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Optimal management of an urban road network with an environmental perspective

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Abstract

Within the framework of numerical simulation and optimal control of partial differential equations, in this work we deal with the mathematical modelling and optimal management of urban road networks. In particular, we are interested in finding the optimal management of the network intersections in order to reduce traffic congestion and atmospheric pollution. So, we consider two different multi-objective control problems (the former from a cooperative viewpoint, the latter within a hierarchical paradigm), propose a complete numerical algorithm to solve them, and, finally, present several numerical tests for a realistic case posed in the Guadalajara Metropolitan Area (Mexico), where the possibilities of our methodology are shown.

Keywords: Traffic flow, Air pollution, Queues modelling, Numerical simulation, Optimal control

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1. Introduction

The development of modern cities has brought as an inherent aspect the appearance or exacerbation of various inconveniences whose complexity requires a scientific study in order to solve them. In particular, two major urban problems can be mentioned: traffic congestion and atmospheric pollution. Both problems have their main cause in urban traffic, being closely related to each other. The first one refers fundamentally to the increase in the necessary time (and the consequent discomfort generated by acoustic pollution and economic cost) for the inhabitants to carry out their normal urban transfers. On the other hand, vehicles emit through their escapes various toxic substances, such as carbon monoxide or nitrogen oxides. The concentration levels of such toxic substances depend, as in the case of acoustic pollution, on traffic flow, but also on weather conditions, especially wind.

Faced with such problems, suitable urban planning becomes a vitally important issue. The design and management of an urban road network can be formulated as an optimization problem, which seeks to minimize the generalized travel cost in the road network, considering multiple variables such as the topology of the network (involving number and location of intersections and roads), the characteristics of the roads (capacity, length, number of lanes, travel times under free flow, operation and construction costs) and the characteristics of the intersections (level crossing or not, existence or not of traffic lights). This issue mainly involves urban planners; however, the effectiveness of the measures adopted will depend on the behaviour of users of the traffic network. The objectives of both are usually opposed, while in the design and planning of road networks the interest is placed on the best functioning of the whole network or on the reduction of pollution, users seek their own benefit, fundamentally related to minimizing individual travel times and costs.

The application of partial differential equations models is usual, both in the analysis of traffic flow in urban networks [14, 6, 8, 12, 11] and in the study of air pollution [1, 9, 17]. Nevertheless, the number of works combining both issues is much more restricted (see, for instance, [16], [4], [10] or [5]), and usually assume a previously known vehicular flow, which restricts the design of a road network optimal in terms of pollution and travel times.

The general aim of the present work is the search for a solution to the problem of the optimal management of an urban road network, applying optimal control techniques. In previous works [2, 3] the authors have proposed a methodology that, coupling a 1D model for vehicular flow with a 2D model for pollutant dispersion, allows to simulate how any change made on the road network influences air pollution. In the current work, and in order to make the model more suitable for real-world situations, we have incorporated the modelling of entrance queues, taking also into account their effects on travel times and atmospheric pollution.

This work can be considered within the framework of the optimal control of partial differential equations, and, with that idea in mind, in Section 2 we reformulate the model initially proposed in [2] in order to set a novel state system that, taking also into account the queues effects, shows in an explicit way the relation between the design variables (controls) of the problem and the corresponding states (traffic flow and air pollution). From this system, and in order to illustrate the possibilities of our approach, in Section 3 we introduce two different well-posed optimal control problems: a multi-objective cooperative (Pareto) problem, and a bi-level non-cooperative (Stackelberg) one, both related to the optimal management of the network intersections, looking for environmental and operational interests. Then, we propose a complete numerical algorithm to solve the problems (Section 4), including adjoint state techniques [15] to evaluate the objective function in an alternative simpler way, and, in final section, we present and discuss some numerical results obtained when applying our proposed methodology to a real-world case posed in the Guadalajara Metropolitan Area (GMA), one of the largest metropolitan areas in Mexico (almost five million inhabitants, more than two million vehicles).

2. The state system

We consider an urban domain $\Omega \subset \mathbb{R}^2$ including a road network composed of N_R unidirectional avenues (segments) meeting at a number N_J of junctions (intersections), such that the endpoints of each segment are either on the boundary of Ω or corresponds to one of the junctions (see Figure 1).

Each segment $A_i \subset \Omega,\ i = 1, \ldots, N_R$, is modelled by an interval $[a_i, b_i]$, and we denote by:

$$\begin{array}{rcll} \sigma_i : & [a_i, b_i] & \longrightarrow & A_i \subset \Omega \\ & s & \longmapsto & \sigma_i(s) = (x_i(s), y_i(s)) \end{array} \tag{1}$$

a parametrization of the segment A_i preserving the sense of motion on the

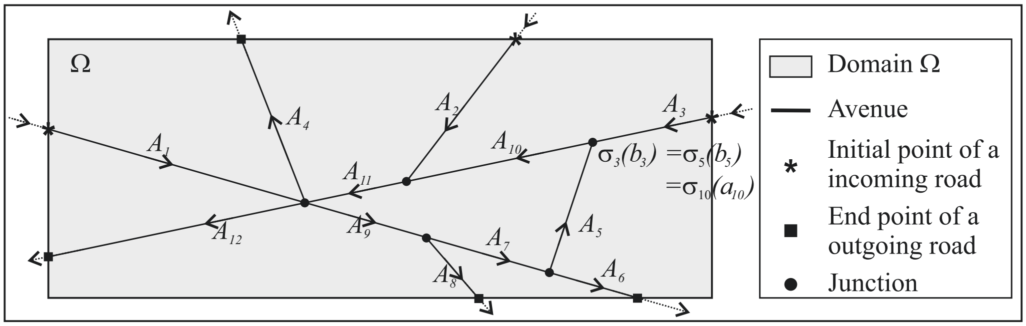

Figure 1: Example of a typical domain Ω for the methodology presented in our study.

avenue. We denote by $\mathcal{I}^{in}$, $\mathcal{I}^{out} \subset \{1 \ldots, N_R\}$ the sets of indices corresponding to incoming and outgoing roads in the network, respectively. Moreover, for each junction $j = 1 \ldots, N_J$, we denote $\mathcal{I}_j^{in}$, $\mathcal{I}_j^{out} \subset \{1 \ldots, N_R\}$ the sets of indices corresponding to avenues incoming and outgoing in that junction, respectively.

