Gibbs-Wilbraham oscillation related to an Hermite interpolation problem on the unit circle

E. Berriochoa[a], A. Cachafeiro[b,*], J. M. García Amor[c]

[a]*Departamento de Matemática Aplicada I, Facultad de Ciencias, Universidad de Vigo, 32004 Ourense, Spain*
[b]*Departamento de Matemática Aplicada I, Escuela de Ingeniería Industrial, Universidad de Vigo, 36310 Vigo, Spain*
[c]*Instituto E. S. Valle Inclán, Xunta de Galicia, 36001 Pontevedra, Spain*

Abstract

The aim of this piece of work is to study some topics related to an Hermite interpolation problem on the unit circle. We consider as nodal points the zeros of the para-orthogonal polynomials with respect to a measure in the Baxter class and such that the sequence of the first derivative of the reciprocal of the orthogonal polynomials is uniformly bounded on the unit circle. We study the convergence of the Hermite-Fejér interpolants related to piecewise continuous functions and we describe the sets in which the interpolants uniformly converge to the piecewise continuous function as well as the oscillatory behavior of the interpolants near the discontinuities, where a Gibbs-Wilbraham phenomenon appears. Finally we present some numerical experiments applying the main results and by considering nodal systems of interest in the theory of orthogonal polynomials.

Keywords: Hermite-Fejér interpolation; approximation; Baxter class; para-orthogonal polynomials; unit circle; Gibbs-Wilbraham phenomenon.
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1. Introduction

Hermite interpolation problems on the unit circle have been thoroughly studied in the last years. In [1] it is given a method to determine the Laurent polynomials of Hermite interpolation in an efficient way. There the nodes are equally spaced on the unit circle and the method is based on the use of the FFT. Later on, in [3] the authors present two different ways of obtaining Laurent polynomials of Hermite interpolation on the unit circle by taking as nodal system the n roots of a complex number with modulus 1. One of them is based on a functional series whose coefficients can be computed efficiently by means of the FFT and the other relies on a barycentric formulation.

In [4] it was obtained nice expressions for the so called fundamental polynomials of the first and the second kind to express the Laurent polynomials of Hermite interpolation. There, the nodes are arbitrary complex numbers with modulus 1 and in the same paper, it was considered the case when the nodal polynomials are the para-orthogonal polynomials with respect to measures in the Szegő class and having Szegő function with analytic extension outside the disk. Under these conditions it was

*Corresponding author
Email addresses: `esnaola@uvigo.es` (E. Berriochoa), `acachafe@uvigo.es` (A. Cachafeiro), `garciaamor@uvigo.es` (J. M. García Amor)

proved that the Hermite-Fejér interpolants related to continuous functions uniformly converge to the continuous functions.

When the nodal system is constituted by the n roots of a complex number with modulus 1 it was also studied in [2], in the case of piecewise continuous functions, the behavior of the Hermite-Fejér and Hermite interpolants near the discontinuities, including the description of Gibbs-Wilbraham phenomenon and the determination of the height of the jump.

Now, in the present paper we study the behavior of the Hermite-Fejér interpolants related to continuous and piecewise continuous functions on the unit circle taking as nodal systems the zeros of the para-orthogonal polynomials with respect to measures in the Baxter class and satisfying an additional condition. We highlight that our hypotheses are weaker than those considered in [4] and our nodal systems are, by far, more general than those considered in [2] and [4]. In the case of piecewise continuous functions we obtain that the Hermite-Fejér interpolants uniformly converge to the function far away from the discontinuity points and we study the oscillatory behavior of the interpolants in a neighborhood of the discontinuities. There we observe a Gibbs-Wilbraham phenomenon that can be compared with the type of oscillation of other processes like Lagrange interpolation and Fourier series, (see [7], [15]). Moreover, following the ideas of Krylov (see [9]) we could correct the perturbation produced by the Gibbs-Wilbraham phenomenon in the Hermite-Fejér process applied to piecewise continuous functions for a wide class of nodal systems.

The organization of the paper is the following. In section 2, after introducing the Laurent polynomial of Hermite interpolation on the unit circle, we present some results related with the classical theory of orthogonal polynomials when the measure of orthogonality belongs to the Baxter class. First we recall a result concerning with the distribution of the zeros of para-orthogonal polynomials, (see [13]), and by assuming an additional condition we obtain some useful bounds for the nodal polynomial and its first derivative. Under our assumptions we obtain the convergence of the Hermite-Fejér interpolants related to continuous functions on the unit circle. Section 3 is devoted to study the convergence of the Hermite-Fejér interpolants related to piecewise continuous functions. For simplicity, first it is considered the characteristic function of an arc and then we consider functions with jump discontinuities. We describe the subsets of the unit circle in which the Hermite-Fejér interpolants uniformly converge to the piecewise continuous function and we describe the error of interpolation near the discontinuities, where a Gibbs-Wilbraham phenomenon appears.

In the last section first we give examples of measures in the Baxter class and satisfying our hypothesis about the uniform boundedness of the first derivative of the reciprocal of the orthogonal polynomials and such that their Szegő functions have not analytic extension outside the disk. Secondly we use these examples to present the first numerical experiment applying our main results. We also show, in a second example, the similarity of the Gibbs-Wilbraham phenomenon for different nodal systems according to our results. Finally we present some conclusions as well as some possible extensions of our work.

2. Laurent polynomial of Hermite interpolation on the unit circle

Let $\{z_{j,n}\}_{j=1}^n$ be a set of complex numbers with $|z_{j,n}| = 1$, $\forall j$ and $z_{j,n} \neq z_{i,n}$ for $j \neq i$, and let $\{u_{j,n}\}_{j=1}^n$, $\{v_{j,n}\}_{j=1}^n$ be two sets of arbitrary complex numbers. In what follows, for simplicity, we will omit the subscript n and we will write z_j, u_j and v_j.
The Hermite interpolation problem in the space of Laurent polynomials $\Lambda = span\{z^k : k \in \mathbb{Z}\}$ with nodal system $\{z_j\}_{j=1}^n$ consists in determining the unique Laurent polynomial $\mathcal{H}_{-n,n-1} \in \Lambda_{-n,n-1} =$

$span\{z^k : -n \leq k \leq n-1\}$ such that

$$\mathcal{H}_{-n,n-1}(z_j) = u_j \text{ and } \mathcal{H}'_{-n,n-1}(z_j) = v_j \text{ for } j = 1, \cdots, n. \tag{1}$$

It is clear that the polynomial $\mathcal{H}_{-n,n-1}$ can be decomposed like $\mathcal{H}_{-n,n-1} = \mathcal{HF}_{-n,n-1} + \mathcal{HD}_{-n,n-1}$, where the Hermite-Fejér polynomial $\mathcal{HF}_{-n,n-1}$ is characterized by satisfying

$$\mathcal{HF}_{-n,n-1}(z_j) = u_j \text{ and } \mathcal{HF}'_{-n,n-1}(z_j) = 0 \text{ for } j = 1, \cdots, n.$$

and the polynomial $\mathcal{HD}_{-n,n-1}$ satisfies

$$\mathcal{HD}_{-n,n-1}(z_j) = 0 \text{ and } \mathcal{HD}'_{-n,n-1}(z_j) = v_j \text{ for } j = 1, \cdots, n.$$

If f is a complex function defined on the unit circle $\mathbb{T} := \{z : |z| = 1\}$, we will denote by $\mathcal{HF}_{-n,n-1}(f,)$ the Hermite-Fejér interpolation polynomial satisfying the conditions

$$\mathcal{HF}_{-n,n-1}(f, z_j) = f(z_j) \text{ and } \mathcal{HF}'_{-n,n-1}(f, z_j) = 0 \text{ for } j = 1, \cdots, n.$$

If we denote by $w_n(z)$ any nodal polynomial, that is, a polynomial of degree n with zeros $\{z_j\}_{j=1}^n$ then, from [4], we have the following expression for the interpolation polynomial defined above

$$\mathcal{HF}_{-n,n-1}(z) = \frac{(w_n(z))^2}{z^n} \sum_{k=1}^{n} \frac{1}{(w'_n(z_k))^2} \left(\frac{z_k^n}{(z-z_k)^2} + \frac{z_k^{n-1}}{z-z_k}(n - \frac{z_k w''_n(z_k)}{w'_n(z_k)}) \right) u_k. \tag{2}$$

In what follows we will take as nodal points the zeros of some kind of polynomials, the para-orthogonal polynomials on the unit circle, (see [8], [12]). To introduce these polynomials we consider a Borel finite and positive measure μ on $[0, 2\pi]$ with infinite support. If $\{\phi_n\}$ is the sequence of monic orthogonal polynomials with respect to the measure μ, we consider as nodal points the zeros of the sequence of para-orthogonal polynomials defined by

$$w_n(z; \tau) = \phi_n(z) + \tau\phi_n^*(z), \tag{3}$$

where τ is a complex number with $|\tau| = 1$ and $\phi_n^*(z)$ is the reciprocal polynomial defined by $\phi_n^*(z) = z^n\overline{\phi_n}(\frac{1}{z})$. Recall that $P^*(z) = z^n\overline{P}(\frac{1}{z})$ if the degree of $P(z)$ is n (see [14]). It is well known that the zeros of the para-orthogonal polynomials $w_n(z; \tau)$ are simple and they belong to the unit circle $\mathbb{T}$, (see [8], [12]). For simplicity, in what follows we will write $w_n(z)$ instead of $w_n(z; \tau)$.

2.1. Some nodal systems on $\mathbb{T}$

Our aim in this paper is to study an interpolation problem on the unit circle by using nodal systems which are far from those constituted by the n roots of a complex number with modulus 1, although the distribution of these points is near the equally spaced. However, the development of the corresponding theory requires very elaborate calculus. Thus we consider some special Borel finite and positive measures μ on $[0, 2\pi]$ with infinite support and we take as nodal points the zeros of the sequence of para-orthogonal polynomials $w_n(z)$ related to the measures μ.

First, in this subsection we present some results related with the classical theory of orthogonal polynomials as well as some technical results that we will use throughout the paper.

We recall that a measure μ belongs to the Szegő class ($\mu \in S$) if it satisfies the Szegő condition, $\log \mu' \in L^1[0, 2\pi]$, where μ' denotes the Radon Nikodym derivative of the measure. Alternatively, these measures can be characterized by several equivalent conditions among which we highlight that the Verblunsky coefficients $\{\phi_n(0)\} \in \ell^2$. We also recall that every measure $\mu \in S$ has associated the so called Szegő function $\pi_\mu(z)$, defined by

$$\pi_\mu(z) = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right).$$

The Szegő function is analytic and nonvanishing in the open unit disk and it has boundary values such that $|\pi_\mu(e^{i\theta})|^2 = \frac{1}{\mu'(\theta)}$ a.e. in $[0, 2\pi]$, (see [12], [14]).
It is very well-known that the asymptotic behavior of the orthogonal polynomials and their reciprocals can be described in terms of the Szegő function as follows. If we denote by $\Pi_\mu(z)$ the normalized Szegő function, that is, $\Pi_\mu(z) = \frac{\pi_\mu(z)}{\pi_\mu(0)}$, then the following asymptotic results hold:

(i) $\lim_{n\to\infty} \phi_n^*(z) = \Pi_\mu(z)$ uniformly on compact subsets of $|z| < 1$.

