1 \le r < \infty$, $1/r = 1/p + 1/q$, such that there exits a kernel $K(x, y, z)$ defined away of the diagonal $x = y = z$ such that

$$|K(x,y,z)| \le c\,\frac{1}{(|x-y| + |x-z|)^{2n}} \quad , \quad |\nabla K(x,y,z)| \le c\,\frac{1}{(|x-y| + |x-z|)^{2n+1}}\,,$$

and

$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z) f(y) g(z)\, dydz\,, \quad x \notin \operatorname{supp} f \cap \operatorname{supp} g$$

where f, g are bounded functions with compact support. See the paper by Grafacos and Torres [25] and the references given there.

Let b, b_1, b_2 be functions in CMO, the closure in BMO of the space of C^∞ functions with compact support. Consider the following bilinear commutators

$$\begin{cases} [T,b]_1(f,g) = T(bf,g) - bT(f,g), \\ [T,b]_2(f,g) = T(f,bg) - bT(f,g), \\ \left[[T,b_1]_1, b_2\right]_2(f,g) = [T,b_1]_1(f, b_2 g) - b_2[T,b_1]_1(f,g). \end{cases} \tag{5.2}$$

It has been shown by Bényi and Torres [4, Theorem 1] that the three commutators acting from $L_p \times L_q \longrightarrow L_r$ are compact. We can now complement this result by showing some other function spaces where the action of the commutators is also compact.

Theorem 5.4. *Let T be a bilinear Calderón-Zygmund operator, let $b, b_1, b_2 \in CMO$ and let S be any of the bilinear commutators defined in (5.2). Suppose $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$.*

(a) *If $1 \le \alpha, \beta < \infty$, $1 \le \gamma \le \infty$ and $1/\alpha + 1/\beta = 1 + 1/\gamma$, then $S : L_{p,\alpha} \times L_{q,\beta} \longrightarrow L_{r,\gamma}$ is compact.*

(b) *If* $1 < s < \infty$ *and* $\delta > (s-1)/s$ *or* $s = 1$ *and* $\delta \geq 0$,
then $S : L_{p,s}(\log L)_\delta \times L_{q,s}(\log L)_\delta \longrightarrow L_{r,s}(\log L)_\delta$ *is compact.*

Proof. Take $\varepsilon > 0$ such that $1 < (1-\varepsilon)\min\{p,q,r\}$ and $\theta = (1+\varepsilon)/2 < 1$. For $u = p, q, r$, write $u_0 = (1-\varepsilon)u$ and $u_1 = (1+\varepsilon)u$. Then $1 < u_0, u_1 < \infty$, $(1-\theta)/u_0 + \theta/u_1 = 1/u$ and $1/p_j + 1/q_j = 1/r_j$ for $j = 0, 1$. Hence, by [4, Theorem 1], $S : L_{p_j} \times L_{q_j} \longrightarrow L_{r_j}$ compactly for $j = 0, 1$. Now, using Theorem 4.8 and having in mind that $(L_{u_0}, L_{u_1})_{\theta,v} = L_{u,v}$, we obtain the case (a).

To establish (b) we take $\Gamma_0 = \Gamma_1 = \Gamma_2 = \ell_s(2^{-\theta m}(1+|m|)^\delta)$. According to [12, Example 2.4 and Corollary 3.9], these Banach sequence lattices satisfy that $\Gamma_0 * \Gamma_1 \hookrightarrow \Gamma_2$. Moreover, we have that

$$(L_{u_0}, L_{u_1})_{\ell_s(2^{-\theta m}(1+|m|)^\delta)} = L_{u,s}(\log L)_\delta$$

by [36, Proposition 6.2]. Consequently, applying Theorem 4.6 we obtain the case (b). $\square$

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