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On the interpolation of the measure of non-compactness of bilinear operators with weak assumptions on the boundedness of the operator

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Abstract

We complete the range of the parameters in the interpolation formula established by Mastyło and Silva for the measure of non-compactness of a bilinear operator interpolated by the real method.

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Dedicated to Professor Hans Triebel on the occasion of his 85th birthday.

1. Introduction

One of the questions considered by Calderón [4] in his seminal paper on the complex method was the interpolation of compact bilinear operators. The counterpart for the real method of Lions and Peetre [18] has been done recently, starting with the papers by Fernandez and Silva [11] and Fernández-Cabrera and Martínez [12, 13]. A motivation for the research has been the fact that compact bilinear operators arise rather naturally in harmonic analysis. Namely, commutators of bilinear Calderón-Zygmund operators and multiplication by functions in the subspace CMO of BMO are compact (see the papers by Bényi and Torres [1], Cobos, Fernández-Cabrera and Martínez [6] and Torres, Xue and Yan [26]). Other results on interpolation of compact bilinear operators can be

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found in the papers by Mastyło and Silva [22] and Cobos, Fernández-Cabrera and Martínez [7, 8].

Quantitative versions in terms of the measure of non-compactness of some of these qualitative results have been established by Mastyło and Silva [21] and Besoy and Cobos [3]. Both papers refer to the real method but they work with different assumptions on the operator. Mastyło and Silva assume that T is defined from $(A_0 \cap A_1) \times (B_0 \cap B_1)$ into $E_0 \cap E_1$ with

$$\|T(a,b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1,$$

and establish the following log-convexity inequality for the measure of non-compactness

$$(1.1)\quad \begin{aligned} &\beta(T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \bar{E}_{\theta,r}) \\ &\qquad \leq C\, \beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^{\theta}, \end{aligned}$$

where $1 \leq p, q < \infty$, $1 < r < \infty$ and $1/p + 1/q = 1 + 1/r$ (see [21, Theorem 3.2]). Besoy and Cobos ask a stronger assumption on T. Namely, they suppose that T is bounded from $(A_0 + A_1) \times (B_0 + B_1)$ into $E_0 + E_1$ and that the restrictions $T : A_j \times B_j \longrightarrow E_j$ are also bounded for $j = 0, 1$. In this case, they show that the log-convexity inequality holds for operators acting among couples of quasi-Banach spaces and for the whole range of the parameters p, q, r (see [3, Theorem 3.5]). More precisely, if (E_0, E_1) is a couple of s-Banach spaces, then the conditions on the parameters are

$$(1.2)\qquad 0 < p, q, r \leq \infty \ \text{ with } \ 1/r = \begin{cases} 1/p + 1/q - 1/s & \text{if } \ p, q \geq s, \\ 1/\max(p,q) & \text{if } \ p < s \text{ or } q < s. \end{cases}$$

Approaches followed in [21] and [3] are completely different.

In applications, the weaker assumption on T is more handy. For this reason, it is important to extend inequality (1.1) to the remaining range of parameters. For couples of Banach spaces, this means the values of (1.2) with $s = 1$. Accordingly, we prove in this paper such a result.

Our techniques are a refinement of those used by Mastyło and Silva [21] based on duality results for bilinear operators introduced by Ramanujan and Schock [24] and the corresponding results established by Cobos, Fernández-Martínez and Martínez [9] and Fernández-Martínez [14] for the measure of non-compactness in the linear case (see also the paper by Edmunds and Teixeira [25]). Duality is the reason why we work with Banach couples.

We start by reviewing in Section 2 the basic results on the real interpolation method. We also establish there the variant of the bilinear interpolation theorem that we will need later. In Section 3 we study the connections between the measure of non-compactness of a bilinear operator and the measure of non-compactness of its adjoint operator, which is a linear operator. Finally, in Section 4, we establish the log-convexity inequality.

2. Preliminaries

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space.