(ii) $\lim_{n\to\infty} \dfrac{\phi_n(z)}{z^n} = \overline{\Pi_\mu(\frac{1}{\bar{z}})}$ uniformly on compact subsets of $|z| > 1$.

For obtaining a suitable distribution of the zeros of the nodal polynomials $w_n(z)$ we will assume that the measure μ belongs to the Baxter class ($\mu \in B$) characterized by satisfying that the Verblunsky coefficients $\{\phi_n(0)\} \in \ell^1$. Notice that it holds $B \subset S$. In this situation the following result holds (see [13]):

Theorem 1. *Let μ be a measure in the Baxter class and $\{z_{j,n}\}_{j=1}^n$, with $z_{j,n} = e^{i\theta_j^{(n)}}$, be the zeros of the para-orthogonal polynomials $w_n(z)$ and $0 \le \theta_j^{(n)} \le \cdots \le \theta_n^{(n)} \le 2\pi$, ($\theta_{n+1}^{(n)} = 2\pi + \theta_1^{(n)}$), then it holds*

$$\lim_{n\to\infty} \sup_{j=1,\cdots,n} n|\theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{2\pi}{n}| = 0.$$

Proof. For the proof one can see [13]. □

Thus for $\mu \in B$ it is easy to deduce that for $\varepsilon > 0$, ε as small as we want, n large enough and $j = 1, \cdots, n$, it holds that

$$\frac{2\pi - \varepsilon}{n} < \theta_{j+1}^{(n)} - \theta_j^{(n)} < \frac{2\pi + \varepsilon}{n}, \tag{4}$$

and therefore the distribution of these zeros is near to the distribution of the equally spaced ones. In what follows, for simplicity, we will omit the subscript n and we will write θ_j and z_j.

When the measure $\mu \in B$ we can obtain the asymptotic behavior of $w_n(z)$ and $|w_n(z)|$ on $\mathbb{T}$. Moreover, if it also holds an additional condition, we can obtain useful bounds for the nodal polynomial $w_n(z)$ and its two first derivatives.

Lemma 1. *Let μ be a measure in the Baxter class B and assume that the sequence $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$. If $m = \min_{z\in\mathbb{T}} |\Pi_\mu(z)|$ and $M = \max_{z\in\mathbb{T}} |\Pi_\mu(z)|$ then for every $z \in \mathbb{T}$ it holds:*

(i)

$$\phi_n'(z) = \frac{n\phi_n(z)}{z} - z^{n-2}\overline{(\phi_n^*)'(z)}. \tag{5}$$

(ii)

$$|w_n(z)|^2 = 2|\Pi_\mu(z)|^2(1 + \Re(\tau\overline{z}^n \frac{\Pi_\mu(z)}{\overline{\Pi_\mu(z)}})) + o(1) \text{ and } |w_n(z)| < 2M + o(1). \tag{6}$$

(iii)

$$m - o(1) < \frac{|w_n'(z)|}{n} < M + o(1), \tag{7}$$

and there exists a constant $C > 0$ such that

$$|nw_n'(z) - zw_n''(z)| < nC. \tag{8}$$

(We denote by $o(1)$ sequences which converge to 0 uniformly on z.)

Proof. (i) Taking derivatives in $\phi_n^*(z) = z^n\overline{\phi_n}(\frac{1}{z})$ we have $(\phi_n^*)'(z) = nz^{n-1}\overline{\phi_n}(\frac{1}{z}) - z^{n-2}\overline{\phi_n'(z)}$. So, $\overline{(\phi_n^*)'(z)} = \frac{n}{z^{n-1}}\phi_n(z) - \frac{1}{z^{n-2}}\phi_n'(z)$ and finally (5). Notice that this property holds for arbitrary polynomials of degree n.

(ii) Since $\mu \in B$ then $\phi_n^*(z)$ uniformly converges to $\Pi_\mu(z)$ and $\frac{\phi_n(z)}{z^n}$ uniformly converges to $\overline{\Pi_\mu(\frac{1}{\overline{z}})}$ on $\mathbb{T}$, (see [13]). Therefore $\phi_n^*(z) = \Pi_\mu(z) + \varepsilon_n(z)$ and $\phi_n(z) = z^n\overline{\Pi_\mu(z)} + z^n\overline{\varepsilon_n(z)}$, where $\varepsilon_n(z)$ converges to 0 uniformly on $z \in \mathbb{T}$. Hence $|w_n(z)| = |\phi_n(z) + \tau\phi_n^*(z)| = |z^n\overline{\Pi_\mu(z)} + z^n\overline{\varepsilon_n(z)} + \tau\Pi_\mu(z) + \tau\varepsilon_n(z)| < 2M + \delta_n(z)$, where $\delta_n(z)$ goes to zero uniformly on $\mathbb{T}$.
In the same way we have $|w_n(z)|^2 = |z^n\overline{\Pi_\mu(z)} + z^n\overline{\varepsilon_n(z)} + \tau\Pi_\mu(z) + \tau\varepsilon_n(z)|^2 = 2|\Pi_\mu(z)|^2 + \overline{\tau}z^n(\overline{\Pi_\mu(z)})^2 + \tau\overline{z}^n(\Pi_\mu(z))^2 + \sigma_n(z)$, where $\sigma_n(z)$ goes to zero uniformly on $\mathbb{T}$. Notice that in order to simplify the notation we write $o(1)$ to denote any sequence which converges to 0 uniformly on z.

(iii) As a consequence of (5) we have

$$w_n'(z) = \frac{n\phi_n(z)}{z} - z^{n-2}\overline{(\phi_n^*)'(z)} + \tau(\phi_n^*)'(z) = \frac{n\phi_n(z)}{z} + \frac{A_n(z)}{z}, \tag{9}$$

where $A_n(z)$ is a sequence of polynomials of degree less or equal than n uniformly bounded on $\mathbb{T}$, so we have (7) . Moreover,

$$zw_n''(z) - nw_n'(z) = z\left(-\frac{n\phi_n(z)}{z^2} + \frac{n\phi_n'(z)}{z} + \left(\frac{A_n(z)}{z}\right)'\right) - n\phi_n'(z) - n\tau(\phi_n^*)'(z) =$$

$$n\left(-\frac{\phi_n(z)}{z} + \frac{z}{n}\left(\frac{A_n(z)}{z}\right)' - \tau(\phi_n^*)'(z)\right). \tag{10}$$

As $\frac{A_n'(z)}{n}$ is bounded due to Bernstein theorem, (see [10]), then (8) is a straightforward consequence of (10). □

Lemma 2. *Let μ be a measure in the Baxter class and assume that the sequence $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$. Then there exists a positive constant K such that*

$$\frac{|w_n(z)|^2}{n^2}\sum_{k=1}^{n}\frac{1}{|z - z_k|^2} < K \text{ for every } z \in \mathbb{T}.$$

Proof. From the proof of Lemma 2 in [4] we can write

$$|w_n(z)|^2 \sum_{k=1}^{n} \frac{1}{|z-z_k|^2} = |zw_n(z)w_n'(z) + z^2(w_n''(z)w_n(z) - (w_n'(z))^2)|$$

and therefore

$$\frac{|w_n(z)|^2}{n^2} \sum_{k=1}^{n} \frac{1}{|z-z_k|^2} \le \frac{|w_n(z)|}{n}\frac{|w_n'(z)|}{n} + |w_n(z)|\frac{|w_n''(z)|}{n^2} + \frac{|w_n'(z)|^2}{n^2}$$

for $z \in \mathbb{T}$. From the preceding lemma we know that $|w_n(z)|$ and $\frac{|w_n'(z)|}{n}$ are uniformly bounded on $\mathbb{T}$ and it is easy to obtain that $\frac{|w_n''(z)|}{n^2}$ is bounded too. Indeed,

$$\frac{|w_n''(z)|}{n^2} = \frac{|zw_n''(z) - nw_n'(z) + nw_n'(z)|}{n^2} \le \frac{1}{n}\frac{|zw_n''(z) - nw_n'(z)|}{n} + \frac{|w_n'(z)|}{n}.$$

□

Remark 1. *We can give a local version of (7) in the preceding Lemma 1 as follows. Since we know that for a fixed number $\varepsilon > 0$ and n large enough it holds that $m - \varepsilon < \frac{|w_n'(z)|}{n} < M + \varepsilon$, then if z belongs to a small arc $\gamma \subset \mathbb{T}$ and $m_\gamma = \min_{z\in\gamma} |\Pi_\mu(z)|$ and $M_\gamma = \max_{z\in\gamma} |\Pi_\mu(z)|$, in such a way that $|M_\gamma - m_\gamma| < \varepsilon$, then for n large enough it holds $M_\gamma - 2\varepsilon < \dfrac{|w_n'(z)|}{n} < M_\gamma + \varepsilon$.*

Remark 2. *Lemma 1 was proved in [4] by assuming the stronger hypotheses that the measure μ belongs to the Szegő class and the Szegő function has analytic extension outside the unit disk ($\mu \in S_A$).*

Indeed if we assume that the measure $\mu \in S_A$, then the hypotheses of Lemma 1 hold. Take into account that for every $z \in \mathbb{T}$ it holds that $(\phi_n^*)'(z) = \Pi'(z) + \mathcal{O}(n^2 r^n)$, for some $r < 1$, from which it follows the uniformly boundedness of $(\phi_n^*)'$ on $\mathbb{T}$, (see [11]).

It seems more interesting to give an example of a measure satisfying the assumptions of Lemma 1 and not belonging to S_A. The example is given in the last section where we also present some numerical experiments about our results.

Another sufficient condition for the uniformly boundedness of $(\phi_n^*)'$ can be given in terms of the coefficients of the monic orthogonal polynomial sequence related to the measure μ.

Proposition 1. *If $\phi_n(z) = \sum_{k=0}^{n} \overline{\alpha_{n-k,n}} z^k$ with $\overline{\alpha_{0,n}} = 1$ and we assume that $|\alpha_{k,n}| < E$, for all k and n, and there exist N, M natural numbers such that for $k \ge N$ and $n \ge M$ $|\alpha_{k,n}| < \frac{1}{k^c}$,($c > 2$) then $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$.*

Proof. Indeed for $z \in \mathbb{T}$ it holds that there exists a positive constant F such that

$$|(\phi_n^*)'(z)| = \Big|\sum_{k=0}^{n} k\alpha_{k,n} z^{k-1}\Big| = \Big|\sum_{k=1}^{N} k\alpha_{k,n} z^{k-1} + \sum_{k=N+1}^{n} k\alpha_{k,n} z^{k-1}\Big| \le$$

$$E\sum_{k=1}^{N} k + \sum_{k=N+1}^{n} \frac{1}{k^{c-1}} \le \frac{EN(N+1)}{2} + H_{c-1} \le F,$$

where H_{c-1} denotes the sum of the armonic series $\sum_{k=1}^{\infty} \frac{1}{k^{c-1}}$. □

In [4] it was also proved the convergence of the Hermite-Fejér interpolants related to continuous functions on $\mathbb{T}$ under the assumptions that the measure belongs to the Szegő class and its Szegő function has analytic extension outside the unit disk and taking as nodal systems the zeros of the para-orthogonal polynomials. Now we can obtain a similar result under weaker hypotheses.