2.1. The traffic model

To model the traffic flow in the road network we are going to consider the classical LRW model, coupled with a simple queue model. We denote by $\rho_i(s,t) \in [0, \rho_i^{max}]$ the density of cars in the avenue A_i, measured in *number of cars/km* (with ρ_i^{max} standing for the maximum allowed value). We suppose that we know functions $f_i : [0, \rho_i^{max}] \to \mathbb{R}$ giving the flow rate [*number of cars/h*] on the avenue A_i in terms of the density ($f_i(\rho_i) = \rho_i v_i$, with v_i [*km/h*] the velocity on the avenue A_i). Function f_i is called the static relation on A_i, and it verifies the following properties (see Figure 2):

1. $f_i : [0, \rho_i^{max}] \to \mathbb{R}$ is Lipschitz continuous and concave.
2. $f_i(0) = f_i(\rho_i^{max}) = 0$.
3. There exists a unique value $\rho_{C_i} \in (0, \rho_i^{max})$ (denoted *critical density*) such that f_i is strictly increasing in $[0, \rho_{C_i})$ and strictly decreasing in $(\rho_{C_i}, \rho_i^{max}]$ (the value $C_i = f_i(\rho_{C_i})$ is usually known as *road capacity*).

For each $y \in \mathcal{I}^{in}$, we denote by $q_y(t) \geq 0$ the queue length (measured in *number of cars*) downstream the avenue A_y, and we suppose that the downstream road capacity C_y^{in} and the desired inflow rate $f_y^{in}(t)$ are known. Finally, we also assume that maximum outflow rates $f_z^{out}(t)$, $z \in \mathcal{I}^{out}$, are

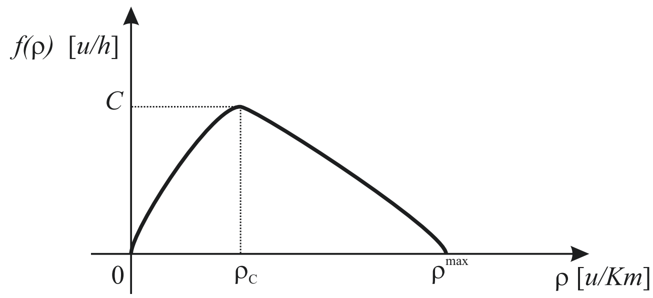

Figure 2: Standard static relation giving flow rate $f(\rho) = \rho v$ as a function of density ρ.

given. Then, the traffic flow in the road network is modelled by the following system: for $i = 1, \ldots, N_R$, $y \in \mathcal{I}^{in}$, $z \in \mathcal{I}^{out}$, $j = 1, \ldots, N_J$, $k \in \mathcal{I}_j^{in}$, and $l \in \mathcal{I}_j^{out}$:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial f_i(\rho_i)}{\partial s} = 0 \quad \text{in } (a_i, b_i) \times (0, T), \tag{2a}$$

$$\rho_i(., 0) = \rho_i^0 \quad \text{in } [a_i, b_i], \tag{2b}$$

$$f_k(\rho_k(b_k, .)) = \sum_{l \in \mathcal{I}_j^{out}} \min\left\{\alpha_{lk}^j D_k(\rho_k(b_k, .)),\, \beta_{kl}^j S_l(\rho_l(a_l, .))\right\} \quad \text{in } (0, T), \tag{2c}$$

$$f_l(\rho_l(a_l, .)) = \sum_{k \in \mathcal{I}_j^{in}} \min\left\{\alpha_{lk}^j D_k(\rho_k(b_k, .)),\, \beta_{kl}^j S_l(\rho_l(a_l, .))\right\} \quad \text{in } (0, T), \tag{2d}$$

$$f_z(\rho_z(b_z, .)) = \min\{f_z^{out}, D_z(\rho_z(b_z, .))\} \quad \text{in } (0, T), \tag{2e}$$

$$f_y(\rho_y(a_y, .)) = \min\{D_y^{in}(q_y, .), S_y(\rho_y(a_y, .))\} \quad \text{in } (0, T), \tag{2f}$$

$$\left.\begin{array}{l} \dfrac{dq_y}{dt} = f_y^{in} - f_y(\rho_y(a_y, .)) \quad \text{in } (0, T), \\ q_y(0) = q_y^0, \end{array}\right\} \tag{2g}$$

where:

- function ρ_i^0 represents the initial density at road A_i.
- terms D_i and S_i denote, respectively, the demand and supply functions: $D_i,\, S_i : [0, \rho_i^{max}] \longrightarrow \mathbb{R}$ given by

$$D_i(\rho) = \left\{ \begin{array}{ll} f_i(\rho) & \text{if } 0 \leq \rho \leq \rho_{C_i}, \\ C_i & \text{if } \rho_{C_i} \leq \rho \leq \rho_i^{max}, \end{array} \right. \tag{3}$$

$$S_i(\rho) = \begin{cases} C_i & \text{if } 0 \leq \rho_i \leq \rho_{C_i}, \\ f_i(\rho) & \text{if } \rho_{C_i} \leq \rho \leq \rho_i^{max}. \end{cases} \tag{4}$$

- parameters α_{lk}^j represent the preferences of drivers arriving to a junction, that is, α_{lk}^j gives the percentage of drivers that, arriving to junction j from the incoming avenue A_k, are going to take the outgoing avenue A_l. Consequently, the following constraints should be verified:

$$0 \leq \alpha_{lk}^j \leq 1 \text{ and } \sum_{l \in \mathcal{I}_j^{out}} \alpha_{lk}^j = 1. \tag{5}$$

- parameters β_{kl}^j represent the ingoing capacities in outgoing avenues, that is, β_{kl}^j gives the percentage of vehicles that, at a junction j and coming from the avenue A_k, can enter the outgoing avenue A_l. In a similar way to previous case, these parameters should verify:

$$0 \leq \beta_{kl}^j \leq 1 \text{ and } \sum_{k \in \mathcal{I}_j^{in}} \beta_{kl}^j = 1. \tag{6}$$

- term $D_y^{in}(q_y, t)$ represents the demand of queue q_y at time t, and it is given by

$$D_y^{in}(q_y, t) = \begin{cases} \min\{f_y^{in}(t), C_y^{in}\} & \text{if } q_y = 0, \\ C_y^{in} & \text{if } q_y > 0. \end{cases} \tag{7}$$

- value $q_y^0 \geq 0$ represents the initial queue length downstream avenue A_y.

