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HARMONIC MEAN CORDIAL LABELING OF SOME GRAPHS

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ABSTRACT. All the graphs considered in this article are simple and undirected. Let *G = (V(G), E(G))* be a simple undirected Graph. A function $f : V(G) \to \{1, 2\}$ is called *Harmonic Mean Cordial* if the induced function $f^* : E(G) \to \{1, 2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A Graph G is called *Harmonic Mean Cordial graph* if it admits Harmonic Mean Cordial labeling. In this article, we have provided some graphs which are not Harmonic Mean Cordial and also we have provided some graphs which are Harmonic Mean Cordial.

Keywords: Harmonic Mean Cordial, cycle, complete bipartite graph, join of two graphs.

AMS Subject Classification: 05C78.

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. The concept of cordial labeling was introduced by Cahit in [2] the year 1987. Recall from [5] that function $f : V(G) \to \{1, 2\}$ is called *Harmonic Mean Cordial* if the induced function $f^* : E(G) \to \{1, 2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A Graph G is called *Harmonic Mean Cordial* if it admits Harmonic Mean Cordial labeling. Let $G = (V(G), E(G))$ be a simple, undirected graph and $\{v_1, v_2, \cdots v_n\} \subseteq V(G)$, we call $v_1, v_2, \cdots v_n$ are in sequence if it forms a path. For the sake of convenience of the reader we use *HMC* for harmonic mean cordial labeling. Motivated by the Results proved in [5] and [4], in this article, we have provided some

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examples of non *HMC* graphs and also we have provided some *HMC* graphs. It is useful to recall some useful definitions of graph theory to make this article self-contained.

Definition 1.1. [1] *A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. It is denoted by K_n.*

Definition 1.2. [1] *A walk in a graph G is a finite alternating sequence of vertices and edges. A walk is called a trail if all the edges are distinct. Cycle is a closed trail in which all the vertices are distinct. It is denoted by C_n.*

Definition 1.3. [1] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ is the graphs whose vertex set is $V_1 \cup V_2$ and egde set is $E_1 \cup E_2$. When G_1 and G_2 are vertex disjoint $G_1 \cup G_2$ is called sum of G_1 and G_2 and it is denoted by $G_1 + G_2$.*

Definition 1.4. [1] *Let G_1 and G_2 be two vertex disjoint graphs. Then the join $G_1 \vee G_2$ of G_1 and G_2 is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is also adjacent to every vertex of G_2.*

Definition 1.5. [1] *A graph is bipartite if is its vertex set can be partitioned into two non empty subsets V_1 and V_2 such that each edge of G has one end in V_1 and other in V_2. The pair (V_1, V_2) is called bipartition of a bipartite graph. It is denoted by $G(V_1, V_2)$. A simple bipartite graph $G(V_1, V_2)$ is complete if each vertex of V_1 is adjacent to all the vertices of V_2. If $G(V_1, V_2)$ is complete with $|V_1| = m$ and $|V_2| = n$ then $G(V_1, V_2)$ is denoted by $K_{m,n}$.*

Definition 1.6. [6] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the Corona of G_1 and G_2 is denoted as $G_1 \odot G_2$ is a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i^th vertex of G_1 to every point in the i^th copy of G_2 .*

Definition 1.7. [1] *The Helm graph H_n is the graph obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle.*

Definition 1.8. [3] *A Closed helm is the graph obtained from a helm by joining each pendent vertex to form a cycle. It is denoted by CH_n.*

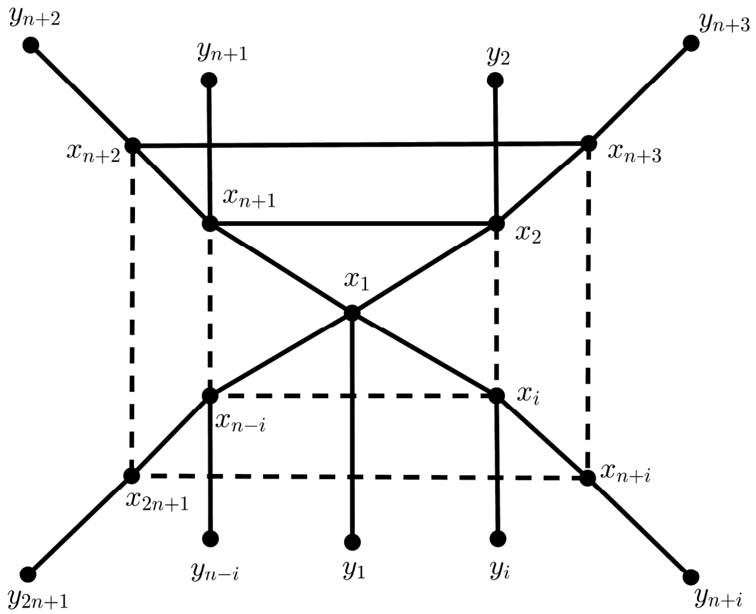
Definition 1.9. [3] *The direct (tensor) product $G \times H$ of two graphs $G = (V(G),\ E(G))$ and $H = (V(H),\ E(H))$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(x, y)(x', y') | xx' \in E(G) \text{ and } yy' \in E(H)\}$.*

Here, we mention the results proved in the Section 2. In Theorem 2.1, we have proved that the $CH_n \odot K_1$ is *HMC*. We have shown in Theroem 2.2 that the tensor Product $P_m \times P_n$ is *HMC*. The Complete bipartite graph $K_{m,n}$ is not *HMC* is proved in theorem 2.3. We have discussed harmonic mean cordial labelling of $K_n \vee C_m$ and we have proved that it is not *HMC* in Corollary 2.1 for any $n \geq 2$, $m \geq 3$ and $n, m \in N$. In Corollary 2.2, we have proved that $C_m \vee C_n$ is not *HMC* for any $n, m \geq 3$ and $n, m \in N$.

2. Main Results

Theorem 2.1. *$CH_n \odot K_1$ is HMC.*

Proof. Note that $|V(CH_n \odot K_1)| = 4n + 2$ and $|E(CH_n \odot K_1)| = 6n + 1$. Let $V(CH_n) = \{x_1, x_2, ..., x_{n+1}\}$ be the vertex set of CH_n with x_1 as an apex vertex and y_i be the pendent vertex, adjacent to x_i in $CH_n \odot K_1$ for $1 \leq i \leq 2n + 1$ as shown in the following figure.