Theorem 2. *Let $\mu \in B$ and assume that the sequence $\{(\phi_n^*)'\}$ is bounded uniformly on $\mathbb{T}$. Let the nodal polynomial $w_n(z)$ be the para-orthogonal polynomial related to the measure μ and let f be a continuous function on $\mathbb{T}$. Then $\mathcal{HF}_{-n,n-1}(f,z)$ uniformly converges to f on $\mathbb{T}$.*

Proof. Due to the preceding lemmas, the hypotheses of Corollary 1 in [4] are satisfied and therefore the result is true. □

3. Hermite-Fejér interpolation polynomial for a piecewise continuous function on $\mathbb{T}$. Convergence.

Throughout this section we study the convergence of the Hermite-Fejér interpolation polynomials related to piecewise continuous functions. Let $A = (a_1, a_2)$ be an arc contained in $\mathbb{T}$ and let χ_A be the characteristic function of the set A defined by $\chi_A(z) = 1$ if $z \in A$ and $\chi_A(z) = 0$ if $z \notin A$. If f is a continuous function on $\mathbb{T}$, we consider the product function $f\chi_A$, which has two jump discontinuity points, a_1 and a_2.

Theorem 3. *Let $\mu \in B$ and assume that the sequence $\{(\phi_n^*)'\}$ is bounded uniformly on $\mathbb{T}$. Let the nodal polynomial $w_n(z)$ be the para-orthogonal polynomial related to the measure μ, f be a continuous function on $\mathbb{T}$ and $A = (a_1, a_2)$ be an arc contained in $\mathbb{T}$. If K is a compact subset of $\mathbb{T}$ such that $a_1, a_2 \notin K$, then $\mathcal{HF}_{-n,n-1}(f\chi_A, z)$ uniformly converges to $f\chi_A$ in K.*

Proof. First we assume that $K \subset A$. Then, from (2) we get

$$\mathcal{HF}_{-n,n-1}(f\chi_A, z) = \mathcal{HF}_{-n,n-1}(f,z) - \frac{(w_n(z))^2}{z^n} \sum_{\substack{k=1 \\ z_k \in \mathbb{T}\setminus A}}^{n} \frac{1}{(w_n'(z_k))^2} \left(\frac{z_k^n}{(z-z_k)^2} + \frac{z_k^{n-1}}{z-z_k}(n - \frac{z_k w_n''(z_k)}{w_n'(z_k)}) \right) f(z_k).$$

By using Theorem 2 we can state that $\mathcal{HF}_{-n,n-1}(f,z)$ uniformly converges to f in $\mathbb{T}$. Thus, it only suffices to prove that the second sum uniformly converges to 0 in K.
Indeed, if $z \in K \subset A$, we apply the preceding Lemma 1 in the following way. For simplicity we can write for all $z \in \mathbb{T}$ and every natural number n

$$|w_n(z)| \le C_1, \ 0 < C_2 < \frac{|w_n'(z)|}{n}, \ |nw_n'(z) - zw_n''(z)| < nC_3,$$

where C_1, C_2 and C_3 are appropriate positive constants. Then

$$\frac{|w_n(z)|^2}{|z^n|} \Big| \sum_{\substack{k=1 \\ z_k \in \mathbb{T}\setminus A}}^{n} \frac{1}{(w_n'(z_k))^2} \left(\frac{z_k^n}{(z-z_k)^2} + \frac{z_k^{n-1}}{z-z_k}(n - \frac{z_k w_n''(z_k)}{w_n'(z_k)}) \right) f(z_k) \Big| \le$$

$$C_1^2 \parallel f \parallel_\infty \sum_{\substack{k=1 \\ z_k \in \mathbb{T}\setminus A}}^{n} \frac{1}{C_2^2 n^2} \left(\frac{1}{|z-z_k|^2} + \frac{C_3}{C_2} \frac{1}{|z-z_k|} \right).$$

If we denote by d the distance between K and $\mathbb{T}\setminus A$, then

$$C_1^2 \parallel f \parallel_\infty \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{C_2^2 n^2}\left(\frac{1}{|z-z_k|^2}+\frac{C_3}{C_2}\frac{1}{|z-z_k|}\right) \leq$$

$$C_1^2 \parallel f \parallel_\infty \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{C_2^2 n^2}\left(\frac{1}{d^2}+\frac{C_3}{C_2 d}\right) \leq C_1^2 \parallel f \parallel_\infty \left(\frac{1}{d^2}+\frac{C_3}{C_2 d}\right)\frac{1}{C_2^2 n},$$

which goes to zero when n tends to ∞.

When $K \subset \mathbb{T}\setminus A$ one can proceed in the same way. □

Corollary 1. *Let $\{A_i\}_{i=1}^p$ with $A_i = (a_i, a_{i+1})$ and $a_{p+1} = a_1$ be a set of disjoint arcs on $\mathbb{T}$ such that $\cup_{i=1}^p \overline{A_i} = \mathbb{T}$ and let $\{f_i\}_{i=1}^p$ be continuous functions on $\mathbb{T}$. If K is a compact subset of $\mathbb{T}$ such that $a_i \notin K, \forall i = 1,\cdots,p$ and $g = \sum_{i=1}^p f_i\chi_{A_i}$, then $\mathcal{HF}_{-n,n-1}(g,z)$ uniformly converges to g in K.*

Let us continue assuming $A = (a_1, a_2)$ is an arc contained in $\mathbb{T}$ like the next Figure 1 shows. Let us consider the characteristic function of A, χ_A, and the Hermite-Fejér interpolation polynomial $\mathcal{HF}_{-n,n-1}(\chi_A, z)$. Since we have studied its behavior on compact subsets of the arc not including the end points, now our aim is to study its behavior near the extreme points of the arc.

In the sequel we use the nodal system constituted by the zeros of the para-orthogonal polynomials with respect to a measure μ satisfying the hypotheses of Lemma 1 and we consider a fixed positive real number ε so that the distribution of the arguments of the zeros fulfills relation (4).

First we establish some notation. If n is large enough, it is clear that there are nodal points in the arc A. We denote by z_1 the nodal point in A which is closer to a_2. The other nodes are numbered in the clockwise sense from z_1 in such a way that the last point z_n is the nodal point in $\mathbb{T}\setminus A$ which is closer to a_2. We denote by ℓ_A the length of A and we consider the following subsets of $\mathbb{T}$ as Figure 1 shows. In order to describe these sets, it is useful to use the angular distance between z and z_1 or z_n by using the parameters d or D.
If ε is a fixed positive real number we define

$$I_{n,\varepsilon} = \{z\in\mathbb{T} : z = z_1 e^{-i\frac{2\pi-\varepsilon}{n}d}, \text{ with } d\in[-\frac{1}{2},\sqrt{n}]\},$$

$$J_{n,\varepsilon} = \{z\in A : z = z_1 e^{-i\frac{2\pi-\varepsilon}{n}d}, \text{ with } d\in(\sqrt{n},\ell_A\frac{n}{2\pi-\varepsilon}-\sqrt{n}]\},$$

$$I'_{n,\varepsilon} = \{z\in\mathbb{T} : z = z_n e^{i\frac{2\pi-\varepsilon}{n}D}, \text{ with } D\in[-\frac{1}{2},\sqrt{n}]\},$$

$$J'_{n,\varepsilon} = \{z\in\mathbb{T}\setminus A : z = z_n e^{i\frac{2\pi-\varepsilon}{n}D}, \text{ with } D\in(\sqrt{n},(2\pi-\ell_A)\frac{n}{2\pi-\varepsilon}-\sqrt{n}]\}.$$

Notice that we can define another two sets but we omit their description because they play the same role of $I_{n,\varepsilon}$ and $I'_{n,\varepsilon}$ changing a_2 by a_1. In order to simplify the notation, in what follows we omit the subscript ε.

It is clear that if $\sqrt{n}\notin\mathbb{N}$, we take the integer part $E[\sqrt{n}]$, but for simplicity we continue writing $\sqrt{n}$.

In the next Lemmas we present some technical results concerning the distance between the points in the unit circle and our nodal points $\{z_j\}_{j=1}^n$. The information given in these lemmas (bounds

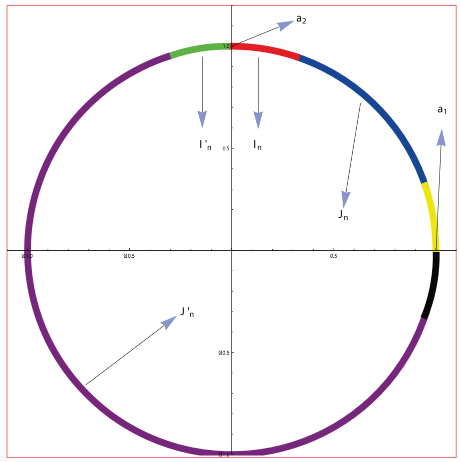

Figure 1: Subsets of $\mathbb{T}$ with $a_1 = 1$ and $a_2 = i$

for several sums associated to the interpolation problem) will be used in the proofs of the two main theorems. We continue assuming that ε is a fixed positive number and we denote by $(\widehat{z - z_j})$ the length of the shortest arc joining z and z_j.

Lemma 3. *For $z \in \mathbb{T}$ it holds that*

(i)

$$\frac{1}{(\widehat{z - z_j})^2} \leq \frac{1}{|z - z_j|^2} \leq \frac{\pi^2}{4} \frac{1}{(\widehat{z - z_j})^2}.$$

(ii) If ε_1 is a positive number, n is large enough and $(\widehat{z - z_j}) < \frac{2(2\pi - \varepsilon)}{\sqrt{n}}$, then

$$\frac{1}{(\widehat{z - z_j})^2} \leq \frac{1}{|z - z_j|^2} \leq (1 + \varepsilon_1) \frac{1}{(\widehat{z - z_j})^2}.$$

Proof. It can be obtained following the ideas given in the proof of Lemma 2 in [2]. ☐

To establish our main results, we have to use the special function PolyGamma of order 1, $\psi_1(z)$, which has the representation $\psi_1(z) = \sum_{\ell=0}^{\infty} \frac{1}{(z+\ell)^2}$, for $z \notin \mathbb{Z}_-$, (see [6]). We recall that the PolyGamma function of order k, $\psi_k(z)$, is defined by $\psi_k(z) = \frac{d^{k+1}}{dz^{k+1}} \ln \Gamma(z)$, $k \in \mathbb{N}$.