We endow $A_0 + A_1 = \Sigma(\bar{A})$ and $A_0 \cap A_1 = \Delta(\bar{A})$ with the norms $K(1,\cdot)$ and $J(1,\cdot)$, respectively, where for $t > 0$

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\},\ a \in \Sigma(\bar{A}),$$

and

$$J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in \Delta(\bar{A}),$$

are the Peetre's functionals.

Let $0 < \theta < 1$ and $0 < q \le \infty$. The *real interpolation space* $(A_0, A_1)_{\theta,q} = \bar{A}_{\theta,q}$ is formed by all $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} < \infty,$$

(the integral should be replaced by the supremum when $q = \infty$). The space $(A_0, A_1)_{\theta,q}$ is a Banach space if $1 \le q \le \infty$ and a quasi-Banach space if $0 < q < 1$ (see [2, 27]).

The real interpolation method has the interpolation property for linear operators: Let $\bar{B} = (B_0, B_1)$ be another Banach couple and let R be a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restrictions $R : A_j \longrightarrow B_j$ are bounded for $j = 0, 1$. Then $R : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}$ is bounded with

$$\|R\|_{\bar{A}_{\theta,q},\bar{B}_{\theta,q}} \le \|R\|_{A_0,B_0}^{1-\theta}\|R\|_{A_1,B_1}^{\theta}.$$

The space $(A_0, A_1)_{\theta,q}$ can also be introduced using series instead of integrals because the functional

$$\|a\|_{\theta,q} = \Big(\sum_{m=-\infty}^{\infty} \left[2^{-\theta m}K(2^m, a)\right]^q\Big)^{1/q},$$

(the sum should be replaced by the supremum if $q = \infty$) is an equivalent norm (respectively, quasi-norm) to $\|\cdot\|_{\bar{A}_{\theta,q}}$ if $1 \le q \le \infty$ (respectively, if $0 < q < 1$).

The real interpolation space can be equivalently described by means of the J-functional. Indeed, $(A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ for which there is $(u_m) \subseteq A_0 \cap A_1$ such that

$$a = \sum_{m=-\infty}^{\infty} u_m \quad \text{(convergence in } A_0 + A_1\text{)}$$

and

$$\Big(\sum_{m=-\infty}^{\infty} \left[2^{-\theta m}J(2^m, u_m)\right]^q\Big)^{1/q} < \infty.$$

Moreover, the functional

$$\|a\|_{\theta,q}^J = \inf\left\{\Big(\sum_{m=-\infty}^{\infty} \left[2^{-\theta m}J(2^m, u_m)\right]^q\Big)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m,\ (u_m) \subseteq A_0 \cap A_1\right\}$$

is equivalent to $\|\cdot\|_{\bar{A}_{\theta,q}}$.

Subsequently, if A is a quasi-Banach space, we put A^* for its dual space. If $A_0 \cap A_1 \subseteq A$, we write A° for the closure of $A_0 \cap A_1$ in A. Since $A_0 \cap A_1 =$

$A_0^\circ \cap A_1^\circ$, it follows from the J-description of the real interpolation space that $(A_0, A_1)_{\theta,q} = (A_0^\circ, A_1^\circ)_{\theta,q}$. The J-description also gives that $A_0 \cap A_1$ is dense in $\bar{A}_{\theta,q}$ if $q < \infty$. Therefore, $\bar{A}_{\theta,q}^\circ = \bar{A}_{\theta,q}$ if $0 < q < \infty$.