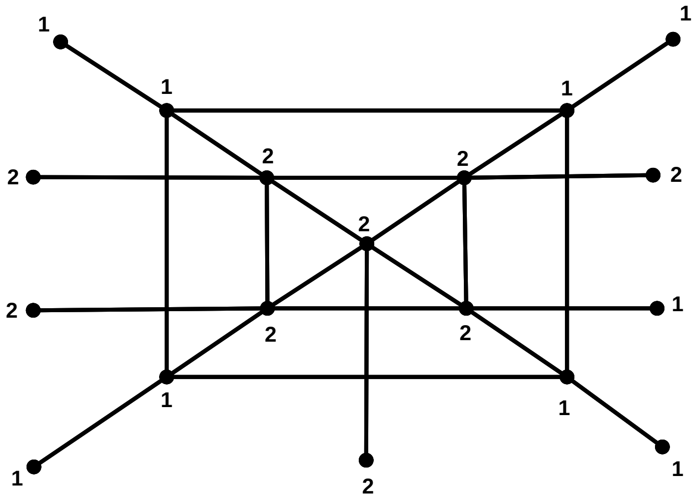


$CH_n \odot K_1$

Let us define the labeling function $f : V(CH_n \odot K_1) \to \{1, 2\}$ as follows
$f(x_i) = 2,\ 1 \le i \le n+1$
$f(x_i) = 1,\ n+1 \le i \le 2n+1$
$f(y_i) = 2,\ 1 \le i \le n$
$f(x_i) = 1,\ n+1 \le i \le 2n+1$
Note that, $v_f(1) = 2n+1 = v_f(2)$ and $e_f(1) = 3n+1$, $e_f(2) = 3n$. Therefore, $CH_n \odot K_1$ is *HMC*.

□

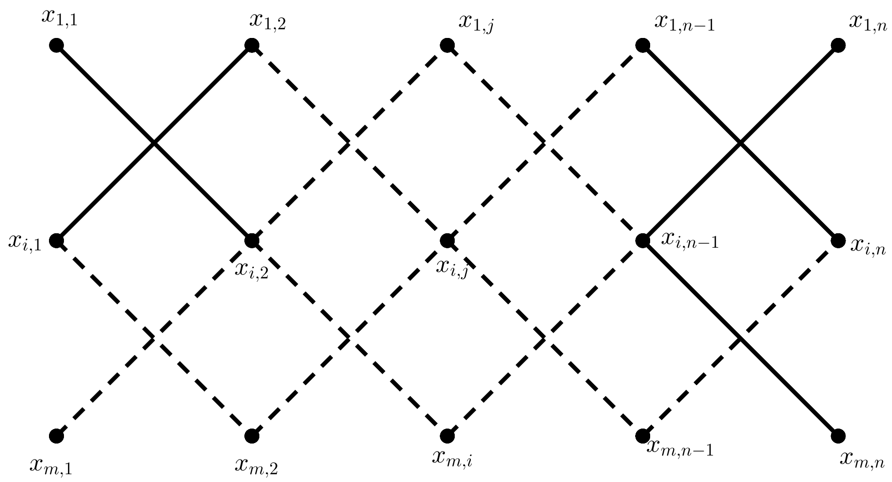
Example 2.1. *HMC labeling of $CH_4 \odot K_1$ is shown in the following figure*



$CH_4 \odot K_1$

Theorem 2.2. *The $P_m \times P_n$ is HMC $\forall\ m, n \in \mathbb{N}$.*

Proof. Let $G = (V, E)$ be the $P_m \times P_n$. Note that $|V| = mn$ and $|E| = 2mn - 2m - 2n + 2$. Let $V = \{x_{1,1}, x_{1,2}, ..., x_{1,n}, x_{2,1}, x_{2,2}, ..., x_{2,n}, ..., x_{m,1}, x_{m,2}, ..., x_{m,n}\}$ be a vertex set of G as shown in the following figure



$P_m \times P_n$

Case 1: mn is even
Define a labeling function $f : V(P_m \times P_n) \to \{1, 2\}$ as follows,

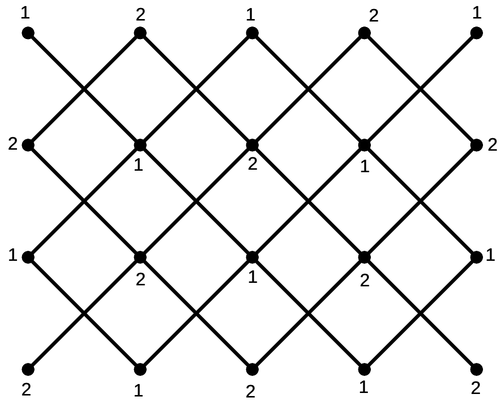
$$f(x_{i,j}) = \begin{cases} 1 & \text{if } 1 \le i \equiv 1 \pmod 2 \le m \text{ and } 1 \le j \equiv 1 \pmod 2 \le n \\ 1 & \text{if } 1 \le i \equiv 0 \pmod 2 \le m \text{ and } 1 \le j \equiv 0 \pmod 2 \le n \\ 2 & \text{if } 1 \le i \equiv 1 \pmod 2 \le m \text{ and } 1 \le j \equiv 0 \pmod 2 \le n \\ 2 & \text{if } 1 \le i \equiv 0 \pmod 2 \le m \text{ and } 1 \le j \equiv 1 \pmod 2 \le n \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{mn}{2}$ and $e_f(1) = e_f(2) = \frac{2mn-2m-2n+2}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$.
Case 2: mn is odd
Define a labeling function $f : V(P_m \times P_n) \to \{1, 2\}$ as follows,

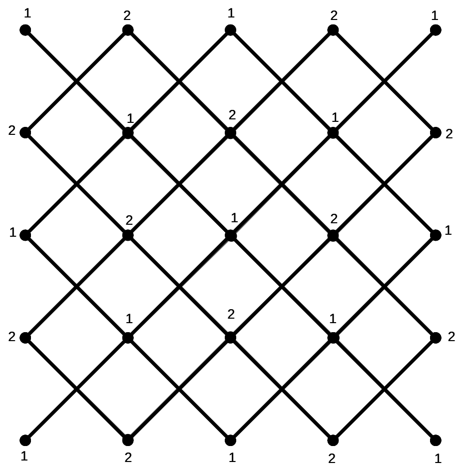
$$f(x_{i,j}) = \begin{cases} 1 & \text{if } 1 \le i \equiv 1 \pmod 2 \le m \text{ and } 1 \le j \equiv 1 \pmod 2 \le n \\ 1 & \text{if } 1 \le i \equiv 0 \pmod 2 \le m \text{ and } 1 \le j \equiv 0 \pmod 2 \le n \\ 2 & \text{if } 1 \le i \equiv 1 \pmod 2 \le m \text{ and } 1 \le j \equiv 0 \pmod 2 \le n \\ 2 & \text{if } 1 \le i \equiv 0 \pmod 2 \le m \text{ and } 1 \le j \equiv 1 \pmod 2 \le n \end{cases}$$