Lemma 4. *If $z \in J_n$ then*

(i)

$$\frac{1}{n^2} \sum_{\substack{k=1 \\ z_k \in \mathbb{T} \setminus A}}^{n} \frac{1}{|z - z_k|^2} \leq \frac{\pi^2}{4(2\pi - \varepsilon)^2} \psi_1(\sqrt{n}),$$

and therefore $\displaystyle\frac{1}{n^2} \sum_{\substack{k=1 \\ z_k \in \mathbb{T} \setminus A}}^{n} \frac{1}{|z - z_k|^2}$ *converges to* 0 *uniformly on* z.

(ii)

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}\leq\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}(H_{\sqrt{n}+n,1}-H_{\sqrt{n}-1,1}),$$

where $H_{n,1}$ *denotes the* n*-th partial sum of the harmonic series, that is,* $H_{n,1}=\sum_{k=1}^{n}\frac{1}{k}$*. Therefore* $\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}$ *converges to* 0 *uniformly on* z*.*

Proof. (i) We assume that z_s is the first node in $\mathbb{T}\setminus A$ in the clockwise sense from J_n. Then if $z\in J_n$ it holds $\widehat{(z-z_s)}\geq\sqrt{n}\frac{2\pi-\varepsilon}{n}$ and by taking into account Lemma 3 we obtain $\frac{1}{|z-z_s|}\leq\frac{\pi}{2}\frac{n}{\sqrt{n}(2\pi-\varepsilon)}$. Proceeding in the same way we get $\frac{1}{|z-z_{s+1}|}\leq\frac{\pi}{2}\frac{n}{(\sqrt{n}+1)(2\pi-\varepsilon)},\cdots,\frac{1}{|z-z_n|}\leq\frac{\pi}{2}\frac{n}{(\sqrt{n}+n-s)(2\pi-\varepsilon)}$. Thus

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{k=s}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{\ell=0}^{n-s}\frac{1}{|z-z_{s+\ell}|^2}\leq$$

$$\frac{\pi^2}{4(2\pi-\varepsilon)^2}\sum_{\ell=0}^{n-s}\frac{1}{(\sqrt{n}+\ell)^2}\leq\frac{\pi^2}{4(2\pi-\varepsilon)^2}\psi_1(\sqrt{n}),$$

which goes to 0 when n tends to infinity.

(ii) For the other sum we proceed in a similar way obtaining

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}=\frac{1}{n^2}\sum_{k=s}^{n}\frac{1}{|z-z_k|}=\frac{1}{n^2}\sum_{\ell=0}^{n-s}\frac{1}{|z-z_{s+\ell}|}\leq$$

$$\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}\sum_{\ell=0}^{n-s}\frac{1}{\sqrt{n}+\ell}\leq\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}\sum_{\ell=0}^{n}\frac{1}{\sqrt{n}+\ell}=\tag{11}$$

$$\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}\sum_{k=\sqrt{n}}^{\sqrt{n}+n}\frac{1}{k}=\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}(H_{\sqrt{n}+n,1}-H_{\sqrt{n}-1,1}).$$

□

Lemma 5. *Let* $z\in I_n$*,* $z=z_1e^{-id(\frac{2\pi-\varepsilon}{n})}$*, with* $d\in[-\frac{1}{2},\sqrt{n}]$*. Then*

$$\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1)\leq\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}\leq\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1),$$

where $o(1)$ *are sequences converging to* 0 *uniformly on* z*.*

Proof. If we assume that z_s is the first node in $\mathbb{T}\setminus A$ in the clockwise sense from J_n then

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{k=s}^{n}\frac{1}{|z-z_k|^2}.\tag{12}$$

We split the preceding sum in two parts as follows,

$$\frac{1}{n^2}\sum_{k=s}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{k=n-\sqrt{n}}^{n}\frac{1}{|z-z_k|^2}+\frac{1}{n^2}\sum_{k=s}^{n-\sqrt{n}-1}\frac{1}{|z-z_k|^2}, \tag{13}$$

where the first sum includes the nodes which are not far from $2\sqrt{n}$ arcs from z.

First, if we assume that $z=z_1e^{-id(\frac{2\pi-\varepsilon}{n})}$, with $d\in[0,\sqrt{n}]$, then

$$\frac{(2\pi-\varepsilon)}{n}(d+1)\le \widehat{z-z_n}\le\frac{(2\pi+\varepsilon)}{n}(d+1).$$

Secondly, if we assume that $z=z_1e^{-id(\frac{2\pi-\varepsilon}{n})}$, with $d\in[-\frac{1}{2},0)$, then

$$\frac{(2\pi-\varepsilon)}{n}(d+1)\le \widehat{z-z_n}\le\frac{(2\pi+\varepsilon)}{n}+\frac{(2\pi-\varepsilon)}{n}d.$$

Thus, in both cases we can write that

$$\frac{(2\pi-\varepsilon)}{n}(d+1)\le \widehat{z-z_n}\le\frac{(2\pi+3\varepsilon)}{n}(d+1).$$

Hence we have for $k=1,\cdots,n-s$

$$\frac{(2\pi-\varepsilon)}{n}(d+k+1)\le \widehat{z-z_{n-k}}\le\frac{(2\pi+3\varepsilon)}{n}(d+k+1).$$

By applying Lemma 3 we get

$$\frac{1}{|z-z_{n-k}|^2}\le(1+\varepsilon)\frac{1}{\left(\widehat{z-z_{n-k}}\right)^2}\le(1+\varepsilon)\frac{n^2}{(2\pi-\varepsilon)^2(d+k+1)^2}$$

and therefore

$$\frac{1}{n^2}\sum_{k=n-\sqrt{n}}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{k=0}^{\sqrt{n}}\frac{1}{|z-z_{n-k}|^2}\le\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\sum_{\ell=0}^{\sqrt{n}}\frac{1}{(d+1+\ell)^2},$$

which converges to $\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)$, when n tends to infinity.
For the second sum we use again Lemma 3 obtaining that

$$\frac{1}{|z-z_{n-k}|^2}\le\frac{\pi^2}{4(2\pi-\varepsilon)^2}\frac{n^2}{(d+k+1)^2}$$

and therefore

$$\frac{1}{n^2}\sum_{k=s}^{n-\sqrt{n}-1}\frac{1}{|z-z_k|^2}\le\frac{\pi^2}{4(2\pi-\varepsilon)^2}\sum_{\ell=\sqrt{n}+1}^{n-s}\frac{1}{(d+1+\ell)^2}, \tag{14}$$

which goes to 0 when n tends to infinity.
Hence for $z\in I_n$

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}$$

is bounded from above by $\frac{(1+\varepsilon)}{(2\pi-\epsilon)^2}\psi_1(1+d)+o(1)$.

To bound (12) from below we take into account that $\frac{1}{|z-z_{n-k}|^2} \geq \frac{1}{(\widehat{z-z_{n-k}})^2} \geq \frac{n^2}{(2\pi+3\varepsilon)^2(d+k+1)^2}$. Then

$$\frac{1}{n^2}\sum_{k=n-\sqrt{n}}^{n}\frac{1}{|z-z_k|^2} \geq \frac{1}{(2\pi+3\varepsilon)^2}\sum_{\ell=0}^{\sqrt{n}}\frac{1}{(d+\ell+1)^2},$$

where the last sequence converges to $\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)$ and we also obtain

$$\frac{1}{n^2}\sum_{k=s}^{n-\sqrt{n}-1}\frac{1}{|z-z_k|^2} \geq \frac{1}{(2\pi+3\varepsilon)^2}\sum_{\ell=\sqrt{n}+1}^{n-s}\frac{1}{(d+\ell+1)^2},$$

where the last sequence goes to zero. Hence

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2} \geq \frac{1}{(2\pi+3\varepsilon)^2}\sum_{\ell=0}^{\sqrt{n}}\frac{1}{(d+\ell+1)^2}+\frac{1}{(2\pi+3\varepsilon)^2}\sum_{\ell=\sqrt{n}+1}^{n-s}\frac{1}{(d+\ell+1)^2}$$

and therefore the lemma is proved. □

Lemma 6. *Let $z\in I_n$, $z=z_1e^{-id(\frac{2\pi-\varepsilon}{n})}$, with $d\in[-\frac{1}{2},\sqrt{n}]$. Then*

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|} \leq \frac{\pi}{2}\frac{1}{(2\pi-\varepsilon)n}(H_{n-s,1}-H_{\sqrt{n},1})+\frac{1}{n}\frac{\sqrt{1+\varepsilon}}{(2\pi-\varepsilon)}\sum_{\ell=0}^{\sqrt{n}-1}\frac{1}{(d+\ell+1)},$$

and therefore $\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}$ *converges to* 0 *uniformly on* z.

Proof. If we assume that z_s is the first node in $\mathbb{T}\setminus A$ in the clockwise sense fron J_n, then

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}=\frac{1}{n^2}\sum_{k=s}^{n}\frac{1}{|z-z_k|}=\frac{1}{n^2}\sum_{k=s}^{n-\sqrt{n}-1}\frac{1}{|z-z_k|}+\frac{1}{n^2}\sum_{k=n-\sqrt{n}}^{n}\frac{1}{|z-z_k|}.$$

On the one hand, it is clear that

$$\sum_{k=s}^{n-\sqrt{n}-1}\frac{1}{|z-z_k|}=\sum_{k=\sqrt{n}+1}^{n-s}\frac{1}{|z-z_{n-k}|} \tag{15}$$

and by using (14) we get

$$\frac{1}{n^2}\sum_{k=\sqrt{n}+1}^{n-s}\frac{1}{|z-z_{n-k}|} \leq \frac{\pi}{2}\frac{1}{(2\pi-\varepsilon)n}\sum_{\ell=\sqrt{n}+1}^{n-s}\frac{1}{(d+\ell+1)} \leq$$

$$\frac{\pi}{2}\frac{1}{(2\pi-\varepsilon)n}\sum_{\ell=\sqrt{n}+1}^{n-s}\frac{1}{\ell}=\frac{\pi}{2}\frac{1}{(2\pi-\varepsilon)n}(H_{n-s,1}-H_{\sqrt{n},1}).$$

On the other hand, as we have seen in the previous lemma

$$\frac{1}{n^2}\sum_{k=n-\sqrt{n}}^{n}\frac{1}{|z-z_k|}\leq\frac{1}{n}\frac{\sqrt{1+\varepsilon}}{(2\pi-\varepsilon)}\sum_{\ell=0}^{\sqrt{n}}\frac{1}{(d+\ell+1)}.$$

Hence the lemma is proved. □

Lemma 7. *If μ is a measure in the Baxter class B then for $z\in I_n$ and n large enough*

$$\Re(\tau\overline{z}^n\frac{\Pi_\mu(z)}{\overline{\Pi_\mu(z)}})=-\cos(2\pi-\varepsilon)d+o(1),$$

where $o(1)$ is a sequence which converges to 0 uniformly on z.