We say that the Banach couple $\bar{A}$ is *regular* if $A_0 \cap A_1$ is dense in A_0 and in A_1. If this is the case, then $\overline{A^*} = (A_0^*, A_1^*)$ is also a Banach couple and the following duality formulae hold with equivalence of norms:

(2.1) If $1 \le q < \infty$, $1/q + 1/q' = 1$ and $0 < \theta < 1$, then

$$(A_0, A_1)_{\theta,q}^* = (A_0^*, A_1^*)_{\theta,q'}.$$

(2.2) If $q = \infty$ and $0 < \theta < 1$, then

$$((A_0, A_1)_{\theta,\infty}^\circ)^* = (A_0^*, A_1^*)_{\theta,1}.$$

(2.3) If $0 < q < 1$ and $0 < \theta < 1$, then

$$(A_0, A_1)_{\theta,q}^* = (A_0^*, A_1^*)_{\theta,\infty}.$$

See [2, Theorem 3.7.1 and Remark in p. 55] and [23, (0.5) in p. 124].

Let A, B, E be Banach spaces and let $T : A \times B \longrightarrow E$ be a bilinear operator. We say that T is *bounded* if

$$\|T\|_{A\times B,E} = \sup\left\{\|T(a,b)\|_E : \|a\|_A \le 1\,,\ \|b\|_B \le 1\right\} < \infty.$$

We write $\mathcal{L}(A\times B, E)$ for the space of all bounded bilinear operators from $A\times B$ into E.

Let $\bar{E} = (E_0, E_1)$ be another Banach couple. By $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ we mean that T is a bilinear operator defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ such that there are constants $M_j > 0$ with

$$\|T(a,b)\|_{E_j} \le M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}),\ j = 0, 1. \tag{2.4}$$

Using (2.4), it is not hard to check that T may be uniquely extended to a bilinear operator $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ with $\|T\|_{A_j^\circ \times B_j^\circ, E_j^\circ} \le M_j$.

The behaviour of bounded linear operators under real interpolation was already considered by Lions and Peetre [18] in their foundational paper on the real method. Their result was extended by Karadzhov [16] to the full range for the parameters (see also the paper by König [17] for a proof). In our later considerations we are going to need the following variant of the bilinear interpolation theorem.

Theorem 2.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let T be a bilinear operator from $(A_0 \cap A_1) \times (B_0 + B_1)$ into $E_0 + E_1$ such that there are positive constants M_j with*

$$\|T(a,b)\|_{E_j} \le M_j \,\|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_j,\ j = 0, 1.$$

Let $0 < \theta < 1$ and $0 < p, q, r \le \infty$ be positive numbers satisfying

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if} \quad p, q \ge 1, \\ \frac{1}{\max(p,q)} & \text{if} \quad p < 1 \text{ or } q < 1. \end{cases}$$

Then there is a constant $C > 0$ independent of T such that

$$\|T(a,b)\|_{\bar{E}_{\theta,r}} \le C\, M_0^{1-\theta} M_1^{\theta}\, \|a\|_{\bar{A}_{\theta,p}} \|b\|_{\bar{B}_{\theta,q}}, \quad a \in \Delta(\bar{A}),\ b \in \bar{B}_{\theta,q}.$$

Moreover, T may be uniquely extended to a bounded bilinear operator

$$T : \bar{A}^{\circ}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \bar{E}_{\theta,r}.$$

Proof. Take $n \in \mathbb{Z}$ such that $2^n \le M_1/M_0 < 2^{n+1}$. Given any $a \in A_0 \cap A_1$, $u \in B_0 \cap B_1$ and $m, k \in \mathbb{Z}$, if $a = a_0 + a_1$ with $a_j \in A_j$, we obtain

$$\begin{aligned} K(2^m, T(a,u)) &\le \|T(a_0,u)\|_{E_0} + 2^m \|T(a_1,u)\|_{E_1} \\ &\le M_0 \|a_0\|_{A_0} \|u\|_{B_0} + 2^m M_1 \|a_1\|_{A_1} \|u\|_{B_1} \\ &\le \max(M_0, 2^{-n} M_1)(\|a_0\|_{A_0} + 2^{m-k}\|a_1\|_{A_1}) J(2^{k+n}, u). \end{aligned}$$

Hence,

$$K(2^m, T(a,u)) \le 2M_0 K(2^{m-k}, a) J(2^{k+n}, u). \tag{2.5}$$

Now take any $b \in \bar{B}_{\theta,q}$ and let $b = \sum_{k=-\infty}^{\infty} u_k$ be any J-representation of b. Since $\sum_{k=-\infty}^{\infty} \|u_k\|_{B_0+B_1} < \infty$, we also have that $b = \sum_{k=-\infty}^{\infty} u_{k+n}$ in $B_0 + B_1$.