Then $v_f(1) = \frac{mn+1}{2}$, $v_f(2) = \frac{mn-1}{2}$ and $e_f(1) = e_f(2) = \frac{2mn-2m-2n+2}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.
Hence, The $P_m \times P_n$ is *HMC*. □

Example 2.2. *HMC labeling of $P_4 \times P_5$ and $P_5 \times P_5$ is shown in the following figure.*



$P_4 \times P_5$



$P_5 \times P_5$

Theorem 2.3. *Complete bipartite graph $K_{m,n}$ is not HMC, where $m, n \geq 2$.*

Proof. Without loss of generality, we may assume $m \geq n$. Let $V(K_{m,n}) = V_1 \cup V_2$. Where, $|V_1| = n$ and $|V_2| = m$. Suppose that $K_{m,n}$ is *HMC*.

Case 1: $n + m$ is even

Since $K_{m,n}$ is *HMC*, we have $v_f(1) = \frac{n+m}{2} = v_f(2)$. Suppose that there exist t vertices with label 1 in V_1. So, we have $n - t$ vertices with label 2 in V_1. Hence, there exists $(\frac{m+n}{2} - t)$ vertices with label 1 in V_2 and $m - (\frac{m+n}{2} - t) = (\frac{m-n}{2} + t)$ vertices with label 2 in V_2. Note that, $e_f(1) = mt + (n-t)(\frac{m+n}{2} - t)$ and $e_f(2) = (n-t)(m - \frac{m+n}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n}{2} - t) - (n-t)(m - \frac{m+n}{2} + t) = mt + (n-t)^2 - nt + t^2 > 2$.

Case 2 : $n + m$ is odd

In this Case we have two possibilities

$(i) v_f(1) = \frac{n+m+1}{2}$ and $v_f(2) = \frac{n+m-1}{2}$

$(ii) v_f(1) = \frac{n+m-1}{2}$ and $v_f(2) = \frac{n+m+1}{2}$

So, we consider the following Cases.

Subase 2.1: $v_f(1) = \frac{n+m+1}{2}$ and $v_f(2) = \frac{n+m-1}{2}$

Suppose that there exist t vertices with label 1 in V_1. So, we have, $(n-t)$ vertices with label 2 in V_1. Hence, there exists $(\frac{m+n+1}{2} - t)$ vertices with label 1 in V_2 and $[m - (\frac{m+n+1}{2} - t)] = (\frac{m-n-1}{2} + t)$ vertices with label 2 in V_2. Note that, $e_f(1) = mt + (n-t)(\frac{m+n+1}{2} - t)$ and $e_f(2) = (n-t)(\frac{m-n-1}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n+1}{2} - t) - (n-t)(\frac{m-n-1}{2} + t) = mt + (n-t)^2 + n - nt + t^2 - t > 2$.

Subase 2.2: $v_f(1) = \frac{n+m-1}{2}$ and $v_f(2) = \frac{n+m+1}{2}$

Suppose that there exist t vertices with label 1 in V_1. So, we have, $(n-t)$ vertices with label 2 in V_1. Hence, there exists $(\frac{m+n-1}{2} - t)$ vertices with label 1 in V_2 and $[m - (\frac{m+n-1}{2} - t)] = (\frac{m-n+1}{2} + t)$ vertices with label 2 in V_2. Note that, $e_f(1) = mt + (n-t)(\frac{m+n-1}{2} - t)$ and $e_f(2) = (n-t)(\frac{m-n+1}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n+1}{2} - t) - (n-t)(\frac{m-n+1}{2} + t) = mt + (n-t)^2 - n + nt > 2$. Hence, $K_{m,n}$ is not *HMC*, where $m, n \geq 2$.

□

Theorem 2.4. *$K_n \vee C_n$ is not HMC, where $n \geq 3$.*

Proof. Suppose that $K_n \vee C_n$ is *HMC*. Note that, $|V(K_n \vee C_n)| = 2n$ and $|E(K_n \vee C_n)| = n\frac{(n-1)}{2} + n + n^2$. Since, $|V(K_n \vee C_n)| = 2n$ and we have assume that $K_n \vee C_n$ is *HMC*. We have $v_f(1) = v_f(2) = n$.

Case 1: All the vertices of label 1 and label 2 are in sequence in C_n

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(n-t)$ vertices of of label 1 in C_n. Hence, we have $(n-t)$ vertices of label 2 in K_n and t vertices of label 2 in C_n. Note that, $e_f(1) = (n-t)t + t\frac{(t-1)}{2} + (n-t)^2 + (n-t+1) + nt$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + t(n-t) + (t-1)$. Now, $e_f(1) - e_f(2) = \frac{n^2}{2} + t^2 + \frac{3n}{2} - 3t + 2$. If $t \geq 3$ then as $n \geq 2$, we have $e_f(1) - e_f(2) > 1$.

If $t = 1$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

If $t = 2$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in C_n

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(n-t)$ vertices of label 1 in C_n. Hence, we have $(n-t)$ vertices of label 2 in K_n and t vertices of label 2 in C_n. Suppose that there exist i no. of vertices with label 2 are not in sequence in C_n. Then, we have $e_f(1) = \frac{t(t-1)}{2} + t(n-t) + tn + (n-t)^2 + (n-t+i+1)$ and $e_f(2) = (t-i-1) + \frac{(n-t-1)(n-t)}{2} + t(n-t)$ Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Case 1. Now, we have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, in this Case $e_f(1) - e_f(2) > 1$.

Case 3: We have n no. of vertices with label 1 in K_n and n no. of vertices with label 2 in C_n

Then, we have $e_f(1) = \frac{n(n-1)}{2} + n^2$ and $e_f(2) = n$. Then, $e_f(1) - e_f(2) = \frac{n(n-1)}{2} + n^2 - n = \frac{3n^2}{2} - \frac{3n}{2} > 1$ as $n^2 > n$.

Case 4: We have n no. of vertices with label 2 in K_n and n no. of vertices with label 1 in C_n

Then we have, $e_f(1) = n^2 + n$ and $e_f(2) = \frac{n(n-1)}{2}$. Then, $e_f(1) - e_f(2) = n^2 + n - \frac{n(n-1)}{2} = \frac{n^2}{2} + \frac{3n}{2} > 1$.