Proof. If $z\in I_n$ then $z=z_1e^{-i\frac{2\pi-\varepsilon}{n}d}$, with $d\in[-\frac{1}{2},\sqrt{n}]$. Since $\mu\in B$ then $\phi_n^*(z)=\Pi_\mu(z)+\varepsilon_n(z)$ and $\phi_n(z)=z^n\overline{\Pi_\mu(z)}+z^n\overline{\varepsilon_n(z)}$, where $\varepsilon_n(z)$ converges to 0 uniformly on $z\in\mathbb{T}$. Taking into account that z_1 is a zero of $w_n(z)$ then we can write

$$\tau=-\frac{\phi_n(z_1)}{\phi_n^*(z_1)}=-\frac{z_1^n\overline{\Pi_\mu(z_1)}+z_1^n\overline{\varepsilon_n(z_1)}}{\Pi_\mu(z_1)+\varepsilon_n(z_1)}=-\frac{z_1^n\overline{\Pi_\mu(z_1)}}{\Pi_\mu(z_1)}+o(1),$$

where $o(1)$ converges to 0 uniformly on $z\in\mathbb{T}$.
Therefore due to the continuity of $\Pi_\mu(z)$ we get for $z\in I_n$ and n large enough

$$\tau\overline{z}^n\frac{\Pi_\mu(z)}{\overline{\Pi_\mu(z)}}=\left(-\frac{z_1^n\overline{\Pi_\mu(z_1)}}{\Pi_\mu(z_1)}+o(1)\right)\frac{1}{z^n}\frac{\Pi_\mu(z)}{\overline{\Pi_\mu(z)}}=-\frac{z_1^n}{z^n}\frac{\overline{\Pi_\mu(z_1)}}{\Pi_\mu(z_1)}\frac{\Pi_\mu(z)}{\overline{\Pi_\mu(z)}}+o(1)=e^{i(2\pi-\varepsilon)d}(-1+o(1)),$$

from which it follows the result. □

Now we are in conditions to describe the behavior of the Hermite-Fejér interpolants related to the characteristic function χ_A near the discontinuities as well as the error of interpolation, that is, we can describe the Gibbs-Wilbraham phenomenon that appears, (see [15]). These results are given in the next theorem.

Theorem 4. *Let μ be a measure in the Baxter class and assume that the sequence $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$. Let ε be a positive fixed number and let $\{z_j\}_{j=1}^n$ be the zeros of the para-orthogonal polynomial $w_n(z)$, with $z_j=e^{i\theta_j}$ and θ_j satisfying (4). If A is an arc $A=(a_1,a_2)\subset\mathbb{T}$, then it holds*

(i) $\mathcal{HF}_{-n,n-1}(\chi_A,z)$ uniformly converges to 1 in $J_n\subset A$.

(ii) If $z\in I_n$, $z=z_1e^{-i\frac{(2\pi-\varepsilon)}{n}d}$, with $d\in[-\frac{1}{2},\sqrt{n}]$, $m=\min_{z\in\mathbb{T}}|\Pi_\mu(z)|$ and $M=\max_{z\in\mathbb{T}}|\Pi_\mu(z)|$ then for n large enough

$$\begin{aligned}\frac{2m^2(1-\cos(2\pi-\varepsilon)d)}{(M+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1)\leq 1-\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z))\leq\\ \frac{2M^2(1-\cos(2\pi-\varepsilon)d)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1)\end{aligned}\tag{16}$$

and $\Im(\mathcal{HF}_{-n,n-1}(\chi_A,z))=o(1)$.

(iii) $\mathcal{HF}_{-n,n-1}(\chi_A, z)$ *uniformly converges to* 0 *in* $J'_n \subset \mathbb{T} \setminus A$.

(iv) *If* $z \in I'_n$, $z = z_n e^{\frac{(2\pi-\varepsilon)}{n} Di}$, *with* $D \in [-\frac{1}{2}, \sqrt{n}]$, $m = \min_{z\in\mathbb{T}} |\Pi_\mu(z)|$ *and* $M = \max_{z\in\mathbb{T}} |\Pi_\mu(z)|$ *then for* n *large enough*

$$2m^2(1-\cos(2\pi-\varepsilon)D)\frac{1}{(M+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+D) - o(1) \le \Re(\mathcal{HF}_{-n,n-1}(\chi_A, z)) \le$$
$$2M^2(1-\cos(2\pi-\varepsilon)D)\frac{1}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+D) + o(1).$$

Proof. (i) First we obtain a more suitable expression of the Laurent polynomial of Hermite interpolation given in (2), for which we rewrite this last expression as follows:

$$\begin{aligned}\mathcal{HF}_{-n,n-1}(z) = \frac{(w_n(z))^2}{z^n}\sum_{k=1}^{n}\frac{1}{(w'_n(z_k))^2}\left(\frac{z_k^n}{(z-z_k)^2}+\frac{z_k^{n-1}}{z-z_k}\right)u_k+\\ \frac{(w_n(z))^2}{z^n}\sum_{k=1}^{n}\frac{1}{(w'_n(z_k))^2}\frac{z_k^{n-1}}{z-z_k}(n-1-\frac{z_k w''_n(z_k)}{w'_n(z_k)})u_k.\end{aligned} \tag{17}$$

After some computations we obtain some useful relations for simplifying the above expression. Since for $z \in \mathbb{T}$ it holds $w_n(z) = \tau z^n \overline{w_n(z)}$, then we have that $\frac{(w_n(z))^2}{z^n} = \tau |w_n(z)|^2$ and

$$(w'_n(z_k))^2 = -\tau z_k^{n-2} |w'_n(z_k)|^2.$$

Moreover, if we also take into account for $z \in \mathbb{T}$ it holds that $(z-z_k)^2 = -z z_k |z-z_k|^2$, then (17) can be written as

$$\begin{aligned}\mathcal{HF}_{-n,n-1}(z) = |w_n(z)|^2\sum_{k=1}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{1}{|z-z_k|^2}u_k-\\ |w_n(z)|^2\sum_{k=1}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{z_k}{z-z_k}(n-1-\frac{z_k w''_n(z_k)}{w'_n(z_k)})u_k.\end{aligned} \tag{18}$$

Now by applying (18) we get

$$\begin{aligned}1 = |w_n(z)|^2\sum_{k=1}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{1}{|z-z_k|^2}-\\ |w_n(z)|^2\sum_{k=1}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{z_k}{z-z_k}(n-1-\frac{z_k w''_n(z_k)}{w'_n(z_k)}),\end{aligned}$$

and therefore

$$\begin{aligned}1-\mathcal{HF}_{-n,n-1}(\chi_A, z) = |w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{1}{|z-z_k|^2}-\\ |w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w'_n(z_k)|^2}\frac{z_k}{z-z_k}(n-1-\frac{z_k w''_n(z_k)}{w'_n(z_k)}),\end{aligned} \tag{19}$$

from which it follows

$$|1-\mathcal{HF}_{-n,n-1}(\chi_A,z)| \le |w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2}+$$

$$|w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{|z-z_k|}\frac{|nw_n'(z_k)-z_kw_n''(z_k)-w_n'(z_k)|}{|w_n'(z_k)|^3}.$$

If n is large enough and we apply Lemma 1 we get

$$\begin{aligned}|1-\mathcal{HF}_{-n,n-1}(\chi_A,z)| &\le \frac{|w_n(z)|^2}{n^2}\Big(\frac{1}{(m-\varepsilon)^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}+\frac{C+M+\varepsilon}{(m-\varepsilon)^3}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}\Big)\le\\ &\frac{(2M+\varepsilon)^2}{n^2}\Big(\frac{1}{(m-\varepsilon)^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2}+\frac{C+M+\varepsilon}{(m-\varepsilon)^3}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}\Big).\end{aligned} \tag{20}$$

Since $z\in J_n$, by applying Lemma 4 it is immediate that (20) goes uniformly to 0 when n tends to infinity.

(ii) Let $z\in I_n$. Proceeding like in (i) we arrive at equation (19) given before. By applying Lemma 1 to the second sum of (19) we get

$$|w_n(z)|^2\Big|\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w_n'(z_k)|^2}\frac{z_k}{z-z_k}(n-1-\frac{z_kw_n''(z_k)}{w_n'(z_k)})\Big|\le$$

$$|w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|}\frac{|nw_n'(z_k)-z_kw_n''(z_k)-w_n'(z_k)|}{|w_n'(z_k)|^3}\le\frac{(2M+\varepsilon)^2}{n^2}\frac{(C+M+\varepsilon)}{(m-\varepsilon)^3}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|},$$

and from Lemma 6 we conclude that the last sequence converges to 0 uniformly on $z\in I_n$. Therefore, from (19), for n large enough and $z\in I_n$ it holds that $1-\mathcal{HF}_{-n,n-1}(\chi_A,z)$ behaves like

$$|w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2}. \tag{21}$$

Indeed, $1-\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z))-|w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2}$ converges to 0 uniformly on $z\in I_n$, which implies that $1-\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z))$ behaves like $|w_n(z)|^2\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2}$ and $\Im(\mathcal{HF}_{-n,n-1}(\chi_A,z))$ converges to 0 uniformly on $z\in I_n$.

Next our aim is to bound (21). First, on the one hand if we apply Lemmas 1, 5 and 7 we get for

$z \in I_n$

$$|w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2} \leq \frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)+o(1)}{(m-\varepsilon)^2}\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2} \leq$$
$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)+o(1)}{(m-\varepsilon)^2}\times\Big(\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1)\Big) \leq$$
$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1).$$

On the other hand, by applying again Lemmas 1, 5 and 7 we have the following inequalities from bellow

$$|w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2} \geq \frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)+o(1)}{(M+\varepsilon)^2}\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n}\frac{1}{|z-z_k|^2} \geq$$
$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)+o(1)}{(M+\varepsilon)^2}\Big(\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1)\Big) \geq$$
$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(M+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1).$$

Therefore

$$\begin{aligned}\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(M+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1) \leq |w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in\mathbb{T}\setminus A}}^{n} \frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2} \leq \\ \frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1).\end{aligned} \tag{22}$$

Hence

$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(M+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)-o(1) \leq 1-\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z)) \leq$$
$$\frac{2|\Pi_\mu(z)|^2(1-\cos(2\pi-\varepsilon)d)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}\psi_1(1+d)+o(1),$$

from which it follows our result. Notice that we denote by $o(1)$ different sequences converging to 0 uniformly on $z \in I_n$.