Remark 1. *Equations (2c) and (2d) represent coupling conditions, and they guarantee the conservation of cars in junctions. In fact, if we sum (2c) on incoming roads and (2d) on outgoing ones, for each intersection $j = 1, \ldots, N_J$, we obtain the classical Rankine-Hugoniot relations (see [6, 8]):*

$$\sum_{k \in \mathcal{I}_j^{in}} f_k(\rho_k(b_k, .)) = \sum_{l \in \mathcal{I}_j^{out}} f_l(\rho_l(a_l, .)) \quad \text{in } (0, T).$$

These equations indicate that, at each intersection j, the cars that enter are only those wishing to do so (intention indicated by the value of α_{lk}^j) and having permission to do it (consent indicated by the value of β_{lk}^j).

Other types of faster intersections can be considered, relaxing this condition and admitting more cars than they have permission, as long as there *is place in the corresponding avenue (for instance, when cars coming from other avenues do not make full use of their quota). Under such hypothesis, in the equations (2c) and (2d), the term $\beta^j_{kl} S_l(\rho_l(a_l,.))$ must be replaced by*

$$\max\left\{\beta^j_{kl} S_l(\rho_l(a_l,.)),\ S_l(\rho_l(a_l,.)) - \sum_{\tilde{k}\in\mathcal{I}^{in}_j,\,\tilde{k}\neq k} \alpha^j_{l\tilde{k}} D_{\tilde{k}}(\rho_{\tilde{k}}(b_{\tilde{k}},.))\right\} \tag{8}$$

These conditions are those required, for instance, in [11] for simpler networks in which all the intersections present one entering avenue and two outgoing ones (off-ramps), or two entering and one outgoing avenues (on-ramps).

2.2. The pollution model

To simulate air pollution due to vehicular traffic, we consider a mathematical model similar to the one proposed in [3]. We denote by $\phi(x,t)$ $[kg/km^2]$ the carbon monoxide (CO) concentration at point $x \in \Omega$ and at time $t \in [0,T]$, and we obtain it by the following initial/boundary value problem:

$$\frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi - \nabla\cdot(\mu\nabla\phi) + \kappa\phi = \sum_{i=1}^{N_R} \xi_{A_i} \quad \text{in } \Omega\times(0,T), \tag{9a}$$

$$\phi(.,0) = \phi^0 \quad \text{in } \Omega, \tag{9b}$$

$$\mu\frac{\partial \phi}{\partial n} - \phi\,\mathbf{v}\cdot\mathbf{n} = \sum_{y\in\mathcal{I}^{in}} \lambda_y q_y \delta_{\sigma_y(a_y)} \quad \text{on } S^-, \tag{9c}$$

$$\mu\frac{\partial \phi}{\partial n} = 0 \quad \text{on } S^+, \tag{9d}$$

where $\mathbf{v}(x,t)$ $[km/h]$ represents the wind velocity field, $\mu(x,t)$ $[km^2/h]$ is the CO molecular diffusion coefficient, $\kappa(x,t)$ $[h^{-1}]$ is the CO extinction rate corresponding to the (first order) reaction term, ϕ^0 is a known function giving the initial CO concentration, $\mathbf{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega = S^- \cup S^+$, where $S^- = \{(x,t) \in \partial\Omega \times (0,T)$ such that $\mathbf{v}\cdot\mathbf{n} < 0\}$ represents the inflow boundary, and $S^+ = \{(x,t) \in \partial\Omega\times(0,T)$ such that $\mathbf{v}\cdot\mathbf{n} \geq 0\}$ represents the outflow boundary. Finally:

- terms ξ_{A_i} $[kg/km^2/h]$ stand for the sources of pollution due to vehicular traffic on the avenues A_i, and correspond to the following Radon measures: for each $t \in [0,T]$, $\xi_{A_i}(t) : \mathcal{C}(\overline{\Omega}) \longrightarrow \mathbb{R}$ is the distribution given, for all $v \in \mathcal{C}(\overline{\Omega})$, by:

$$\langle \xi_{A_i}(t), v \rangle = \int_{a_i}^{b_i} \left(\gamma_i f_i(\rho_i(s,t)) + \eta_i \rho_i(s,t)\right) v(\sigma_i(s)) \, \|\sigma_i'(s)\| \, ds,$$

where σ_i is the parametrization of avenue A_i, ρ_i is given by the traffic flow model (2), and γ_i and η_i are weight parameters representing contamination rates.

- terms $\lambda_y q_y \delta_{\sigma_y(a_y)}$ correspond to the sources of pollution due to queues downstream roads entering by the inflow boundary, where $\delta_{\sigma_y(a_y)}$ represents the Dirac measure at point $\sigma_y(a_y) \in S^- \subset \partial\Omega$, and λ_y is a contaminant rate parameter.

3. Two optimal control problems

When managing traffic in a road network, classic objectives are related to traffic problems, such as travel time, congestion and so on. However, recent problems of atmospheric pollution around big cities have made that mitigation of this problem also becomes another major objective in the optimal management of road networks.

In this paper, two different objectives will be simultaneously considered, one of each type. Regarding the optimization of traffic flow, following [11], it is intended that the total travel time is minimized and the outflow of the system is maximized, i.e., the following functional is minimized:

$$J_T = \int_0^T \left(\sum_{y \in \mathcal{I}^{in}} \epsilon_y^q q_y(t) + \sum_{i=1}^{N_R} \epsilon_i \int_{a_i}^{b_i} \rho_i(x,t) dx - \sum_{z \in \mathcal{I}^{out}} \epsilon_z^{out} f_z(\rho_z(b_z,t)) \right) dt, \quad (10)$$

where ϵ_y^q, ϵ_i, $\epsilon_z^{out} \geq 0$ are weight parameters.

With regard to contamination, it is sought that the mean CO concentration is as low as possible, that is, we are interested in minimizing the functional

$$J_P = \frac{1}{T\,|\Omega|} \int_0^T \int_\Omega \phi(x,t) \, dx \, dt, \quad (11)$$

where $|\Omega|$ denotes the Euclidean measure of set Ω.

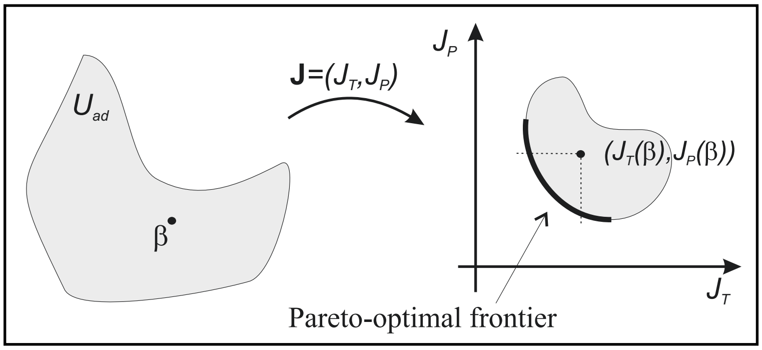

Figure 3: Geometrical interpretation of the Pareto-optimal frontier.