Hence, $K_n \vee C_n$ is not *HMC* where, $n \geq 3$. □

Theorem 2.5. *$K_n \vee C_m$ is not HMC, where $m + n$ is even and $n \geq 2$, $m \geq 3$.*

Proof. Note that, $|V(K_n \vee C_m)| = m + n$. Suppose that $K_n \vee C_m$ is *HMC*. Then we have, $|v_f(1)| = \frac{m+n}{2} = |v_f(2)|$.

Case 1: All the vertices with label 1 and label 2 are in sequence in C_m

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n}{2} - t)$ vertices with label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n}{2} - t) = (\frac{m-n}{2} + t)$ vertices with label 2 in c_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + tm + (n-t)(\frac{m+n}{2} - t) + (\frac{m+n}{2} - t + 1) + t(n-t)$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n}{2} + t) + (\frac{m-n}{2} + t - 1)$. Then, $e_f(1) - e_f(2) = mt + \frac{n^2}{2} - nt + \frac{3n}{2} + t^2 - 3t + 2 = (t-n)^2(\frac{1}{2}) + \frac{t^2}{2} + \frac{3n}{2} + 2 + t(m-3)$. If $m \geq 3$, then $e_f(1) - e_f(2) > 1$.

If $m = 2$, then $e_f(1) - e_f(2) = (t-n)^2(\frac{1}{2}) + \frac{t^2}{2} + \frac{n}{2} + (n-t) + 2$.

Now, $n > t$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices with label 2 are not in sequence in C_m

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n}{2} - t)$

vertices with label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n}{2} - t) = (\frac{m-n}{2} + t)$ vertices with label 2 in C_m. Suppose that there exist i no. of vertices from $(\frac{m-n}{2} + t)$ with label 2 are not in sequence in C_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + t(n-t) + tm + (n-t)(\frac{m+n}{2} - t) + (\frac{m+n}{2} - t + i + 1)$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n}{2} + t) + (\frac{m-n}{2} + t - i - 1)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Case 1. Now, we have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Case 3: $m < n$

Subcase 3.1: All the vertices in C_m are with label 1.

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(n-t)$ vertices with label 2 in K_n. Then we have, $e_f(1) = \frac{t(t-1)}{2} + mn + m + t(n-t)$ and $\frac{(n-t)(n-t-1)}{2}$. Then, $e_f(1) - e_f(2) = mn + m + 2nt + \frac{n}{2} - t^2 - t - \frac{n^2}{2}$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(1) - e_f(2) = \frac{3mn}{2} + \frac{m}{2} + (\frac{n^2}{4} - \frac{m^2}{4})$. We know that $n > m$. So, $e_f(1) - e_f(2) > 1$.

Subcase 3.2: All the vertices in C_m are with label 2.

Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(n-t)$ vertices with label 2 in K_n. Then we have, $e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t)$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + m(n-t) + m$. Then, $e_f(1) - e_f(2) = -m(n-t) + mt - m - \frac{n^2}{2} + 2nt + \frac{n}{2} - t^2 - t$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{3m^2}{4} + \frac{n^2}{4} - \frac{3m}{2}$. As $m \geq 2$. So, $e_f(1) - e_f(2) > 1$.

Case 4: $m > n$

Subcase 4.1: All the vertices in K_n are with label 1.

Suppose that we have t no. of vertices with label 1 in C_m. So, we have $(m-t)$ vertices with label 2 in C_m.

Subsubcase 4.1.1: All the vertices with label 2 are in sequence in C_m.

Then we have, $e_f(1) = \frac{n(n-1)}{2} + (t+1) + nm$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = \frac{n(n-1)}{2} + (t+1) + nm - m + t + 1$. We know that, $nm > m$. So, $e_f(1) - e_f(2) > 1$.

Subsubcase 4.1.2: Some of the vertices with label 2 are not in sequence in C_m.

Suppose that we have i no. of vertices with label 2 are not in sequence in C_m. Suppose that i no. of vertices are not i sequence. Then we have, $e_f(1) = \frac{n(n-1)}{2} + nm + (m + i + 1)$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subsubcase 4.1.2 $\leq e_f(2)$ in Subsubcase 4.1.1 and $e_f(1)$ in Subsubcase 4.1.2 $\geq e_f(1)$ in Subsubcase 4.1.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subsubcase 4.1.1. Now, we have already proved in Subsubcase 4.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Subcase 4.2: All the vertices in K_n are with label 2.

Suppose that we have t no. of vertices with label 1 in C_m. So, we have $(m-t)$ vertices with label 2 in C_m.

Subsubcase 4.2.1: All the vertices with label 2 are in sequence in C_m.

Then we have, $e_f(1) = (t+1) + nt$ and $e_f(2) = \frac{n(n-1)}{2} + (m-t-1) + n(m-t)$. Then, $e_f(1) - e_f(2) = (t+1) + tn - \frac{n(n-1)}{2} - m + t + 1 - mn + tn$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} + 2 > 1$.

Subsubcase 4.2.2: Some of the vertices with label 2 are not in sequence in C_m.

Suppose that i no. of vertices are not i sequence. Then we have, $e_f(1) = tn$ and $e_f(1) = \frac{n(n-1)}{2} + (m - t - i - 1) + n(m-t)$. Now, $e_f(2)$ in Subsubcase 4.2.2 $\leq e_f(2)$ in Subsubcase 4.2.1 and $e_f(1)$ in Subsubcase 4.2.2 $\geq e_f(1)$ in Subsubcase 4.2.1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subsubcase 4.2.1. Now, we have already proved in Subsubcase

4.2.1 that $e_f(2) - e_f(2) > 1$. Hence, $e_f(2) - e_f(2) > 1$ in this Case.
Hence, $K_n \vee C_m$ is not *HMC*, where $m + n$ is even and $n \geq 2$, $m \geq 3$. □

Theorem 2.6. *$K_n \vee C_m$ is not HMC, where $m + n$ is odd and $n \geq 2$, $m \geq 3$.*