(iii) If $z \in J_n'$, by using (18), we get

$$|\mathcal{HF}_{-n,n-1}(\chi_A,z)| \leq |w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in A}}^{n} \frac{1}{|w_n'(z_k)|^2}\frac{1}{|z-z_k|^2}+$$
$$|w_n(z)|^2 \sum_{\substack{k=1\\ z_k\in A}}^{n} \frac{1}{|z-z_k|}\frac{|nw_n'(z_k)-z_kw_n''(z_k)-w_n'(z_k)|}{|w_n'(z_k)|^3},$$

and for n is large enough, if we apply Lemma 1, we get

$$|\mathcal{HF}_{-n,n-1}(\chi_A,z)| \leq \frac{(2M+\varepsilon)^2}{n^2}\Big(\frac{1}{(m-\varepsilon)^2}\sum_{\substack{k=1\\ z_k\in A}}^{n}\frac{1}{|z-z_k|^2}+\frac{C+M+\varepsilon}{(m-\varepsilon)^3}\sum_{\substack{k=1\\ z_k\in A}}^{n}\frac{1}{|z-z_k|}\Big). \tag{23}$$

To prove our thesis, next we study the two last sums. We continue assuming that z_s is the first node in $\mathbb{T}\setminus A$ in the clockwise sense from J_n. Indeed, if $z\in J'_n$, then $\frac{1}{|z-z_1|}\leq\frac{\pi}{2}\frac{n}{\sqrt{n}(2\pi-\varepsilon)}$ and by proceeding in the same way we get $\frac{1}{|z-z_2|}\leq\frac{\pi}{2}\frac{n}{(\sqrt{n}+1)(2\pi-\varepsilon)},\cdots,\frac{1}{|z-z_{s-1}|}\leq\frac{\pi}{2}\frac{n}{(\sqrt{n}+s-2)(2\pi-\varepsilon)}$. Therefore

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in A}}^{n}\frac{1}{|z-z_k|^2}=\frac{1}{n^2}\sum_{k=1}^{s-1}\frac{1}{|z-z_k|^2}\leq\frac{\pi^2}{4(2\pi-\varepsilon)^2}\sum_{k=1}^{s-1}\frac{1}{(\sqrt{n}+k-1)^2}=$$

$$\frac{\pi^2}{4(2\pi-\varepsilon)^2}\sum_{k=0}^{s-2}\frac{1}{(\sqrt{n}+k)^2}\leq\frac{\pi^2}{4(2\pi-\varepsilon)^2}\psi_1(\sqrt{n}),$$

which converges to 0. Proceeding in a similar way

$$\frac{1}{n^2}\sum_{\substack{k=1\\ z_k\in A}}^{n}\frac{1}{|z-z_k|}=\frac{1}{n^2}\sum_{k=1}^{s-1}\frac{1}{|z-z_k|}\leq\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}\sum_{k=1}^{s-1}\frac{1}{(\sqrt{n}+k-1)}=$$

$$\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}\sum_{k=0}^{s-2}\frac{1}{(\sqrt{n}+k)}\leq\frac{\pi}{2(2\pi-\varepsilon)}\frac{1}{n}(H_{\sqrt{n}+s-2,1}-H_{\sqrt{n}-1,1}),$$

which converges to 0. Therefore (23) goes to 0 uniformly on z.

(iv) One has to proceed like in the proof of (ii). □

Remark 3. *In relation with (ii) in the preceding theorem we have to do some considerations. For n large enough $o(1)$ goes to zero uniformly on z and m and M can be substituted by m_γ and M_γ, where γ represents an small arc in which z moves in such a way as $|M_\gamma-m_\gamma|<\varepsilon$. Since we can assume that I_n is included in the small arc, then equation (16) can be rewritten like*

$$\frac{2(M_\gamma-\varepsilon)^2(1-\cos(2\pi-\varepsilon)d)}{(M_\gamma+\varepsilon)^2}\frac{1}{(2\pi+3\varepsilon)^2}\psi_1(1+d)\leq 1-\limsup\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z))\leq$$

$$1-\liminf\Re(\mathcal{HF}_{-n,n-1}(\chi_A,z))\leq\frac{2M_\gamma^2(1-\cos(2\pi-\varepsilon)d)}{(M_\gamma-2\varepsilon)^2}\frac{(1-\varepsilon)}{(2\pi-\epsilon)^2}\psi_1(1+d).$$

Remark 4. *The situation considered in [2] corresponds to the case in which the nodal points are the n roots of a complex number λ of modulus 1. Then $\varepsilon=0, w_n(z)=z^n+\lambda, \Pi_\mu(z)=1$ and $m=M=1$. Now it is clear that in these conditions, if we rewrite Theorem 4, we obtain the result given in [2].*

The preceding theorem can be generalized to the case of piecewise continuous functions on $\mathbb{T}$ as follows.

Theorem 5. *Let μ be a measure in the Baxter class and assume that the sequence $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$. Let ε be a positive fixed number and let $\{z_j\}_{j=1}^n$ be the zeros of the para-orthogonal polynomial $w_n(z)$, with $z_j = e^{i\theta_j}$ and θ_j satisfying (4). Let f and g be continuous functions on $\mathbb{T}$ and let us consider the piecewise continuous function $F = f\chi_A + g\chi_{\mathbb{T}\setminus A}$. If A is an arc $A = (a_1, a_2) \subset \mathbb{T}$, and we assume that $\ell_1 = \lim_{z\to a_2} f(z) \neq \lim_{z\to a_2} g(z) = \ell_2$ then it holds*

(i) $\mathcal{HF}_{-n,n-1}(F,z)$ uniformly converges to f in $J_n \subset A$.

(ii) If $z \in I_n$, $z = z_1 e^{-i\frac{(2\pi-\varepsilon)}{n}d}$, with $d \in [-\frac{1}{2}, \sqrt{n}]$, $m = \min_{z\in\mathbb{T}} |\Pi_\mu(z)|$ and $M = \max_{z\in\mathbb{T}} |\Pi_\mu(z)|$ then for n large enough

$$\begin{aligned}\frac{2m^2(1-\cos(2\pi-\varepsilon)d)}{(M+\varepsilon)^2}\frac{|\ell_1-\ell_2|}{(2\pi+3\varepsilon)^2}\psi_1(1+d) - o(1) \leq |f - \mathcal{HF}_{-n,n-1}(F,z)| \leq \\ \frac{2M^2(1-\cos(2\pi-\varepsilon)d)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}|\ell_1-\ell_2|\psi_1(1+d) + o(1).\end{aligned} \tag{24}$$

(iii) $\mathcal{HF}_{-n,n-1}(F,z)$ uniformly converges to g in $J'_n \subset \mathbb{T}\setminus A$.

(iv) If $z \in I'_n$, $z = z_n e^{\frac{(2\pi-\varepsilon)}{n}Di}$, with $D \in [-\frac{1}{2}, \sqrt{n}]$, $m = \min_{z\in\mathbb{T}} |\Pi_\mu(z)|$ and $M = \max_{z\in\mathbb{T}} |\Pi_\mu(z)|$ then for n large enough

$$\begin{aligned}\frac{2m^2(1-\cos(2\pi-\varepsilon)D)}{(M+\varepsilon)^2}\frac{|\ell_1-\ell_2|}{(2\pi+3\varepsilon)^2}\psi_1(1+D) - o(1) \leq |g - \mathcal{HF}_{-n,n-1}(F,z)| \leq \\ \frac{2M^2(1-\cos(2\pi-\varepsilon)D)}{(m-\varepsilon)^2}\frac{(1+\varepsilon)}{(2\pi-\varepsilon)^2}|\ell_1-\ell_2|\psi_1(1+D) + o(1).\end{aligned} \tag{25}$$

Proof. By applying (18) it is easy to obtain for every $z \in \mathbb{T}$

$$\begin{aligned}\mathcal{HF}_{-n,n-1}(F,z) = \mathcal{HF}_{-n,n-1}(f,z) - \\ \underbrace{|w_n(z)|^2 \sum_{z_k\in\mathbb{T}\setminus A} \frac{(f(z_k)-g(z_k))}{|w'_n(z)|^2}\Big(\frac{1}{|z-z_k|^2} - \frac{z_k}{(z-z_k)}(n-1-z_k\frac{w''_n(z_k)}{w'_n(z_k)})\Big)}_{*}.\end{aligned} \tag{26}$$

Since $\mathcal{HF}_{-n,n-1}(f,z)$ converges uniformly to f on $\mathbb{T}$, we have to study the behavior of $(*\,)$ depending on where z belongs in order to obtain the behavior of $\mathcal{HF}_{-n,n-1}(F,z)$.

(i) If $z \in J_n$ and we take into account that $|f(z_k) - g(z_k)| \leq \| f - g \|_\infty$, then proceeding like in (i) of the preceding Theorem 4, we get that $(*)$ converges to 0 and therefore we have the result.

(ii) We have to study the behavior of $(*\,)$ for $z \in I_n$. Thus, if we denote by $h(z) = f(z) - g(z) - \ell_1 + \ell_2$ we have that $\lim_{z\to a_2} h(z) = 0$ and therefore, given $\varepsilon_1 > 0$ it holds that $|h(z)| < \varepsilon_1$ for $z \in I'_n$ with n large enough. For convenience we rewrite $(*)$ as follows

$$\begin{aligned}|w_n(z)|^2 \sum_{z_k\in\mathbb{T}\setminus A} \frac{(\ell_1-\ell_2)}{|w'_n(z)|^2}\frac{1}{|z-z_k|^2} + |w_n(z)|^2 \sum_{z_k\in\mathbb{T}\setminus A} \frac{h(z_k)}{|w'_n(z)|^2}\frac{1}{|z-z_k|^2} - \\ |w_n(z)|^2(\ell_1-\ell_2) \sum_{z_k\in\mathbb{T}\setminus A} \frac{z_k}{(z-z_k)}\frac{1}{|w'_n(z)|^2}(n-1-z_k\frac{w''_n(z_k)}{w'_n(z_k)}) - \\ |w_n(z)|^2 \sum_{z_k\in\mathbb{T}\setminus A} h(z_k)\frac{z_k}{(z-z_k)}\frac{1}{|w'_n(z)|^2}(n-1-z_k\frac{w''_n(z_k)}{w'_n(z_k)}).\end{aligned} \tag{27}$$

The first sum in the previous expression has been studied in (ii) of Theorem 4 obtaining that it satisfies the inequalities given in (22).
The second sum can be bounded as follows:

$$|w_n(z)|^2\Big|\sum_{z_k\in\mathbb{T}\setminus A}\frac{h(z_k)}{|w_n'(z)|^2}\frac{1}{|z-z_k|^2}\Big|\le$$

$$|w_n(z)|^2\Big(\sum_{z_k\in I_n'}\frac{|h(z_k)|}{|w_n'(z)|^2}\frac{1}{|z-z_k|^2}+\sum_{z_k\in(\mathbb{T}\setminus A)\setminus I_n'}\frac{|h(z_k)|}{|w_n'(z)|^2}\frac{1}{|z-z_k|^2}\Big)\le$$

$$|w_n(z)|^2\varepsilon_1 K\frac{1}{n^2}\sum_{z_k\in I_n'}\frac{1}{|z-z_k|^2}+|w_n(z)|^2\parallel h\parallel_\infty\frac{1}{n^2}\sum_{z_k\in(\mathbb{T}\setminus A)\setminus I_n'}\frac{1}{|z-z_k|^2},$$

for some appropriate constant K and n large enough. Thus, if we proceed like in the preceding theorem we can conclude that $\frac{1}{n^2}\sum_{z_k\in I_n'}\frac{1}{|z-z_k|^2}$ is bounded and $\frac{1}{n^2}\sum_{z_k\in(\mathbb{T}\setminus A)\setminus I_n'}\frac{1}{|z-z_k|^2}$ goes to zero uniformly for $z\in I_n$, which implies that the second sum in (27) converges to zero uniformly for $z\in I_n$.