Finally, for the design variables that can be managed within the network (controls), there exist many possibilities: incoming fluxes [11], driver preferences [13], network expansions [3], etc. In this work, we will focus exclusively on the optimal management of the intersections, trying to look for the values of the parameters that are the most suited to our objectives. So, we propose two different approaches below.

3.1. A multi-objective cooperative problem

Let us suppose first of all that the preferences of the drivers (parameters α^j_{lk}) are known, and that they do not vary even if the entry/exit quotas are modified at the intersections (parameters β^j_{kl}). This may happen, for instance, if we admit that drivers make their route choices based on distances and not on travel times. In this case, the control of the problem will be the vector

$$\beta = (\beta^j_{kl}), \quad j = 1, \ldots, N_J,\ k \in \mathcal{I}^{in}_j,\ l \in \mathcal{I}^{out}_j,$$

and, admitting that there is a unique organization managing the entire network, we will try to solve the multi-objective problem:

$$\begin{array}{l} \min \mathbf{J}(\beta) = (J_T(\beta), J_P(\beta)) \\ \text{subject to (6)} \end{array} \qquad (12)$$

from a cooperative point of view, that is, looking for the Pareto-optimal solutions as defined below (see also Figure 3):

Definition 1. *Let us consider the admissible set* $U_{ad} = \{\beta = (\beta^j_{kl})$ *satisfying*

(6)}. We say that $\beta^* \in U_{ad}$ is a Pareto-optimal solution of problem (12), if there does not exist any $\beta \in U_{ad}$ such that

1. *$J_T(\beta) \leq J_T(\beta^*)$ and $J_P(\beta) \leq J_P(\beta^*)$.*
2. *$J_T(\beta) < J_T(\beta^*)$ or $J_P(\beta) < J_P(\beta^*)$.*

If β^* is a Pareto-optimal solution, the corresponding objective vector $\mathbf{J}(\beta) \in \mathbb{R}^2$ is also called Pareto-optimal. The set of Pareto-optimal solutions is called Pareto-optimal set, and the set of Pareto-optimal objective vectors is known as Pareto-optimal frontier.

3.2. A bi-level (non-cooperative) problem

Let us suppose now that the preferences of the drivers (parameters α^j_{lk}) vary if the input/output quotas at the intersections are modified (parameters β^j_{kl}). Let us also assume that these drivers' preferences always try to minimize the functional J_T, while the selection of the quotas (made by the organization managing the network) is done in order to try to minimize air pollution. In this case, we deal with a bi-level problem, where the follower problem reads: for a given β verifying (6), solve:

$$\begin{array}{l} \min J_T(\alpha, \beta) \\ \text{subject to (5)} \end{array} \tag{13}$$

with

$$\alpha = (\alpha^j_{lk}), \quad j = 1, \ldots, N_J,\ k \in \mathcal{I}^{in}_j,\ l \in \mathcal{I}^{out}_j.$$

The leader problem is then written as:

$$\begin{array}{l} \min J_P(\alpha_\beta, \beta) \\ \text{subject to (6)} \end{array} \tag{14}$$

where α_β is the solution of the follower problem (13) corresponding to β.

In this case, the objective is related to obtaining a Stackelberg strategy for the bi-level problem (13)-(14), as given in the following definition:

Definition 2. *We say that (α^*, β^*) is a Stackelberg strategy, solution of the bi-level problem (13)-(14), if and only if:*

1. *α^* is the best response of the follower to the leader choice β^*, that is, α^* is the solution of the problem (13) for β^*, or, equivalently, $\alpha^* = \alpha_{\beta^*}$.*
2. *β^* is the best choice of the leader, that is, β^* is the solution of the problem (14).*

4. Numerical resolution

In general, both the multi-objective problem (12) and the bi-level problem (13)-(14) are non-convex and therefore many local solutions are expected. So, in this work we are going to use derivative free global methods. The only difficulty to apply this type of methods lies in having effective numerical algorithms that allow evaluating the objective functions J_T and J_P in an efficient way. We devote this section to detail the numerical algorithms used for the evaluation of each one of these functionals.

4.1. Numerical computation of traffic objective

The value of the functional J_T given by (10) can be obtained through numerical integration, once known the functions ρ_i and q_y, solutions of the system (2). To solve this system, we propose an explicit numerical method, obtained by combining a classical first order numerical method for solving (2a)-(2f), which computes flows between cells by the supply-demand method, with the forward Euler scheme for solving (2g). To introduce this method, for each road A_i, the spatial domain $I_i = [a_i, b_i]$ is divided into M_i cells $I_{i,h} = [s_{i,h-\frac{1}{2}}, s_{i,h+\frac{1}{2}}],\ h = 1, \ldots, M_i$, of length $\Delta s_i > 0$, where the midpoint of each cell is denoted by $s_{i,h} = (s_{i,h-\frac{1}{2}} + s_{i,h+\frac{1}{2}})/2$. The time interval $[0, T]$ is also divided into $N \in \mathbb{N}$ subintervals of length $\Delta t = T/N$, and the discrete times $t^n = n\Delta t,\ n = 0, \ldots, N$, are defined. Then, the approximate solution of system (2) at mesh points, values $\rho^n_{i,h} \approx \rho_i(s_{i,h}, t^n)$ and $q^n_y \approx q_y(t^n)$, are computed as described in the following algorithm:

Algorithm 1. *(Numerical resolution of the traffic system)*

- **Step 0.** `Input:` *geometric data of the network, necessary parameters for discretization, and vectors* α *and* β *determining coupling conditions at junctions.*

- **Step 1.** `For` $i = 1, \ldots, N_R$ `take` q^0_i *(initial condition) and* `compute`

$$\rho^0_{i,h} = \rho^0_i(s_{i,h}), \quad h = 1, \ldots, M_i.$$

- **Step 2.** `For` $n = 0, \ldots, N-1$ *and* `for` $i = 1, \ldots, N_R$

  – *Step 2.1 (Flow computation between cells at time* t^n*).*

`Compute`

$$f^n_{i,h+\frac{1}{2}} = \min\left\{D_i(\rho^n_{i,h}), S_i(\rho^n_{i,h+1})\right\}, \quad h = 1,\ldots,M_i - 1.$$