Proof. Note that, $|V(K_n \vee C_m)| = m + n$. Suppose that $K_n \vee C_m$ is *HMC*.
Case 1: All the vertices with label 1 and label 2 in C_m are in sequence in C_m
In this Case we have two possibilities
(i) $v_f(1) = \frac{m+n+1}{2}$ and $v_f(2) = \frac{m+n-1}{2}$
(ii) $v_f(1) = \frac{m+n-1}{2}$ and $v_f(2) = \frac{m+n+1}{2}$
So, we consider the following cases.
Subcase 1.1: $v_f(1) = \frac{m+n+1}{2}$ and $v_f(2) = \frac{m+n-1}{2}$
Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n+1}{2} - t)$ vertices of label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n+1}{2} - t) = (\frac{m-n-1}{2} + t)$ vertices with label 2 in C_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + tn + t(n-t) + \frac{m+n+1}{2} - t + 1 + (n-t)(\frac{m+n+1}{2} - t)$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n-1}{2} + t) + \frac{m-n-1}{2} + t - 1$. Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{5n}{2} + t^2 - 4t + 3 = (t-1)^2 + \frac{n^2}{2} + 2 + \frac{n}{2} + 2(n-t) > 1$ as $n > t$.
Subcase 1.2: $v_f(1) = \frac{m+n-1}{2}$ and $v_f(2) = \frac{m+n+1}{2}$
Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n-1}{2} - t)$ vertices of label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n-1}{2} - t) = (\frac{m-n+1}{2} + t)$ vertices with label 2 in C_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + \frac{m+n-1}{2} - t + 1 + (n-t)(\frac{m+n-1}{2} - t)$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n+1}{2} + t) + \frac{m-n+1}{2} + t - 1$. Then, $e_f(1) - e_f(2) = t^2 + \frac{n^2}{2} + \frac{n}{2} + mt + 1 - 2t - nt = \frac{(n-1)^2}{2} + (t-1)^2 + mt + \frac{n}{2} - \frac{1}{2} > 1$ as $n \geq 2$.
Case 2: Some of the vertices with label 2 are not in sequence in C_m
Subcase 2.1: Suppose that $v_f(1) = \frac{m+n+1}{2}$ and $v_f(2) = \frac{m+n-1}{2}$
Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n+1}{2} - t)$ vertices of label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n+1}{2} - t) = (\frac{m-n-1}{2} + t)$ vertices with label 2 in C_m. Suppose that there exist i no. of vertices from $(\frac{m-n-1}{2} + t)$ with label 2 are not in sequence in C_m. Then we have,$e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + (\frac{m+n+1}{2} - t + i + 1) + (n-t)(\frac{m+n+1}{2} - t)$ and $e_f(2) = (\frac{m-n-t}{2} - t - i - 1) + \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n-1}{2} + t)$. Now, $e_f(2)$ in Subcase 2.1 $\leq e_f(2)$ in Subcase 1.1 and $e_f(1)$ Subcase 2.1 $\geq e_f(1)$ in Subcase 1.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subcase 1.1. Now, we have already proved in Subcase 1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.
Subcase 2.2: $v_f(1) = \frac{m+n-1}{2}$ and $v_f(2) = \frac{m+n+1}{2}$
Suppose that we have t no. of vertices with label 1 in K_n. So, we have $(\frac{m+n-1}{2} - t)$ vertices of label 1 in C_m. Hence, we have $(n-t)$ vertices with label 2 in K_n and $m - (\frac{m+n-1}{2} - t) = (\frac{m-n+1}{2} + t)$ vertices with label 2 in C_m. Suppose that there exist i no. of vertices from $(\frac{m-n+1}{2} + t)$ with label 2 are not in sequence in C_m. Then we have,$e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + (\frac{m+n-1}{2} - t + i + 1) + (n-t)(\frac{m+n-1}{2} - t)$ and $e_f(2) = (\frac{m-n+1}{2} + t - i - 1) + \frac{(n-t)(n-t-1)}{2} + (n-t)(\frac{m-n+1}{2} + t)$. Now, $e_f(2)$ in Subcase 2.2 $\leq e_f(2)$ in Subcase 1.2. and $e_f(1)$ Subcase 2.2 $\geq e_f(1)$ in Subcase 1.2. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subcase 2.1. Now, we have already proved in Subcase 2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.
Case 3: $m < n$
Subcase 3.1: All the vertices in C_m are with label 1 and some vertices with label 1 are

in K_n.
Suppose that there exist t no. of vertices with label 1 in K_n. So, there exists $(n-t)$ vertices with label 2 in K_n. Suppose that we have m no. of vertices with label 1 in C_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + t(n-t) + mn + m$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2}$. Then, $e_f(1) - e_f(2) = mn + m + 2nt + \frac{n}{2} - t - t^2 - \frac{n^2}{2}$.
In this Case we have two possibilities
(i) $m + t = \frac{m+n+1}{2}$
(ii) $m + t = \frac{m+n-1}{2}$
So, we consider the following cases.
Subsubcase 3.1.1: $m + t = \frac{m+n+1}{2}$.
Therefore, $t = \frac{n-m+1}{2}$. Then, $e_f(1) - e_f(2) = \frac{mn}{2} + (2m - \frac{3}{4}) + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{n}{2} > 1$ as $m < n$ and $2m > \frac{3}{4}$ as $m \geq 2$.
Subsubcase 3.1.2: $m + t = \frac{m+n-1}{2}$.
Therefore, $t = \frac{n-m-1}{2}$. Then, $e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{n}{2}) + m + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{1}{4} > 1$ as $n > m$.
Subcase 3.2: All the vertices in C_m are with label 2 and some vertices with label 2 are in K_n.
Suppose that there exist t no. of vertices with label 1 in K_n. So, there exists $(n-t)$ vertices with label 2 in K_n. Suppose that we have m no. of vertices with label 2 in C_m. Then we have, $e_f(1) = \frac{t(t-1)}{2} + t(n-t) + tm$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + m(n-t) + m$. Then, $e_f(1) - e_f(2) = 2mt - mn - \frac{n^2}{2} + 2nt + \frac{n}{2} - t^2 - t - m$.
Subsubcase 3.2.1: $t = \frac{m+n+1}{2}$.
Then, $e_f(1) - e_f(2) = \frac{3m^2}{4} + (\frac{mn}{2} - m) + (\frac{n^2}{4} - \frac{3}{4}) + \frac{n}{2} > 1$ as $m, n \geq 2$.
Subsubcase 3.2.2: $t = \frac{m+n-1}{2}$.
Then, $e_f(1) - e_f(2) = \frac{3m^2}{4} + \frac{mn}{2} - 2m + \frac{n^2}{4} + \frac{1}{4} - \frac{n}{2} = (\frac{n^2}{4} - \frac{n}{2}) + m(\frac{3m}{4} + \frac{n}{2} - 2) + \frac{1}{4} > 1$ as $m, n \geq 2$.
Case 4: $m > n$ **and all the vertices with label 2 are in sequence in** C_m
Subcase 4.1: All the vertices in K_n are with label 1 and some vertices with label 1 are in C_m.
Suppose that there exist t no. of vertices with label 1 in C_m. So, there exists $(m-t)$ vertices with label 2 in C_m. Suppose that we have n no. of vertices with label 1 in K_n. Then we have, $e_f(1) = mn + (t+1) + n\frac{(n-1)}{2}$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = (mn - m) + 2 + 2t + (\frac{n^2}{2} - \frac{n}{2}) > 1$ as $mn > m$ and $\frac{n^2}{2} > \frac{n}{2}$, where, $m, n \geq 2$.
Subcase 4.2: All the vertices in K_n are with label 2 and some vertices with label 2 are in C_m.
Suppose that there exist t no. of vertices with label 1 in C_m. So, there exists $(m-t)$ vertices with label 2 in C_m. Suppose that we have n no. of vertices with label 2 in K_n. Then we have, $e_f(1) = tn + (t+1)$ and $e_f(2) = \frac{n(n-1)}{2} + n(m-t) + (m-t-1)$. Then, $e_f(1) - e_f(2) = 2t + 2 + 2nt - \frac{n^2}{2} + \frac{n}{2} - mn - m$.
Subsubcase 4.2.1: $t = \frac{m+n+1}{2}$.
Then, $e_f(1) - e_f(2) = \frac{5n}{2} + \frac{n^2}{2} + 3 > 1$.
Subsubcase 4.2.2: $t = \frac{m+n-1}{2}$.
Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{n}{2} + 1 > 1$.
Case 5: $m > n$ **and Suppose that some of the vertices with label 2 are not in sequence in** C_m
Subcase 5.1: All the vertices in K_n are with label 1 and some vertices with label 1 are in C_m.