Finally, taking into account that $|h(z)|\le\parallel h\parallel_\infty$ and by applying Lemma 6 like in (ii) of the previous theorem, we obtain the two last sums in (27) go to zero uniformly for $z\in I_n$. Hence, by applying Lemmas 1 and 7 we obtain (ii).

(iii) By applying (18) it is easy to obtain for every $z\in\mathbb{T}$

$$\mathcal{HF}_{-n,n-1}(F,z)=\mathcal{HF}_{-n,n-1}(g,z)-$$

$$\underbrace{|w_n(z)|^2\sum_{z_k\in A}\frac{(f(z_k)-g(z_k))}{|w_n'(z)|^2}\Big(\frac{1}{|z-z_k|^2}-\frac{z_k}{(z-z_k)}(n-1-z_k\frac{w_n''(z_k)}{w_n'(z_k)})\Big)}_{**}. \quad (28)$$

Since $\mathcal{HF}_{-n,n-1}(g,z)$ converges uniformly to g on $\mathbb{T}$ and $|f(z_k)-g(z_k)|<\parallel f-g\parallel_\infty$, if we proceed like in (iii) of the preceding Theorem 4 we get that $(**)$ converges to 0 uniformly for $z\in J_n'$ and therefore (iii) is proved.

(iv) It can be obtained proceeding like in (ii). □

Remark 5. *Wilbraham-Gibbs phenomenon is an important oscillation in a small region, which can be described in a successful way when we use a dilatation of order n. Notice that we are not doing anything new; this is well-known in Fourier theory, see [7]. Obviously different Wilbraham-Gibbs phenomena have different oscillations and these must be well-known if we want to understand or correct them.*

4. Examples and numerical experiments

In order to present some numerical experiments about our main results given in the preceding theorems, first we give examples of monic orthogonal polynomial sequences $\{\phi_n(z)\}$ such that the corresponding measures $\mu\in B$, $\mu\notin S_A$ and the hypothesis in Lemma 1 is satisfied: $\{(\phi_n^*)'\}$ is uniformly bounded on $\mathbb{T}$.

We consider the monic orthogonal polynomial sequences generated by the Verblunsky parameters satisfying $|\phi_1(0)|<1, \phi_2(0)=0$, and for $n\ge 3$

$$\phi_n(0)=\begin{cases}\frac{1}{(2^k-1)^c}, & \text{if } n=2^k-1,\\ 0, & \text{if } n\ne 2^k-1,\end{cases} \quad (29)$$

for $k \geq 2$, with $c > 2$.

The measures of orthogonality μ belong to the Baxter class but they have not analytic extension outside the unit disk, that is, $\mu \notin S_A$. Indeed it is clear that the following conditions hold

$$\sum_{n=2}^{\infty} |\phi_n(0)| = \sum_{k=2}^{\infty} \frac{1}{(2^k-1)^c} \leq \sum_{k=1}^{\infty} \frac{1}{k^c} < \infty,$$

and

$$\limsup_{n\to\infty} \sqrt[n]{|\phi_n(0)|} = \limsup_{k\to\infty} \sqrt[2^k-1]{|\phi_{2^k-1}(0)|} = \limsup_{k\to\infty} \frac{1}{(2^k-1)^{c/(2^k-1)}} = 1.$$

In what follows we prove that the sequences $\{\phi_n(z)\}$ generated by the Verblunsky parameters given above satisfy the hypotheses of Lemma 1.

Lemma 8. *Let $\{\phi_n(z)\}$ be a sequence given by (29). Then the coefficients of $\phi_n(z)$, which are different of the leading coefficient, have modulus less than* 1.

Proof. First we apply the well known recurrence relation $\phi_n(z) = z\phi_{n-1}(z) + \phi_n(0)\phi^*_{n-1}(z)$, with $\phi_{-1}(z) = 0, \phi_0(z) = 1, n \geq 1$, obtaining that the sequence is given by

$$\phi_1(z) = z + \phi_1(0), \phi_2(z) = z\phi_1(z),$$

and for $k \geq 2$

$$\phi_{2^k-1+i}(z) = z^i\phi_{2^k-1}(z),\ i = 1, \cdots, 2^k - 1.$$

Then it is immediate to conclude that $\phi^*_1(z) = \phi^*_2(z)$ and for $k \geq 2$ it holds

$$\phi^*_{2^k-1+i}(z) = \phi^*_{2^k-1}(z),\ i = 1, \cdots, 2^k - 1.$$

Since $\phi_3(z) = z^3 + \phi_1(0)z^2 + \overline{\frac{\phi_1(0)}{3^c}}z + \frac{1}{3^c}$, then the coefficients of $\phi_3(z)$, which are different of the leading coefficient, have modulus less than 1.

Proceeding by induction, if we assume that the coefficients of $\phi_{2^k-1}(z)$, which are different of the leading coefficient, have modulus less than 1, next we prove the property for $\phi_{2^{k+1}-1}(z)$.

Indeed, by using the properties satisfied by these sequences we have

$$\phi_{2^{k+1}-1}(z) = z\phi_{2^{k+1}-2}(z) + \phi_{2^{k+1}-1}(0)\phi^*_{2^{k+1}-2}(z) = z^{2^k}\phi_{2^k-1}(z) + \phi_{2^{k+1}-1}(0)\phi^*_{2^k-1}(z).$$

Since the polynomials $z^{2^k}\phi_{2^k-1}(z)$ and $\phi^*_{2^k-1}(z)$ have no common monomials of the same degree and their coefficients are those of $\phi_{2^k-1}(z)$ or their conjugates, the property is proved. $\square$

Lemma 9. *Let $\{\phi_n(z)\}$ be a sequence given by (29). Then the sequence $\{(\phi^*_n)'\}$ is bounded uniformly on $\mathbb{T}$.*

Proof. We proceed by using induction. Firstly, $(\phi^*_1)'(z) = (\phi^*_2)'(z) = \phi_1(0)$,
Since $(\phi^*_3)'(z) = (\phi^*_4)'(z) = (\phi^*_5)'(z) = (\phi^*_6)'(z) = 3\phi_3(0)z^2 + 2\phi_1(0)\phi_3(0)z + \overline{\phi_1(0)}$, then for $z \in \mathbb{T}$

$$|(\phi^*_3)'(z)| \leq 1 + \frac{2}{3^c} + \frac{3}{3^c} \leq 1 + \frac{1}{2^{c-1}} + \frac{1}{3^{c-1}} = H_{3,c-1} < H_{c-1},$$

where $H_{3,c-1}$ is the 3rd partial sum and H_{c-1} is the sum of the armonic series $\sum_{n=1}^{\infty} \frac{1}{n^{c-1}}$.

Now if we assume that $|(\phi^*_{2^k-1})'(z)| < H_{2^k-1,c-1}$ then we prove that $|(\phi^*_{2^{k+1}-1})'(z)| < H_{2^{k+1}-1,c-1}$.

Since $\phi^*_{2^{k+1}-1}(z) = \phi^*_{2^{k+1}-2}(z) + \phi_{2^{k+1}-1}(0) z^{2^k} \phi_{2^k-1}(z)$, then by taking derivatives $|(\phi^*_{2^{k+1}-1})'(z)| \leq |(\phi^*_{2^{k+1}-2})'(z)| + |\phi_{2^{k+1}-1}(0)||(z^{2^k}\phi_{2^k-1}(z))'|$.

By the hypothesis of induction $|(\phi^*_{2^{k+1}-1})'(z)| = |(\phi^*_{2^k-1})'(z)| \leq H_{2^k-1,c-1}$. Moreover, $|\phi_{2^{k+1}-1}(0)| = \frac{1}{(2^{k+1}-1)^c}$ and to bound $(z^{2^k}\phi_{2^k-1}(z))'$ we proceed as follows.

If $\phi_{2^k-1}(z) = \sum_{j=0}^{2^k-2} a_j z^j + z^{2^k-1}$ then $(z^{2^k}\phi_{2^k-1}(z))' = \sum_{j=0}^{2^k-2} a_j (2^k+j) z^{2^k+j-1} + (2^{k+1}-1) z^{2^{k+1}-2}$. Thus, for $z \in \mathbb{T}$ we have $|(z^{2^k}\phi_{2^k-1}(z))'| \leq \sum_{j=0}^{2^k-1}(2^k+j)$. Therefore $|(\phi^*_{2^{k+1}-1})'(z)| \leq H_{2^k-1,c-1} + \frac{1}{(2^{k+1}-1)^c}\sum_{j=0}^{2^k-1}(2^k+j) \leq H_{2^k-1,c-1} + \sum_{j=2^k}^{2^{k+1}-1}\frac{1}{j^{c-1}} = H_{2^{k+1}-1,c-1} < H_{c-1}$.

□

4.1. Numerical experiments

We have designed some numerical experiments in order to observe our results in a graphical way.

Example 1. *First we consider the Baxter measures presented at the beginning of this section. In particular we use the measure associated to the sequence of orthogonal polynomials generated by the Verblunsky coefficients given in (29) by taking $c = 4$ and $\phi_1(0) = 0$. By applying the recurrence relations given in Lemma 8 we compute the orthogonal polynomials and we obtain the para-orthogonal polynomials $w_n(z)$ defined in (3) with $\tau = -1$. To obtain the Hermite-Fejér interpolants we take $n = 2^k - 1$ with $k = 5, 7, 10$, and we use the barycentric expression corresponding to (18). This type of expressions are very interesting for their numerical stability and they are obtained in a standard way as can be seen in [5]. In all the cases we compute the nodal points, which are the zeros of $w_n(z)$ with $n = 31, 127, 1023$, as well as the coefficients of the corresponding barycentric formula.*

We consider the test function $F(z)$ defined by $e^{\frac{z+1/z}{2}}$ in the shortest arc between 1 and $-i$ and by 0 in the complementary arc, that is, between $-i$ and 1. Taking into account the previous lemmas, then the hypotheses of Theorem 5 are satisfied. Notice that at the point $a_2 = 1$ we have that $|\ell_1 - \ell_2| = e$.

In the next figures we have plotted the function, the interpolants and the approximants of the interpolants stated in Theorem 5. For the sake of simplicity we only present the cases corresponding to take three values of n and the key question is the evolution when n grows.

As we can see in Figure 2 the function $F(z)$, in green, is interpolated by $\mathcal{HF}_{-31,30}(F,z)$, in red, and the approximants suggested in (24) and (25) given by $e(1 - \frac{1}{\pi^2}\sin^2(\pi d)\psi_1(1+d))$ and $e\frac{1}{\pi^2}\sin^2(\pi d)\psi_1(1+d)$, with $d = -5, \cdots, 5$, are plotted in blue.

The second graphic (Figure 3) corresponds to $n = 127$ and the third one (Figure 4) is for $n = 1023$. It is clear that in the last case the function is essentially constant in our region and in a very graphical way the interpolant is indistinguishable with the approximation of the interpolant suggested in (24) and (25). Thus, in this first example we highlight the relation between the function and the interpolants in a neighborhood of the discontinuities and their evolution when n grows.