– *Step 2.2 (Computation of downstream flow at time t^n).*

`If` $i \in \mathcal{I}^{out}_j$ *for any* $j = 1,\ldots,N_J$, `compute`

$$f^n_{i,\frac{1}{2}} = \sum_{k\in\mathcal{I}^{in}_j} \min\left\{\alpha^j_{ik} D_k(\rho^n_{k,M_k}),\ \beta^j_{ki} S_i(\rho^n_{i,1})\right\},$$

`else` *(if $i \in \mathcal{I}^{in}$)*, `compute`

$$f^n_{i,\frac{1}{2}} = \min\{D^{in}_i(q^n_i), S_i(\rho^n_{i,1})\}.$$

– *Step 2.3 (Computation of upstream flow at time t^n).*

`If` $i \in \mathcal{I}^{in}_j$ *for any* $j = 1,\ldots,N_J$, `compute`

$$f^n_{i,M_i+\frac{1}{2}} = \sum_{l\in\mathcal{I}^{out}_j} \min\left\{\alpha^j_{li} D_i(\rho^n_{i,M_i}),\ \beta^j_{il} S_l(\rho^n_{l,1})\right\},$$

`else` *(if $i \in \mathcal{I}^{out}$)*, `compute`

$$f^n_{i,M_i+\frac{1}{2}} = \min\{f^{out}_i(t^n), D_i(\rho^n_{i,M_i})\}.$$

– *Step 2.4 (Computation of solution at time t^{n+1}).*

`Compute`

$$\rho^{n+1}_{i,h} = \rho^n_{i,h} - \frac{\Delta t}{\Delta s_i}\left(f^n_{i,h+\frac{1}{2}} - f^n_{i,h-\frac{1}{2}}\right), \quad h = 1,\ldots,M_i,$$

`If` $i \in \mathcal{I}^{in}$, `compute`

$$q^{n+1}_i = q^n_i + \Delta t\left(f^{in}_i(t^n) - f^n_{i,\frac{1}{2}}\right).$$

As above commented, once the system (2) has been solved, the computation of J_T is done by means of numerical integration, as shown in the following algorithm:

Algorithm 2. *(Numerical computation of J_T)*

- **Step 0** *(Input data).*

  `Input:` *necessary parameters for discretization and vectors α and β determining coupling conditions at junctions.*

- **Step 1** *(Numerical resolution of traffic system).*

  `Compute`*, by using Algorithm 1,*

$$\rho_{i,h}^n \text{ and } q_i^{n+1}, \text{ for } \quad n = 0, \ldots, N-1, \quad \begin{array}{l} i = 1, \ldots, N_R, \\ h = 1, \ldots, M_i. \end{array}$$

- **Step 2** *(Numerical integration).*

  `Compute`

$$J_T^\Delta = \Delta t \sum_{n=1}^{N} \left(\sum_{y \in \mathcal{I}^{in}} \epsilon_y^q q_y^n + \sum_{i=1}^{N_R} \epsilon_i \Delta s_i \sum_{h=1}^{M_i} \rho_{i,h}^n - \sum_{z \in \mathcal{I}^{out}} \epsilon_z^{out} f_z(\rho_{z,M_z}^n) \right).$$

4.2. Numerical evaluation of the pollution objective

To evaluate the pollution objective J_P, adjoint techniques can be very advantageous in order to simplify the numerical computations. By introducing the following adjoint system:

$$-\frac{\partial g}{\partial t} - \mathbf{v} \cdot \nabla g - \nabla \cdot (\mu \nabla g) + \kappa g = \frac{1}{T\,|\Omega|} \quad \text{in } \Omega \times (0,T), \tag{15a}$$

$$g(x,T) = 0 \quad \text{in } \Omega, \tag{15b}$$

$$\mu \frac{\partial g}{\partial n} = 0 \quad \text{on } S^-, \tag{15c}$$

$$\mu \frac{\partial g}{\partial n} + g\, \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } S^+, \tag{15d}$$

we can prove (see [3]) that the cost functional J_P, related to minimizing CO pollution, admits the following alternative equivalent expression:

$$J_P = \sum_{i=1}^{N_R} \int_0^T \int_{a_i}^{b_i} \left(\gamma_i f_i(\rho_i(s,t)) + \eta_i \rho_i(s,t)\right) \, g(\sigma_i(s),t) \, \|\sigma_i'(s)\| \, ds \, dt$$
$$+ \sum_{y \in \mathcal{I}^{in}} \int_0^T \lambda_y q_y(t) g(\sigma_y(a_y),t) \chi_{S^-}(\sigma_y(a_y),t) \, dt + \int_\Omega \phi^0(x) \, g(x,0) \, dx, \quad (16)$$

where χ_{S^-} denotes the characteristic function of the set $S^- \subset \partial\Omega \times (0,T)$. This expression is very useful, since adjoint state g is fully independent of any traffic variable and, consequently, it only needs to be computed once in a previous initial step. Then, once obtained g, for each numerical computation of J_P we only have to solve the traffic system (2) and apply numerical integration in (16).

To obtain g, we propose to solve the adjoint system (15) by a numerical method which combines characteristics for the time discretization with Lagrange P_1 finite elements for the space discretization (see [9]). For self-containedness, this numerical method is completely detailed in the following algorithm:

Algorithm 3. *(Numerical resolution of the adjoint system)*

- **Step 0** *(Initial data).*

  `Input:` *a polygonal approximation Ω_h of Ω, and an admissible triangulation τ_h of it, with vertices $\{x_j, \ j = 1, ..., N_v\}$ satisfying that vertices on the boundary $\partial\Omega_h$ also lie in the boundary $\partial\Omega$, $\sigma_y(a_y)$, $\sigma_z(b_z) \in \partial\Omega_h$, for all $y \in \mathcal{I}^{in}$, $z \in \mathcal{I}^{out}$, and that, for $n = 0, \ldots, N-1$, each edge of $\partial\Omega_h$ is contained in $(S_h^n)^- = \{x \in \partial\Omega_h \ \text{ such that } \ \mathbf{v} \cdot \mathbf{n} \leq 0\}$ or in $(S_h^n)^+ = \{x \in \partial\Omega_h \ \text{ such that } \ \mathbf{v} \cdot \mathbf{n} \geq 0\}$.*

  `Take:` *$\mathcal{B}_{V_h} = \{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{N_v}\}$ the nodal basis of the finite element space $V_h = \{v_h \in C(\bar{\Omega}_h) \ \text{ such that } \ v_h|_\mathcal{T} \in P_1, \forall \mathcal{T} \in \tau_h\}$, that is, the set of N_v functions in V_h satisfying that $\tilde{v}_i(x_j) = \delta_j^i$.*

  `Take` $\mathbf{g}_h^N = \mathbf{0} \in \mathbb{R}^{N_v}$.