Suppose that $p < 1$ and $p \le q$, so $r = q$. We have

$$K(2^m, T(a,b)) \le \sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n})) \le \Big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^p \Big)^{1/p}.$$

Combining this inequality with (2.5) and using Young's inequality with parameters $p/q = 1 + p/q - 1$, we derive

$$\begin{aligned} \|T(a,b)\|_{\bar{E}_{\theta,q}} &\le 2M_0 \Big\| \Big(\sum_{k=-\infty}^{\infty} 2^{-\theta(m-k)p} K(2^{m-k}, a)^p 2^{-\theta k p} J(2^{k+n}, u_{k+n})^p \Big) \Big\|_{\ell_{q/p}}^{1/p} \\ &\le 2M_0 \big\| (2^{-\theta m} K(2^m, a))^p \big\|_{\ell_1}^{1/p} \big\| (2^{-\theta k} J(2^{k+n}, u_{k+n})) \big\|_{\ell_q} \\ &\le 2M_0 2^{\theta n} \|a\|_{\theta,p} \big\| (2^{-\theta k} J(2^k, u_k)) \big\|_{\ell_q}. \end{aligned}$$

Taking the infimum over all J-representations of b, we derive that

$$\|T(a,b)\|_{\bar{E}_{\theta,q}} \le C\, M_0^{1-\theta} M_1^{\theta}\, \|a\|_{\bar{A}_{\theta,p}} \|b\|_{\bar{B}_{\theta,q}}, \quad a \in \Delta(\bar{A}),\ b \in \bar{B}_{\theta,q}.$$

Finally, since $A_0 \cap A_1$ is dense in $\bar{A}_{\theta,p}$, the operator T may be uniquely extended to a bounded bilinear operator $T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \bar{E}_{\theta,q}$.

If $q < p \le 1$, then $r = p$ and we can proceed as before but using now that

$$K(2^m, T(a,b)) \le \Big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^q \Big)^{1/q}$$

and Young's inequality with parameters $q/p = q/p + 1 - 1$. The cases $q < 1$ and $q \le p$, and $p < q \le 1$ are similar. Finally, if $1 \le p, q$ then $1/r = 1/p + 1/q - 1$ and we can proceed directly with Young's inequality. □

Others results on interpolation of bilinear operators can be found, for example, in the papers by Janson [15], Mastyło [19, 20] and Cobos, Fernández-Cabrera and Martínez [6] .

3. Measure of non-compactness

Let A, B be Banach spaces. We write U_A for the closed unit ball of A and define U_B similarly. Given any bounded linear operator $R \in \mathcal{L}(A, B)$, the (ball) measure of non-compactness $\beta(R) = \beta(R : A \longrightarrow B)$ is defined to be the infimum of the set of all $\sigma > 0$ for which there is a finite subset $\{b_1, \cdots, b_s\} \subseteq B$ such that

$$R(U_A) \subseteq \bigcup_{k=1}^{s} \{b_k + \sigma U_B\},$$

(see [5, 10]).

Clearly, the operator R is compact if and only if $\beta(R) = 0$.

Let $R^* \in \mathcal{L}(B^*, A^*)$ be the adjoint operator of R. A well-known result of Schauder says that R is compact if and only if R^* is compact. If the operator R is not compact, then the following inequalities hold for the measure of non-compactness

$$\frac{1}{2}\,\beta(R : A \longrightarrow B) \leq \beta(R^* : B^* \longrightarrow A^*) \leq 2\,\beta(R : A \longrightarrow B), \tag{3.1}$$

(see [10, Corollary 2.10, p. 12]).