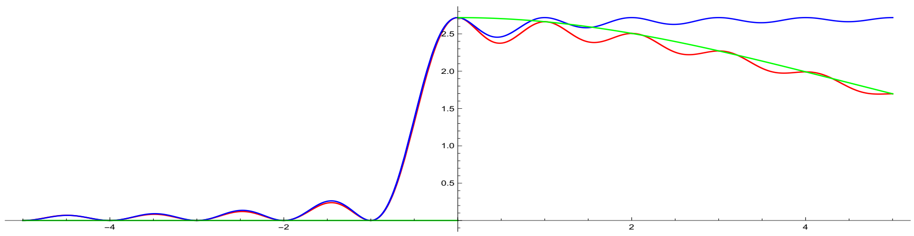

Figure 2: $F(z)$, $\mathcal{HF}_{-31,30}(F,z)$ and its approximation

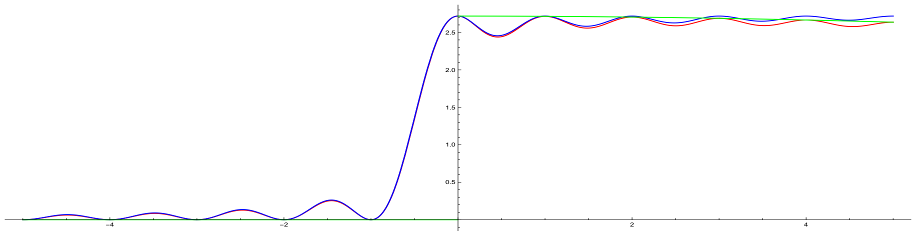

Figure 3: $F(z)$, $\mathcal{HF}_{-127,126}(F,z)$ and its approximation

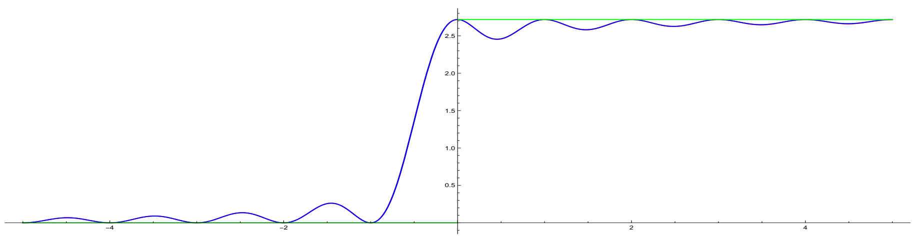

Figure 4: $F(z)$, $\mathcal{HF}_{-1023,1022}(F,z)$ and its approximation

We have also done some others numerical experiments, which are in concordance with the previous one and with the theoretical results.

Example 2. *In this experiment we take as test function* $F(z)$ *defined by* $2+2(1-\Re(z))\sin\frac{1}{1-\Re(z)}$ *for* z *belonging to the shortest arc between* 1 *and* $-i$ *and* 0 *in the complementary arc. Moreover, we use two different measures in order to obtain two different nodal systems of* 511 *nodes and to construct their corresponding interpolants. The first interpolant is related to the measure used in the previous example and the second one is related with a Roger-Szegő (RS) measure; we use* $\tau=-1$ *in both para-orthogonal polynomials. We recall that the RS measures (or wrapped Gaussian measures) are one of the classical examples of measures on the unit circle. These type of measures are analytical weights on the unit circle and satisfy the conditions of Lemma 1 because of Remark 2. A detailed description of these measures, which depend on a parameter* q*, can be found in [12]. In particular we use* $q=0.05$ *for this example.*

We denote by $\mathcal{HF}_{-511,510}(F,z)$ *the Hermite-Fejér polynomial corresponding to the first measure and by* $\mathcal{HFRS}_{-511,510}(F,z)$ *the Hermite-Fejér polynomial related to the Roger-Szegő measure.*

The graphic in Figure 5 shows the behaviour on $\mathbb{T}$ of the functions $F(z)$, in green, $\mathcal{HF}_{-511,510}(F,z)$, in blue, and $\mathcal{HFRS}_{-511,510}(F,z)$, in red. We use the relationship $z=e^{-i\frac{2\pi d}{n}}$, with $-\frac{511}{2}\le d\le\frac{511}{2}$, $n=$

511, to obtain the representation as before. So, the shortest arc between 1 and $-i$ corresponds to values of $d \in [0, 127.75]$.

On the one hand Theorem 5 states that our interpolants and the function must be quite similar when we are far away the discontinuities. On the other hand Theorem 5 states that the interpolants and the function must be quite different when we are close to the discontinuities. These two asserts can be seen in the picture.

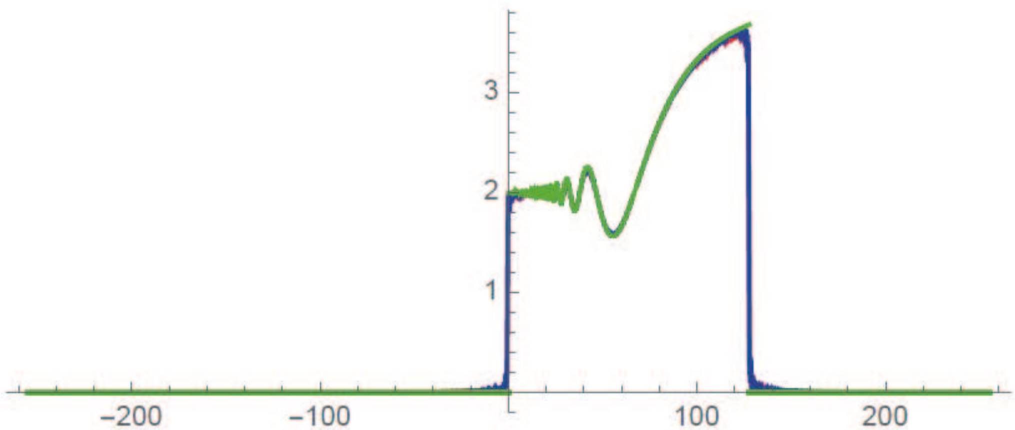

Figure 5: $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$

Notice that Theorem 5 establishes the shape of $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ near the discontinuities. Their shapes must be similar to the shape of $|\ell_1 - \ell_2| \frac{1}{\pi^2} \sin^2(\pi d)\psi_1(1+d)$ starting on the node closest to the discontinuity. This phenomenon can be observed in the Figure 6 which represents on the left hand side $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$, $\mathcal{HFRS}_{-511,510}(F,z)$ and the approximation given in Theorem 5 near 1. On the right hand side of the same figure we have plotted $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ near $-i$. In this last graphic it is clear that the initial nodes are different but the shapes are essentially the same.

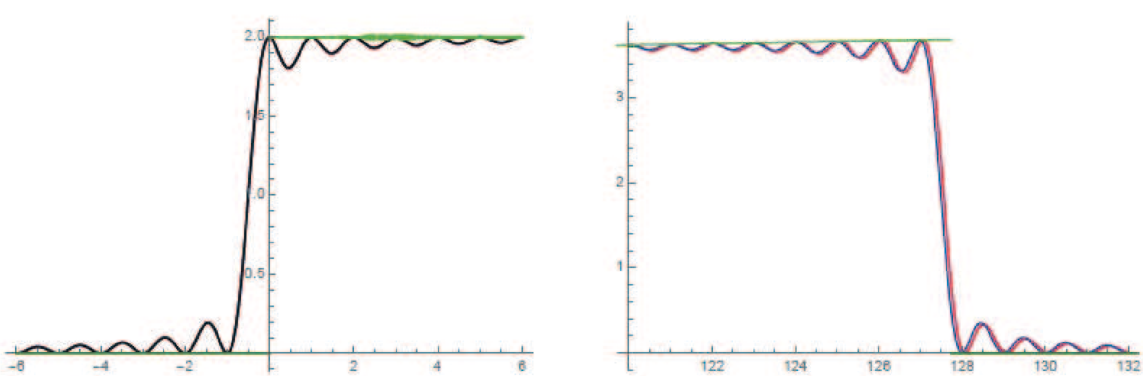

Figure 6: $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ near 1 and $-i$ respectively

Finally, Figure 7 is devoted to give some extra details of $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ for $d \in [0.8, 2.2]$ and also the difference between $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ near 1.

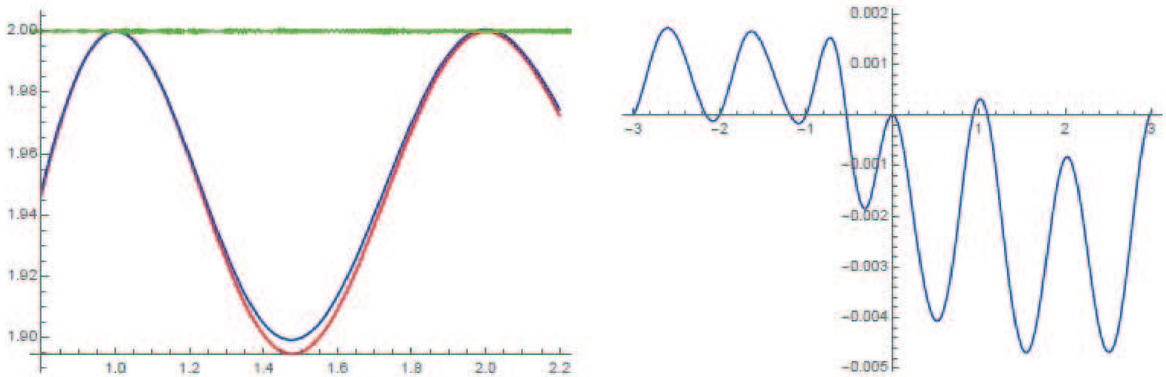

Figure 7: $F(z)$, $\mathcal{HF}_{-511,510}(F,z)$ and $\mathcal{HFRS}_{-511,510}(F,z)$ and their difference near 1

4.2. Final considerations

In this piece of work we have studied the relevant aspects of the Hermite-Fejér interpolation when we approximate piecewise continuous functions related to nodal systems with good properties. In this case, the distribution of the nodal points is closely related to the measures. As a natural continuation of this work our aim is to study similar problems for general nodal systems that could correspond to the zeros of para-orthogonal polynomials with respect to measures with weaker properties. A quite different point of view could be to establish sufficient conditions for choosing nodal systems with good properties. In particular such type of distribution of points could be provided by some mechanic models like the position of a mobile moving with constant velocity on $\mathbb{T}$ modified by an oscillatory movement of high frecuency and low elongation. The position of this mobile on $\mathbb{T}$ during equal periods of time would correspond to a distribution like the studied in this paper.

Other possible future lines of research could be the study of the convergence of the interpolants, for piecewise continuous functions, by considering other interpolatory processes like Lagrange interpolation with the nodal systems considered in this paper. Finally, another possible extension could be the study of the associated problems in $[-1,1]$ through the Szegő transformation.

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