- **Step 1** *(Computation of time-independent terms).*

`Compute`, *by using the vertex quadrature rule, the* $N_v \times N_v$ *matrix* A_h, *and the vector* $b_h \in \mathbb{R}^{N_v}$, *given by*

$$(A_h)_{ij} = \left(\frac{1}{\Delta t} + \kappa\right) \int_{\Omega_h} \tilde{v}_i \tilde{v}_j \, dx + \int_{\Omega_h} \mu \nabla \tilde{v}_i \nabla \tilde{v}_j \, dx,$$

$$(b_h)_i = \frac{1}{T\,|\Omega|} \int_{\Omega_h} \tilde{v}_i \, dx.$$

- **Step 2.** `For` $n = N-1, \ldots, 0$
  - *Step 2.1 (Computation of characteristics terms).*

    `For` $j = 1, \ldots, N_v$ `compute` *the value of* $X_j^{n+1} = X^{n+1}(x_j)$, *where* $X^{n+1}(x) \approx X(x, t^n; t^{n+1})$ *gives the position at instant* t^{n+1} *of particle that is at point* x *at instant* t^n. X_j^{n+1} *is computed by solving the initial value problem:*

    $$\begin{cases} \dfrac{dX}{d\tau} = \mathbf{v}\left(X(x_j, t^n; \tau), \tau\right), \\ X(x_j, t^n; t^n) = x_j, \end{cases}$$

    with the backward Euler scheme.
  - *Step 2.2 (Computation of time-dependent matrices).*

    `Compute`, *by using the vertex quadrature rule, the* $N_v \times N_v$ *matrices*

    $$(A_h^n)_{ij} = \int_{(S_h^n)^+} \tilde{v}_i \tilde{v}_j \, \mathbf{v} \cdot \mathbf{n} \, d\Gamma,$$

    $$(B_h^{n+1})_{ij} = \frac{1}{\Delta t} \int_{\Omega_h} (\tilde{v}_j \circ X^{n+1}) \tilde{v}_i \, dx.$$
  - *Step 2.3 (Solution of the linear system).*

    `Compute` *the vector* $\mathbf{g}_h^n \in \mathbb{R}^{N_v}$, *giving the approximations* $(g_h^n)_j \approx g(x_j, t^n)$, *by solving the linear system*

    $$(A_h + A_h^n)\, \mathbf{g}_h^n = b_h + B_h^{n+1} \mathbf{g}_h^{n+1}.$$

Finally, the numerical computation of pollution objective J_P is detailed in the following algorithm:

Algorithm 4. *(Numerical computation of J_P)*

- **Step 0** *(Input data).*

  `Input:` *a polygonal approximation Ω_h of Ω, a triangulation τ_h of Ω_h (with vertices $\{x_j,\ j = 1,...,N_v\}$), and the vector $\mathbf{g}_h^n \in \mathbb{R}^{N_v}$, obtained by Algorithm 3, giving the approximations $(g_h^n)_j \approx g(x_j, t^n)$.*

- **Step 1** *(Numerical resolution of traffic system).*

  `Compute`*, by using Algorithm 1,*

$$\rho_{i,h}^n \textit{ and } q_y^n, \textit{ for } \begin{array}{ll} n = 1, \ldots, N, & i = 1, \ldots, N_R, \\ h = 1, \ldots, M_i, & y \in \mathcal{I}^{in}. \end{array}$$

- **Step 2** *(Numerical integration).*

  `Compute`

$$J_P^\Delta = \Delta t \sum_{n=1}^{N} \sum_{i=1}^{N_R} \sum_{h=1}^{M_i} \Delta s_i \left(\gamma_i f_i(\rho_{i,h}^n) + \eta_i \rho_{i,h}^n\right) \, g_h^n(\sigma_i(s_{i,h})) \, \|\sigma_i'(s_{i,h})\|$$
$$+ \sum_{n=1}^{N} \sum_{\substack{y \in \mathcal{I}^{in} \\ \sigma_y(a_y) \in (S_h^n)^-}} \lambda_y q_y^n g_h^n(\sigma_y(s_{y,1})) + \frac{1}{3} \sum_{\mathcal{T} \in \tau_h} |\mathcal{T}| \sum_{x_j \in \mathcal{T}} \Phi^0(x_j) g_j^0,$$

  where $g_h^n \in V_h$ is given by $g_h^n(x) = \sum_{j=1}^{N_v} g_j^n \tilde{v}_j(x)$.

5. Numerical results

In this section we present and discuss some numerical results obtained when applying our proposed methodology to a real-world case posed in the Guadalajara Metropolitan Area (GMA), in Mexico, for the case of the multi-objective cooperative problem (12). To continue with the experiments developed in [2] and [3], the domain Ω shown in Figure 4 has been taken, and a main road network given by $N_R = 15$ avenues with $N_J = 9$ junctions has been studied. A simulation for a time interval of $T = 24\,h$ has been carried