Let E be another Banach spaces and let $T : A \times B \longrightarrow E$ be a bounded bilinear operator. The (ball) *measure of non-compactness* $\beta(T) = \beta(T : A \times B \longrightarrow E)$ of T is the infimum of all $\sigma > 0$ for which there exists a finite subset $\{z_1, \cdots, z_s\} \subseteq E$ such that

$$T(U_A, U_B) = \big\{T(a, b) \;:\; a \in U_A,\, b \in U_B\big\} \subseteq \bigcup_{k=1}^{s} \{z_k + \sigma U_E\}.$$

The operator T is said to be *compact* if for any bounded sets $V \subseteq A$, $W \subseteq B$ we have that the closure of $T(V, W)$ is compact in E. Again, the operator T is compact if and only if its measure of non-compactness is 0.

Following Ramanujan and Schock [24], the *adjoint operator* $T^\times$ of T is the linear map

$$T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})$$

defined by $(T^\times f)(a, b) = f[T(a, b)]$. Here $\mathbb{K}$ is the scalar field.

It turns out that $\|T\|_{A\times B,E} = \|T^\times\|_{E^*,\mathcal{L}(A\times B,\mathbb{K})}$. Moreover, T is compact if and only if $T^\times$ is compact (see [24, Theorem 2.6]).

Next we study inequalities of the type (3.1) in the bilinear setting. For operators acting among Banach couples, this question has been studied in [21, Lemma 3.1],

Theorem 3.1. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then we have*

$$\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})) \leq 4\,\beta(T : A \times B \longrightarrow E).$$

Proof. Given any $\varepsilon > 0$, we can find finite subsets $\{a_1, \cdots, a_s\} \subseteq U_A$ and $\{b_1, \cdots, b_s\} \subseteq U_B$ such that for any $(a, b) \in U_A \times U_B$ there exits $1 \leq k \leq s$ with

$$\|T(a, b) - T(a_k, b_k)\| \leq 2\,\beta(T) + \varepsilon. \tag{3.2}$$

Let $S : E^* \longrightarrow \mathbb{K}^s$ be the linear operator assigning to each $f \in E^*$ the s-tuple $Sf = (f[T(a_1, b_1)], \cdots, f[T(a_s, b_s)])$. Since S has finite rank, S is compact.

Therefore, there exits a finite subset of functionals $\{f_1, \cdots, f_m\} \subseteq U_{E^*}$ such that for any $f \in U_{E^*}$ there is $1 \le n \le m$ with $\|Sf - Sf_n\|_{\mathbb{K}^s} \le \varepsilon$. Whence,

$$|f[T(a_k, b_k)] - f_n[T(a_k, b_k)]| \le \varepsilon \text{ for any } 1 \le k \le s. \tag{3.3}$$

Then, given any $f \in U_{E^*}$ if we take n satisfying (3.3), and for any $(a, b) \in U_A \times U_B$ we choose k satisfying (3.2), we obtain

$$\begin{aligned}
|(T^\times f - T^\times f_n)(a, b)| &= |f[T(a, b)] - f_n[T(a, b)]| \\
&\le |f[T(a, b)] - f[T(a_k, b_k)]| + |f[T(a_k, b_k)] - f_n[T(a_k, b_k)]| \\
&\quad + |f_n[T(a_k, b_k)] - f_n[T(a, b)]| \\
&\le \|T(a, b) - T(a_k, b_k)\|_E + \varepsilon + \|T(a_k, b_k) - T(a, b)\|_E \le 4\,\beta(T) + 3\,\varepsilon.
\end{aligned}$$

This yields that $\|T^\times f - T^\times f_n\|_{A\times B,\mathbb{K}} \le 4\,\beta(T) + 3\,\varepsilon$. Consequently,

$$\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})) \le 4\,\beta(T : A \times B \longrightarrow E).$$

