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Reset control systems: The zero-crossing resetting law[☆]

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ABSTRACT

A novel representation of reset control systems with a zero-crossing resetting law, in the framework of hybrid inclusions, is postulated. The well-posedness and stability issues of the resulting hybrid dynamical system are investigated, with a strong focus on how non-deterministic behavior is implemented in control practice. Several stability conditions have been developed by using the eigenstructure of matrices related to the periods of the reset interval sequences and by using Lyapunov function-based conditions.

Notation: $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is a column vector; $\|\mathbf{x}\|$ is the euclidean norm. For a matrix $A \in \mathbb{R}^n \times \mathbb{R}^m$, $\|A\|$ is its spectral norm. $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$ centered at the origin. $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$ is the unit n-sphere. The set of $n \times n$ (positive definite) symmetric matrices is denoted by $\mathbb{S}^n$. $\mathcal{S}_{\mathcal{H}}(\xi)$ is the set of maximal solutions ϕ to the hybrid system $\mathcal{H}$ with $\phi(0,0) = \xi$. dom stands for domain, and $\setminus$ denotes sets difference. $\mathcal{O}^n = \mathbb{R}^n \times \{1, -1\}$. $I_{n\times n}$ is the identity matrix of dimension $n \times n$, and $0_{n_1\times n_2}$ is the null matrix of dimension $n_1 \times n_2$. LTI stands for linear and time-invariant. n_ρ is the number of controller states to be reset at a jump.

1. Introduction

Informally speaking, a reset controller is any controller, usually referred to as the *base controller*, that is equipped with a mechanism for zeroing some of its states, when some event occurs in the control system. Although the term was coined in the late 90s by Hollot, Chait and coworkers [1], specifically to describe "an LTI system with mechanisms and laws to reset their states to zero", the concept was devised much earlier, in the seminal works of Clegg [2] and Horowitz and coworkers [3,4]. Since then, reset control has considerably evolved by using different resetting laws: the original zero-crossing of the error [1,5–9], sector-based resetting [10–14], error band [15,16], reset at fixed instants [17,18], Lyapunov function-based resetting [19], and somehow relaxing the original concept, both including nonlinear base systems and reset to non-zero values in some cases. This has lead to a fecund research area that has been successfully applied in many practical applications, and that has opened many relevant topics in control theory and practice.

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In this work, the focus is on reset control with emphasis on the original concept, using an LTI controller that zeroes its state (fully or partially) when the closed-loop error signal is zero. The main motivation has been to update and formalize previous work by the authors, developed in the framework of impulsive dynamical systems (IDS), by using the hybrid dynamical systems framework (HI) of [20,21]. Note that in the IDS framework, resetting laws are based on the exact crossing of the zero error hypersurface, and there is some fragility in detecting a zero-crossing, especially if measurement noise is present. Although this robustness problem has been alleviated for a specific class of exogenous signals [8], it is acknowledged the HI framework is more conclusive for equipping reset control systems with good structural properties (especially when considering exogenous signals with jump discontinuities) such as continuous-dependence on initial conditions, closeness of perturbed (due for example to measurement noise) and unperturbed solutions, asymptotic stability is preserved under small perturbations, etc.

There exist already several relevant works about reset control in the HI framework, most of them based on a sector-based resetting law (see for example [10–14]). It is important to emphasize that, in general, this resetting law produces different control solutions in comparison with the zero-crossing resetting law (see Example 3.4 of this manuscript). Here, it is not argued that one resetting law is superior to the others; they simply are different solutions that may properly work in control practice. Thus, the zero-crossing law (which has not been previously addressed in the HI framework) demands a specific and detailed study on formalization, basic hybrid properties and specific stability criteria.