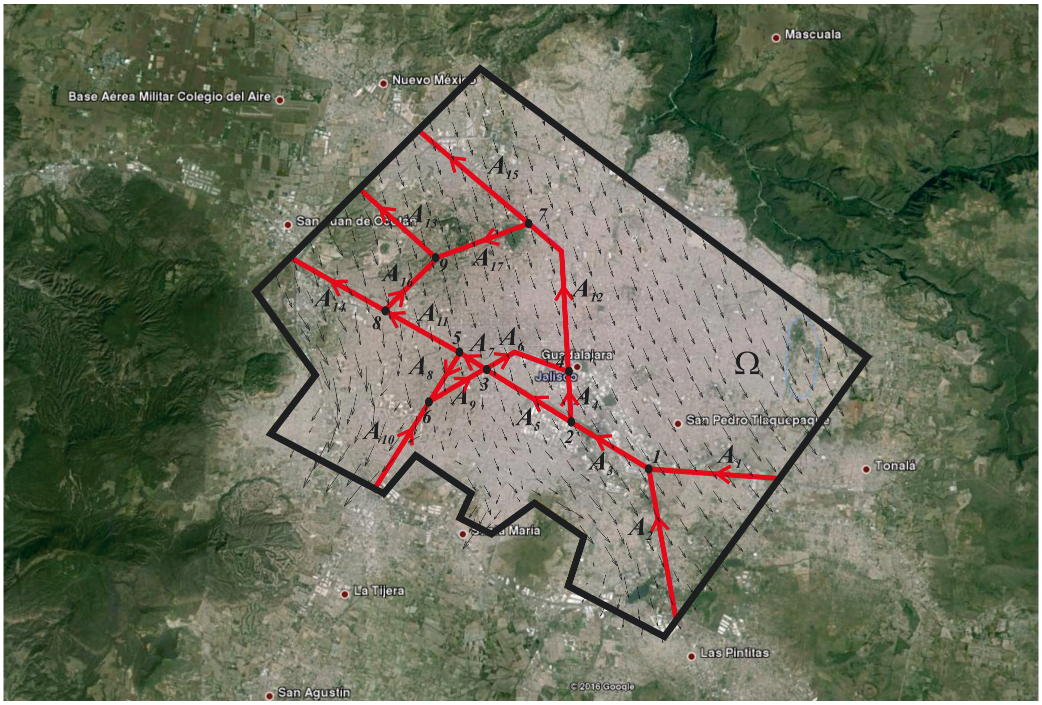

Figure 4: Satellite photo of the Guadalajara Metropolitan Area (Mexico). The domain Ω considered for pollution simulation is drawn in black, the road network is sketched in red, and vectors show the wind velocity field corresponding to the numerical experiments.

out, assuming that at initial time $t = 0$ there are neither queues ($q_y^0 = 0$), nor cars in the network ($\rho_i^0 = 0$), nor pollution in the domain Ω ($\phi^0 = 0$).

In relation to the traffic model, it has been assumed that all the avenues of the network present the same characteristics, so that the theoretical flow is equal for all of them, and it is given by the static relation:

$$f(\rho) = \begin{cases} 50\,\rho & \text{if } \; 0 \leq \rho \leq \rho_C = \dfrac{6000}{149}, \\ \dfrac{10^5}{33}\left(1 - \dfrac{\rho}{120}\right) & \text{if } \; \rho_C < \rho \leq \rho^{max} = 120. \end{cases}$$

Concerning boundary conditions, we have taken the same downstream road capacity for the three incoming roads ($C_1^{in} = C_2^{in} = C_{10}^{in} = 2.013\,10^3$), the same desired inflow rate (corresponding with a typical weekday, with two peak hours), and also equal maximum outflow rates for the three outgoing

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$\mathcal{I}_j^{in}$	$\{1,2\}$	$\{3\}$	$\{5,9\}$	$\{4,6\}$	$\{7\}$	$\{8,10\}$	$\{12\}$	$\{11\}$	$\{16,17\}$
$\mathcal{I}_j^{out}$	$\{3\}$	$\{4,5\}$	$\{6,7\}$	$\{12\}$	$\{8,11\}$	$\{9\}$	$\{15,17\}$	$\{14,16\}$	$\{13\}$
α_{lk}^j	$(1\ 1)$	$\begin{pmatrix}0.4\\0.6\end{pmatrix}$	$\begin{pmatrix}0.4 & 0.4\\0.6 & 0.6\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.7\\0.3\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.7\\0.3\end{pmatrix}$	$\begin{pmatrix}0.7\\0.3\end{pmatrix}$	$(1\ 1)$
$(\beta_T)_{kl}^j$	$\begin{pmatrix}0.46\\0.54\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.56 & 0.76\\0.44 & 0.24\end{pmatrix}$	$\begin{pmatrix}0.41\\0.59\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.46\\0.54\end{pmatrix}$	$(1\ 1)$	$(1\ 1)$	$\begin{pmatrix}0.39\\0.71\end{pmatrix}$
$(\beta_P)_{kl}^j$	$\begin{pmatrix}0.00\\1.00\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}1.00 & 1.00\\0.00 & 0.00\end{pmatrix}$	$\begin{pmatrix}0.59\\0.41\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.27\\0.73\end{pmatrix}$	$(1\ 1)$	$(1\ 1)$	$\begin{pmatrix}0.60\\0.40\end{pmatrix}$
$(\beta_C)_{kl}^j$	$\begin{pmatrix}0.41\\0.59\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.99 & 0.99\\0.01 & 0.01\end{pmatrix}$	$\begin{pmatrix}0.52\\0.48\end{pmatrix}$	$(1\ 1)$	$\begin{pmatrix}0.27\\0.73\end{pmatrix}$	$(1\ 1)$	$(1\ 1)$	$\begin{pmatrix}0.67\\0.33\end{pmatrix}$

Table 1: Values of fixed parameters α_{lk}^j (showing drivers' preferences when arriving at each junction $j = 1, \ldots, 9$), and three Pareto-optimal solutions obtained for the problem (12): the best from the point of view of the traffic (β_T), the best from the pollution viewpoint (β_P), and a compromise solution chosen as a balance between these two previous extreme situations (β_C), whose graphic representations are shown in Figure 5.

roads ($f_z^{13} = f_z^{14} = f_z^{15} = 2.013\,10^3$). Finally, we have considered for weight parameters the values $\epsilon_i = 0.5$, $\epsilon_y^q = 0.1$ and $\epsilon_z^{out} = 0.5$.

With respect to the contamination model, a characteristic wind of the zone (see Figure 4) has been considered, and typical values for CO in molecular diffusion ($\mu = 3.5\,10^{-8}\,km^2/h$), extinction rate ($\kappa = 0.6\,10^{-2}\,h^{-1}$), and emissions ($\gamma_i = 10^{-6} kg/\textit{number of cars}/km$ and $\eta_i = 3.16\,10^{-5} kg/\textit{number of cars}/h$) have been taken. Finally, regarding the sources of pollution due to entrance queues, they have not been taken into account since none of the incoming avenues are located on the inflow boundary S^-.

In the results shown below, it is assumed that the parameters α_{lk}^j (which give the preferences of the drivers when arriving at an intersection) are known (fixed) as given in Table 1. Consequently, the main objective here is to find the Pareto-optimal frontier of the problem (12), for which an elitist genetic algorithm has been used (a variant of the multi-objective non-dominated sorting-based evolutionary algorithm NSGA-II [7], included in the Global Optimization Toolbox of MATLAB R2017a).