□

Theorem 3.2. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then we have*

$$\beta(T : A \times B \longrightarrow E) \le 8\,\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})).$$

Proof. Put $W = \overline{T^\times(E^*)}$ which is a Banach space with the induced norm from $\mathcal{L}(A \times B, \mathbb{K})$. Let $\{R_1, \dots, R_s\} \subseteq \mathcal{L}(A \times B, \mathbb{K})$ and $\sigma > 0$ satisfy that

$$T^\times(U_{E^*}) \subseteq \bigcup_{k=1}^{s} \{R_k + \sigma U_{\mathcal{L}(A\times B,\mathbb{K})}\}.$$

We may assume that $W \cap \{R_k + \sigma U_{\mathcal{L}(A\times B,\mathbb{K})}\} \ne \varnothing$ for $k = 1, \dots, s$. Pick $S_k \in W \cap \{R_k + \sigma U_{\mathcal{L}(A\times B,\mathbb{K})}\}$. Then

$$T^\times(U_{E^*}) \subseteq \bigcup_{k=1}^{s} \{S_k + 2\sigma U_W\}.$$

This yields that

$$\beta(T^\times : E^* \longrightarrow W) \le 2\,\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})).$$

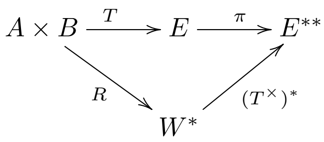
Now consider the adjoint operator $(T^\times)^* : W^* \longrightarrow E^{**}$ and let $R : A \times B \longrightarrow W^*$ the operator defined by $R(a, b) = g_{(a,b)}$ where

$$g_{(a,b)}(T^\times f) = f[T(a, b)].$$

The operator R is bilinear and has norm less than or equal to 1 because

$$\begin{aligned}
\|R(a, b)\|_{W^*} &= \sup\{|g_{(a,b)}(T^\times f)| \,:\, \|T^\times f\|_{A\times B,\mathbb{K}} \le 1\} \\
&= \sup\{|f[T(a, b)]| \,:\, \|T^\times f\|_{A\times B,\mathbb{K}} \le 1\} \\
&= \sup\{|(T^\times f)(a, b)| \,:\, \|T^\times f\|_{A\times B,\mathbb{K}} \le 1\} \le \|a\|_A \|b\|_B.
\end{aligned}$$

The following diagram is useful



where $\pi(w) = \widehat{w}$ is the natural embedding from E into its bidual E^{**}. Note that the diagram commutes because

$$\begin{aligned}\big((T^{\times})^{*}R(a,b)\big)(f) &= R(a,b)(T^{\times}f) = g_{(a,b)}(T^{\times}f)\\ &= f[T(a,b)] = \widehat{T(a,b)}(f) = \pi(T(a,b))(f).\end{aligned}$$

Consequently, using the diagram and (3.1), we obtain

$$\begin{aligned}\beta(T : A\times B \longrightarrow E) &\leq 2\,\beta(\pi T : A\times B \longrightarrow E^{**})\\ &\leq 2\,\|R\|_{A\times B, W^{*}}\beta((T^{\times})^{*} : W^{*} \longrightarrow E^{**})\\ &\leq 4\,\beta(T^{\times} : E^{*} \longrightarrow W)\\ &\leq 8\,\beta(T^{\times} : E^{*} \longrightarrow \mathcal{L}(A\times B, \mathbb{K})).\end{aligned}$$

□

4. Interpolation of the measure of non-compactness

We start with an auxiliary result which complements an embedding of [7, p. 5] (see also [21, Theorem 2.1]).