Suppose that there exist t no. of vertices with label 1 in C_m. So, there exists $(m-t)$ vertices with label 2 in C_m. Suppose that we have n no. of vertices with label 1 in K_n. Suppose that we have i no. of vertices with label 2 are not in sequence in C_m. Then, $e_f(1) = \frac{n(n-1)}{2} + mn + (t+i+1)$ and $e_f(2) = m-t-i-1$. Now, $e_f(2)$ in Subcase 5.1 $\leq e_f(2)$ in Subcase 4.1 and $e_f(1)$ in Subsubcase 5.1 $\geq e_f(1)$ in Subsubcase 4.1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subcase 4.1. Now, we have already proved in Subcase 4.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Subcase 5.2: All the vertices in K_n are with label 2 and some vertices with label 2 are in C_m.

Suppose that there exist t no. of vertices with label 1 in C_m. So, there exists $(m-t)$ vertices with label 2 in C_m. Suppose that we have n no. of vertices with label 2 in K_n. Suppose that we have i no. of vertices with label 2 are not in sequence in C_m. Then $e_f(1) = nt + (t+i+1)$ and $e_f(2) = \frac{n(n-1)}{2} + (m-t-i-1)$. Now, $e_f(2)$ in Subcase 5.2 $\leq e_f(2)$ in Subcase 4.2 and $e_f(1)$ in Subsubcase 5.2 $\geq e_f(1)$ in Subsubcase 4.2. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subcase 4.2. Now, we have already proved in Subcase 4.2 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case. Hence, $K_n \vee C_m$ is not *HMC*, where $m+n$ is odd and $n \geq 2$, $m \geq 3$. □

Corollary 2.1. *$K_n \vee C_m$ is not HMC, where $n \geq 2$,$m \geq 3$, $m, n \in \mathbf{N}$*

Proof. Proof follows from Theorems 2.4, 2.5 and 2.6. □

Theorem 2.7. *$C_m \vee C_n$ is not HMC, where $m = n$ and $m \geq 3$.*

Proof. Suppose that $C_m \vee C_n$ is *HMC* for $m = n$. Note that, $|V(C_m \vee C_n)| = 2n$ and $|E(C_m + C_n)| = n + m + nm = 2n + n^2$ as $n = m$. Since, $|V(C_m \vee C_n)| = m + n = 2n$ as $n = m$. We have assume that $C_m \vee C_n$ is *HMC* for $n = m$. We have $v_f(1) = v_f(2) = n$.

Case 1: All the vertices of label 1 are in sequence in C_m and C_n

Then, it is clear that all the vertices of label 2 are in sequence in C_m and C_n. Suppose that we have t no. of vertices with label 1 in C_m. So, we have $(n-t)$ vertices of of label 1 in C_n. Hence, we have $(m-t)$ vertices of label 2 in C_m and t vertices of label 2 in C_n. Note that, $e_f(1) = (t+1)+(n-t+1)+tn+(n-t)n$ and $e_f(2) = (n-t-1)+(t-1)+t(n-t)$. Then, $e_f(1) - e_f(2) = 2n + 2 + tn + n^2 - t^2$. We know that, $n > t$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in C_m and C_n

Suppose that we have t no. of vertices with label 1 in C_m. So, we have $(n-t)$ vertices of label 1 in C_n. Hence, we have $(m-t)$ vertices of label 2 in C_m and t vertices of label 2 in C_n. Suppose that there exist i no. of vertices with label 2 are not in sequence in C_m and j no. of vertices with label 2 are not in sequence in C_n. Note that, $e_f(1) = (t+i+1)+(n-t+j+1)+tn+(n-t)m$ and $e_f(2) = (n-t-i-1)+(t-j-1)+t(n-t)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ We have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Case 3: We have m no. of vertices with label 1 in C_m and n no. of vertices with label 2 in C_n

Note that, $e_f(1) = mn + m$ and $e_f(2) = n$. Then, $e_f(1) - e_f(2) = mn + m - n = mn > 1$ as $n = m$.

Case 4: We have m no. of vertices with label 2 in C_m and n no. of vertices with label 1 in C_n

Note that, $e_f(1) = mn + n$ and $e_f(2) = m$. Then, $e_f(1) - e_f(2) = mn + n - m > 1$ as $n = m$. Hence, $C_m \vee C_n$ is not *HMC*, where $m = n$ and $m \geq 3$.

□

Theorem 2.8. *$C_m \vee C_n$ is not HMC, where $m+n$ is even and $m,n \geq 3$.*

Proof. Note that, $|V(C_n \vee C_m)| = n+m$. Suppose that $C_n \vee C_m$ is *HMC*. Since we have $v_f(1) = \frac{n+m}{2} = v_f(2)$.