The Pareto-optimal frontier obtained can be seen in Figure 5, in which three remarkable points have been highlighted, corresponding to three solutions chosen for its analysis (see Table 1): β_T, the best solution from the viewpoint of traffic flows and travel times; β_P, the best solution for pollution minimization; and β_C, an intermediate compromise solution chosen between both.

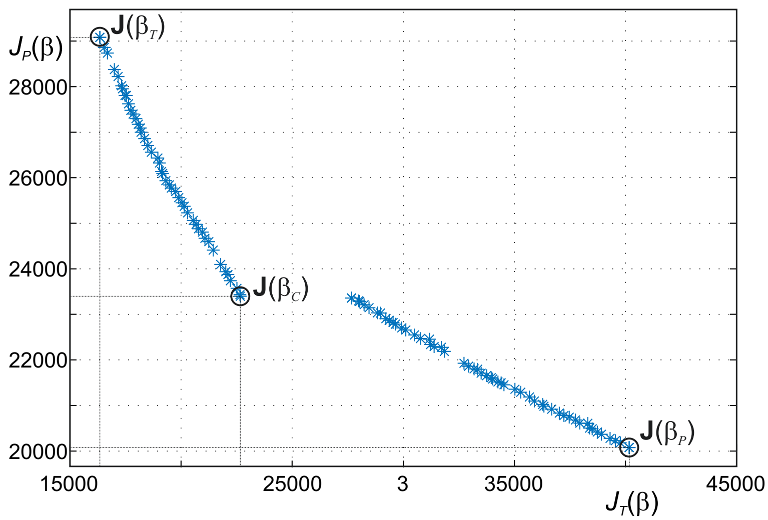

Figure 5: Pareto-optimal frontier obtained for the problem (12), where three particular solutions are emphasized: the optimal solution for the optimization of traffic flow (β_T), the optimal solution for the minimization of pollution (β_P), and a compromise solution corresponding to an intermediate balance choice (β_C).

Figure 6 shows the pollution levels (contour lines at the end of the simulation period $T = 24\,h$), corresponding to these three solutions, where we can observe as CO concentration increases slightly from β_P to β_C, but significantly from β_C to β_T. In addition, Figure 7 compares, for these three Pareto-optimal solutions, the mean values (along the whole simulation time interval) of the main variables for the traffic model: car densities and flow rates throughout the whole network, queue lengths at entrance roads, and outflow rates at exit avenues.

It can be noticed that, as expected, the best solution for traffic flow optimization results in a greater flow rate in the network and at the outgoing avenues, and a smaller number of vehicles in the queues for the incoming roads. On the contrary, the best solution from the environmental viewpoint causes lower CO emissions (since car density and flow rate are lower), but queue lengths increase in a remarkable way. Finally, the chosen compromise solution reaches an intermediate situation that, under satisfactory traffic conditions, reduces pollution to acceptable levels.

Numerical results for the case of the bi-level, non-cooperative problem (13)-(14) will be presented and analyzed by the authors in a forthcoming

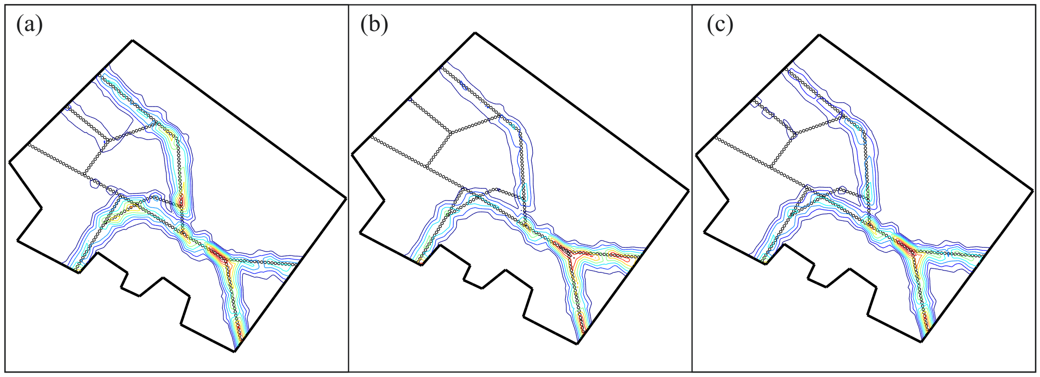

Figure 6: Pollution isolines for final time $T = 24\,h$, corresponding to the Pareto-optimal solutions given in Table 1: (a) optimal solution for traffic flow optimization β_T, (b) optimal solution for pollution minimization β_P, and (c) intermediate compromise solution β_C.

paper, where optimal solutions for Pareto and Stackelberg frameworks will be also compared.

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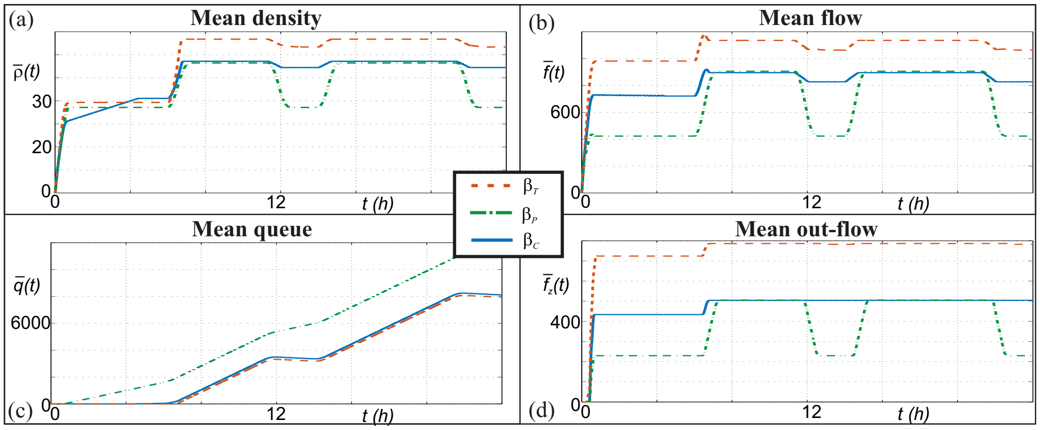

Figure 7: Mean values for the flow variables corresponding to the three Pareto-optimal solutions β_T, β_P and β_C given in Table 1: (a) car density throughout the whole network, (b) flow rate, (c) length of entrance queues, and (d) outflow rates for exit roads.

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