Subsequently, for $0 < r \leq \infty$ we put

$$\widetilde{r} = \begin{cases} \infty & \text{if} \quad 0 < r \leq 1,\\ r'\,,\ 1/r + 1/r' = 1 & \text{if} \quad 1 < r \leq \infty. \end{cases}$$

Lemma 4.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be regular Banach couples. For $j = 0, 1$, put $X_j = \mathcal{L}(A_j \times B_j, \mathbb{K})$ and let $0 < \theta < 1$ and $0 < p, q, r \leq \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \textit{if} \quad p, q \geq 1,\\ \frac{1}{\max(p,q)} & \textit{if} \quad p < 1 \ \textit{or}\ q < 1. \end{cases}$$

Then the following continuous embedding holds

$$(X_0, X_1)_{\theta,\widetilde{r}} \hookrightarrow \mathcal{L}(\bar{A}^{\circ}_{\theta,p} \times \bar{B}^{\circ}_{\theta,q}, \mathbb{K}).$$

Proof. The pair $\bar{X} = (X_0, X_1)$ is a Banach couple because $X_j \hookrightarrow \mathcal{L}(\Delta(\bar{A}) \times \Delta(\bar{B}), \mathbb{K})$, $j = 0, 1$. Let

$$\Phi : \Delta(\bar{A}) \times \mathcal{L}(\Delta(\bar{A}) \times \Delta(\bar{B}), \mathbb{K}) \longrightarrow \Delta(\bar{B})^{*}$$

be the bilinear operator defined by $\Phi(a, R)(b) = R(a, b)$. Since

$$|R(a,b)| \leq \|R\|_{\Delta(\bar{A})\times\Delta(\bar{B}),\mathbb{K}}\|a\|_{\Delta(\bar{A})}\|b\|_{\Delta(\bar{B})},$$

the operator Φ has norm less than or equal to 1. Moreover, if $a \in \Delta(\bar{A})$ and $R \in X_j$, we get

$$|R(a,b)| \le \|R\|_{X_j} \|a\|_{A_j} \|b\|_{B_j}.$$

So

$$\Phi : (\Delta(\bar{A}), \|\cdot\|_{A_j}) \times X_j \longrightarrow B_j^*, \quad j = 0, 1,$$

is also bounded, with norm less than or equal to 1.

By the duality formulae (2.1), (2.2), (2.3), we know that

$$((B_0, B_1)^\circ_{\theta,q})^* = (B_0^*, B_1^*)_{\theta,\widetilde{q}} = \overline{B^*}_{\theta,\widetilde{q}}.$$

Moreover, we have

$$\frac{1}{\widetilde{q}} = \begin{cases} \frac{1}{p} + \frac{1}{\widetilde{r}} - 1 & \text{if} \quad p \ge 1, \\ \frac{1}{\widetilde{r}} & \text{if} \quad p < 1. \end{cases}$$

Whence, applying Theorem 2.1, we get that Φ may be uniquely extended to a bounded bilinear operator

$$\Phi : \bar{A}^\circ_{\theta,p} \times \bar{X}_{\theta,\widetilde{r}} \longrightarrow \overline{B^*}_{\theta,\widetilde{q}} = ((B_0, B_1)^\circ_{\theta,q})^*.$$

Consequently, there is a constant $C > 0$ such that for any $R \in \bar{X}_{\theta,\widetilde{r}}$ and any $a \in \Delta(\bar{A})$, $b \in \Delta(\bar{B})$, we have

$$|R(a,b)| = |\Phi(a,R)(b)| \le \|\Phi(a,R)\|_{\overline{B^*}_{\theta,\widetilde{q}}} \|b\|_{\bar{B}_{\theta,q}} \le C\, \|R\|_{\bar{X}_{\theta,\widetilde{r}}} \|a\|_{\bar{A}_{\theta,p}} \|b\|_{\bar{B}_{\theta,q}}.$$

This yields that $R \in \mathcal{L}(\bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q}, \mathbb{K})$ and that the embedding

$$\bar{X}_{\theta,\widetilde{r}} \hookrightarrow \mathcal{L}(\bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q}, \mathbb{K})$$

is bounded with norm less than or equal to C. $\square$

Now we are ready to establish the announced result on interpolation of the measure of non-compactness of bilinear operators.