Case 1: All the vertices of label 1 and 2 are in sequence in C_n and C_m

Suppose that we have t no. of vertices with label 1 in C_n. So, we have $(n-t)$ vertices of of label 2 in C_n. Hence, we have $\frac{n+m}{2} - t$ vertices of label 1 in C_m and $m - \frac{n+m}{2} + t$ vertices of label 2 in C_m. Note that, $e_f(1) = (t+1) + (\frac{n+m}{2} - t + 1) + tm + (\frac{n+m}{2} - t)(n-t)$ and $e_f(2) = (n-t-1) + (m - \frac{n+m}{2} + t - 1) + (n-t)(m - \frac{n+m}{2} + t)$. Then, $e_f(1) - e_f(2) = mt + 4 + n^2 - 3nt + 2t^2$. We know that $t = \frac{m+n}{2}$, So, we have $e_f(1) - e_f(2) = m^2 + 4 > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in C_n and C_m

Suppose that we have t no. of vertices with label 1 in C_n. So, we have $\frac{n+m}{2} - t$ vertices of label 1 in C_m. Hence, we have $(n-t)$ vertices of label 2 in C_n and $(m - \frac{n+m}{2} + t)$ vertices of label 2 in C_m. Suppose that there exist i no. of vertices with label 2 are not in sequence in C_n and j no. of vertices with label 2 are not in sequence in C_m. Note that, $e_f(1) = (t+i+1) + (\frac{n+m}{2} - t + j + 1) + tm + (n-t)(\frac{n+m}{2} - t)$ and $e_f(2) = (n-t-i-1) + (m - \frac{n+m}{2} + t - j - 1) + (n-t)(m - \frac{n+m}{2} + t)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Case 1. Now, we have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, in this Case $e_f(1) - e_f(2) > 1$.

Case 3: $m > n$

Subcase 3.1: All the vertices in C_n are with label 1.

So, we have n no. of vertices with label 1 in C_n. Suppose that we have t no. of vertices with label 1 in C_m. So, there exist $m-t$ no. of vertices with label 2 in C_m.

Subsubcase 3.1.1: All the vertices in c_m are in sequence.

Then, $e_f(1) = n + (t+1) + mn + tn$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = (mn - m) + n + 2t + tn + 2 > 1$ as $mn > m$.

Subsubcase 3.1.2: All the vertices with label 2 are not in sequence in c_m.

Suppose that we have i no. of vertices from $(m-t)$ no. of vertices are not in sequence in c_m. Then, $e_f(1) = n + (t+i+1) + mn$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subsubcase 3.1.2 $\leq e_f(2)$ in Subsubcase 3.1.1 and $e_f(1)$ in Subsubcase 3.1.2 $\geq e_f(1)$ in Subsubcase 3.1.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subsubcase 3.1.1. Now, we have already proved in Subsubcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Subcase 3.2: All the vertices in C_n are with label 2.

So, we have n no. of vertices with label 2 in C_n. Suppose that we have t no. of vertices with label 1 in C_m. So, there exist $m-t$ no. of vertices with label 2 in C_m.

Subsubcase 3.2.1: All the vertices in c_n are in sequence.

Then, $e_f(1) = t+1+tn$ and $e_f(2) = n+m-t-1$. Then, $e_f(1) - e_f(2) = nt - n - m + 2t + 2$. We know that $t = \frac{n+m}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} + 2 > 1$.

Subsubcase 3.2.2: All the vertices in c_n are not in sequence.

Suppose that we have i no. of vertices from $(n-t)$ no. of vertices are not in sequence in c_m. Then, $e_f(1) = t + i + 1 + tn$ and $e_f(2) = m - t - i - 1 + n + n(m-t)$. Now, $e_f(2)$ in Subsubcase 3.2.2 $\leq e_f(2)$ in Subsubcase 3.2.1 and $e_f(1)$ in Subsubcase 3.2.2 $\geq e_f(1)$ in Subsubcase 3.2.1. so, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subsubcase 3.2.1. Now, we have already proved in Subsubcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case. Hence, $C_m \vee C_n$ is not *HMC*, where $n+m$ is even and $m, n \geq 3$.