Theorem 4.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$. Let $0 < \theta < 1$ and $0 < p, q, r \le \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if} \quad p, q \ge 1, \\ \frac{1}{\max(p,q)} & \text{if} \quad p < 1 \text{ or } q < 1. \end{cases}$$

Then

$$\beta(T : \bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q} \longrightarrow \bar{E}^\circ_{\theta,r}) \\ \le C\, \beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^{\theta},$$

Here C is a constant independent of T.

Proof. According to the bilinear interpolation theorem (see, for example [13, Theorem 4.1]), we know that T may be uniquely extend to a bounded bilinear operator $T : \bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q} \longrightarrow \bar{E}^\circ_{\theta,r}$. Hence, by Theorem 3.2, we have

$$\beta(T : \bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q} \longrightarrow \bar{E}^\circ_{\theta,r}) \le 8\, \beta(T^\times : (\bar{E}^\circ_{\theta,r})^* \longrightarrow \mathcal{L}(\bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q}, \mathbb{K})). \tag{4.1}$$

Let $X_j = \mathcal{L}(A_j^\circ \times B_j^\circ, \mathbb{K})$, $j = 0, 1$. Since $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$, is bounded, the restrictions $T^\times : (E_j^\circ)^* \longrightarrow X_j$ are also bounded and, by Theorem 3.1, we have

$$\beta(T^\times : (E_j^\circ)^* \longrightarrow X_j) \le 4\,\beta(T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ), \quad j = 0, 1. \tag{4.2}$$

Using the formula for the measure of non-compactness of a linear operator interpolated by the real method of Cobos, Fernández-Martínez and Martínez [9, Theorem 1.2] (Banach case) and Fernández-Martínez [14, Theorem 3.1] (quasi-Banach case) and (4.2) we derive that

$$\begin{aligned} \beta\big(T^\times &: ((E_0^\circ)^*, (E_1^\circ)^*)_{\theta,\widetilde{r}} \longrightarrow (X_0, X_1)_{\theta,\widetilde{r}}\big) \\ &\le C_1\,\beta(T^\times : (E_0^\circ)^* \longrightarrow X_0)^{1-\theta}\beta(T^\times : (E_1^\circ)^* \longrightarrow X_1)^\theta \\ &\le 4\,C_1\,\beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta}\beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^\theta. \end{aligned} \tag{4.3}$$

The duality formulae (2.1), (2.2), (2.3) imply that

$$((E_0^\circ)^*, (E_1^\circ)^*)_{\theta,\widetilde{r}} = (E_0^\circ, E_1^\circ)^*_{\theta,r}.$$

Hence, it follows from Lemma 4.1 and (4.3) that

$$\begin{aligned} \beta\big(T^\times &: (E_0^\circ, E_1^\circ)^*_{\theta,r} \longrightarrow \mathcal{L}(\bar{A}^\circ_{\theta,p} \times \bar{B}^\circ_{\theta,q}, \mathbb{K})\big) \\ &\le C_2\,\beta\big(T^\times : (E_0^\circ, E_1^\circ)^*_{\theta,r} \longrightarrow (X_0, X_1)_{\theta,\widetilde{r}}\big) \\ &\le C_3\;\beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta}\beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^\theta. \end{aligned}$$

This estimate combined with (4.1) complete the proof. □

As a direct consequence of Theorem 4.2 we derive the following compactness result:

Theorem 4.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ such that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ compactly for $j = 0$ or $j = 1$. Let $0 < \theta < 1$ and $0 < p, q, r \le \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \textit{if} \quad p, q \ge 1, \\ \frac{1}{\max(p,q)} & \textit{if} \quad p < 1 \textit{ or } q < 1. \end{cases}$$

Then T may be uniquely extended to a compact bilinear operator from $(A_0, A_1)^\circ_{\theta,p} \times (B_0, B_1)^\circ_{\theta,q}$ to $(E_0, E_1)^\circ_{\theta,r}$.

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