□

Theorem 2.9. *$C_m \vee C_n$ is not HMC, where $m+n$ is odd and $m, n \geq 3$.*

Proof. Note that, $|V(C_n \vee C_m)| = n+m = 2k+1$. Suppose that $C_n \vee C_m$ is *HMC*. Without loss of generality we may assume that $m > n$.
In this Case we have two possibilities.
$(i) v_f(1) = \frac{m+n+1}{2}$ and $v_f(2) = \frac{m+n-1}{2}$
$(ii) v_f(1) = \frac{m+n-1}{2}$ and $v_f(2) = \frac{m+n+1}{2}$
So, we consider the following cases.
Case 1: $v_f(1) = \frac{n+m+1}{2} = k+1$ **and** $v_f(2) = \frac{n+m-1}{2} = k$
Subcase 1.1: All the vertices of label 1 are in sequence in C_n and C_m
Then, it is clear that all the vertices of label 2 are in sequence in C_n and C_m. Suppose that we have t no. of vertices with label 1 in C_n. So, we have $(n-t)$ vertices of of label 2 in C_n. Hence, we have $(k+1-t)$ vertices of label 1 in C_m and $(k-n+t)$ vertices of label 2 in C_m. Note that, $e_f(1) = (t+1) + (k+2-t) + tm + (k+1-t)(n-t)$ and $e_f(2) = (n-t-1) + (k-n+t-1) + (n-t)(k-n+t)$. Then, $e_f(1) - e_f(2) = (n-t)^2 + 5 + tm + (n-t)(1-t) = (n-t)(n+1-2t) + tm + 5$. Now, $e_f(1) - e_f(2) > 1$ if $n+1 \geq 2t$. If $n+1 < 2t$, then $\frac{(n+1)}{2} < t$. Now, $t + k = \frac{m+n+1}{2} > \frac{(n+t)}{2} + k$.Therefore, $m > k$. Suppose that $t = \frac{(n+1)}{2} + l$. Then, $e_f(1) - e_f(2) = 2l^2 + 2l + \frac{1}{2} + lm + 5 > 1$.
Subcase 1.2: Some of the vertices of label 2 are not in sequence in C_n and C_m
Suppose that we have t no. of vertices with label 1 in C_n. So, we have $(n-t)$ vertices of label 2 in C_n. Hence, we have $(k-t)$ vertices of label 1 in C_m and $(k+1-n+t)$ vertices of label 2 in C_m. Suppose that there exist i no. of vertices with label 2 are not in sequence in C_n and j no. of vertices with label 2 are not in sequence in C_m. Note that, $e_f(1) = (t+i+1) + (k-t+j+2) + tm + (n-t)(k+1-t)$ and $e_f(2) = (n-t-i-1) + (k-n+t-j-1) + (n-t)(k-n+t)$. Now, $e_f(2)$ in Subcase 1.2 $\leq e_f(2)$ in Subcase 1.1 and $e_f(1)$ in Subcase 1.2 $\geq e_f(1)$ in Subcase 1.1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subcase 1.1. Now, we have already proved in Subcase 1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case .
Case 2: $v_f(1) = \frac{n+m-1}{2} = k$ **and** $v_f(2) = \frac{n+m+1}{2} = k+1$
Subcase 2.1: All the vertices of label 1 are in sequence in C_n and C_m
Then, it is clear that all the vertices of label 2 are in sequence in C_n and C_m. Suppose that we have t no. of vertices with label 1 in C_n. So, we have $(n-t)$ vertices of label 2 in C_n. Hence, we have $(k+1-t)$ vertices of label 1 in C_m and $(k-n+t)$ vertices of label 2 in C_m. Note that, $e_f(1) = (t+1) + (k+1-t) + tm + (k-t)(n-t)$ and $e_f(2) = (n-t-1) + (k-n+t) + (n-t)(k-n+t+1)$. Then, $e_f(1) - e_f(2) = (n-t)^2 + 3 + tm + (t-n)(1+t) = (n-t)(n-1-2t) + tm + 3$. Now, $e_f(1) - e_f(2) > 1$ if $n \geq 1 + 2t$. If $n < 1 + 2t$, then $\frac{(n-1)}{2} < t$. Now, $t + k = \frac{m+n-1}{2} > \frac{(n-t)}{2} + k$.Therefore, $m > k$. Suppose that $t = \frac{(n-1)}{2} + l$. Then, $e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3 + l(m-n-1) > 1$, if $m \geq n+1$. Suppose that $m \leq n+1$. Then since, $m \geq n$, we have $m = n+1$. So, we have $e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3 + l(m-n-1) = (\frac{mn}{2} - \frac{m}{2}) + 3 > 1$.
Subcase 2.2: Some of the vertices of label 2 are not in sequence in C_n and C_m
Suppose that we have t no. of vertices with label 1 in C_n. So, we have $(n-t)$ vertices of label 2 in C_n. Hence, we have $(k-t)$ vertices of label 1 in C_m and $(k+1-n+t)$ vertices of label 2 in C_m. Suppose that there exist i no. of vertices with label 2 are not in sequence in C_n and j no. of vertices with label 2 are not in sequence in C_m. Note that, $e_f(1) = (t+i+1) + (k-t+j+1) + tm + (n-t)(k-t)$ and $e_f(2) = (n-t-i-1) + (k-n+t-j) + (n-t)(k-n+t+1)$. Now, $e_f(2)$ in Subcase 2.2 $\leq e_f(2)$ in Subcase 2.1 and $e_f(1)$ in Subcase 2.2 $\geq e_f(1)$ in Subcase 2.1. So, $e_f(1) - e_f(2)$ in this

Case is $\geq e_f(1) - e_f(2)$ in Subcase 2.1. Now, we have already proved in Subcase 2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Case 3: $m > n$

Subcase 3.1: All the vertices in C_n are with label 1.

So, we have n no. of vertices with label 1 in C_n. Suppose that we have t no. of vertices with label 1 in C_m. So, there exist $m - t$ no. of vertices with label 2 in C_m.

Subsubcase 3.1.1: All the vertices in C_m are in sequence.

Then, $e_f(1) = n + (t + 1) + mn + tn$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = (mn - m) + n + 2t + tn + 2 > 1$ as $mn > m$.

Subsubcase 3.1.2: All the vertices with label 2 are not in sequence in C_m.

Suppose that we have i no. of vertices from $(m - t)$ no. of vertices are not in sequence in C_m. Then, $e_f(1) = n+(t+i+1)+mn$ and $e_f(2) = m-t-i-1$. Now, $e_f(2)$ in Subsubcase 3.1.2 $\leq e_f(2)$ in subsubcase 3.1.1 and $e_f(1)$ in Subsubcase 3.1.2 $\geq e_f(1)$ in Subsubcase 3.1.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in Subsubcase 3.1.1. Now, we have already proved in Subsubcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Subcase 3.2: All the vertices in C_n are with label 2.

So, we have n no. of vertices with label 2 in C_n. Suppose that we have t no. of vertices with label 1 in C_m. So, there exist $m - t$ no. of vertices with label 2 in C_m.

Subsubcase 3.2.1: All the vertices in C_n are in sequence.

Then, $e_f(1) = t+1+tn$ and $e_f(2) = n+m-t-1$. Then, $e_f(1)-e_f(2) = nt-n-m+2t+2$. In this Case we have two possibilities.

$(i) t = \frac{n+m+1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} + \frac{n}{2} + 3 > 1$

$(ii) t = \frac{n+m-1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + (\frac{n^2}{2} - \frac{n}{2}) + 3 > 1$.

Subsubcase 3.2.2: All the vertices in C_n are not in sequence.

Suppose that we have i no. of vertices from $(n - t)$ no. of vertices are not in sequence in C_m. Then, $e_f(1) = t + i + 1 + tn$ and $e_f(2) = m - t - i - 1 + n + n(m - t)$. Now, $e_f(2)$ in Subsubcase 3.2.2 $\leq e_f(2)$ in Subsubcase 3.2.1 and $e_f(1)$ in Subsubcase 3.2.2 $\geq e_f(1)$ in Subsubcase 3.2.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in Subsubcase 3.2.1. Now, we have already proved in Subsubcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case. Hence, $C_m \vee C_n$ is not *HMC*, where $n + m$ is odd and $m, n \geq 3$.

□

Corollary 2.2. *$C_m \vee C_n$ is not HMC, where $n, m \in \mathbf{N}$, $m, n \geq 3$.*

Proof. Proof follows from Theorems 2.7, 2.8 and 2.9. □

3. Conclusion

In this article we have proved that $CH_n \odot K_1$ and the tensor product $P_m \times P_n$ are *HMC*. Also we have proved that Complete bipartite graphs $K_{m,n}$, $K_n \vee C_m$ and $C_n \vee C_m$ are not *HMC* graphs.

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