1.1. Background: Hybrid dynamical systems

This work follows the hybrid system framework developed in [21] (and references therein), that following [22], has been referred to as the Hybrid Inclusions (HI) framework, and the reader is referred to [20,21] for a detailed exposition of it (see also [23] where hybrid systems with inputs are explicitly analyzed). A hybrid system $\mathcal{H}_{\mathbf{w}}$, with state $\mathbf{x} \in \mathbb{R}^n$ and input $\mathbf{w} \in \mathbb{R}^m$, is given by

$$\mathcal{H}_{\mathbf{w}} : \begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{w}), & (\mathbf{x}, \mathbf{w}) \in \mathcal{C}, \\ \mathbf{x}^+ = g(\mathbf{x}, \mathbf{w}), & (\mathbf{x}, \mathbf{w}) \in \mathcal{D}. \end{cases} \tag{1}$$

and is defined by the following data: (i) the *flow set* $\mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^m$, (ii) the *flow mapping* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, (iii) the *jump set* $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$, and (iv) the *jump mapping* $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Hybrid signals are defined as functions on hybrid time domains [21,24]. A hybrid arc $\mathbf{x}$: dom $\mathbf{x} \mapsto \mathbb{R}^n$ is a hybrid signal in which $\mathbf{x}(\cdot, j)$ is locally absolutely continuous for each j. A hybrid input $\mathbf{w}$: dom $\mathbf{w} \mapsto \mathbb{R}^m$ s a hybrid signal in which $\mathbf{w}(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A solution to (1) is defined as a pair $(\mathbf{x}, \mathbf{w})$, consisting of a hybrid arc and a hybrid input with dom $\mathbf{x}$ = dom $\mathbf{w}$, that satisfies the dynamics of the hybrid system $\mathcal{H}_{\mathbf{w}}$ (see [2–4,7,9,10,12,14,17–21,23–31] for details about solution pairs to hybrid systems with inputs). Note that the jump set $\mathcal{D}$ enables jumps but does not force them if there are points in which it is also possible to flow (a similar argument applies to the flow set $\mathcal{C}$); and thus if $\mathcal{C}$ are $\mathcal{D}$ are not disjoint then for a point $(\xi, \psi) \in \mathcal{C} \cap \mathcal{D}$ there may be several solutions pairs $(\mathbf{x}, \mathbf{w})$ to $\mathcal{H}_{\mathbf{w}}$ with $\mathbf{x}(0, 0) = \xi$, for any hybrid input $\mathbf{w}$ with $\mathbf{w}(0, 0) = \psi$.

For $\mathcal{H}_{\mathbf{w}}$, the so-called *hybrid basic conditions* defined in [21] (see also regularity conditions in [23]) are satisfied if $\mathcal{C}$ and $\mathcal{D}$ are both closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$, and if f and g are continuous functions. These hybrid basic conditions guarantee that $\mathcal{H}_{\mathbf{w}}$ (without inputs, that is with $\mathbf{w} = \mathbf{0}$) is well posed in the sense that their solution sets inherit several good structural properties: upper-semicontinuous dependence with respect to initial conditions; closeness of perturbed (due to measurement noise, for example) and unperturbed solutions; asymptotic stability is preserved under small enough perturbations [21], etc.

1.2. Organization of the manuscript

This work is mainly devoted to developing a representation of reset control systems, with a zero-crossing resetting law and an LTI base system, in the HI framework. The focus is on well-posedness and stability, with a strong motivation to obtain HI models that capture key properties in control practice. In Section 2, starting with a new Clegg integrator model equipped with an input zero-crossing detection (ZCD) mechanism, a reset controller model based on it is postulated. Section 3 analyzes closed-loop reset control systems, resulting from the feedback connection between a reset controller (with the ZCD mechanism) and an LTI plant (using plant output measurement). Some basic properties of closed-loop hybrid systems like well-posedness, existence of solutions, and flow persistence (a concept introduced to guarantee the existence of solutions that are unbounded in the t-direction) will be investigated. Also, a deep analysis of how solutions to the closed-loop hybrid system are related to operation in control practice. To avoid the existence of defective solutions, a standard approach based on time regularization is used. Finally, a reset control system in the HI framework with a zero-crossing resetting law and time-regularized is postulated. An example, based on a classical case analyzed by Horowitz, is investigated with the proposed model; also, a comparison with a time-regularized reset control system with a sector-based resetting law is performed. In Section 4, the stability of the proposed reset control system is investigated. A basic result will be a reformulation of previous stability results in the HI framework, relating the stability of the closed-loop hybrid system with the stability of a discrete-time system obtained as a Poincaré-like map. Two different stability approaches are then investigated: one based on the analysis of the reset interval sequences periods that results in analyzing the eigenvalues of different matrices associated with those periods; and the other based on the use of Lyapunov functions that finally results in LMIs conditions whose feasibility determines the stability of the reset control system. Moreover, several examples, that illustrate the applicability of the proposed results, are developed.

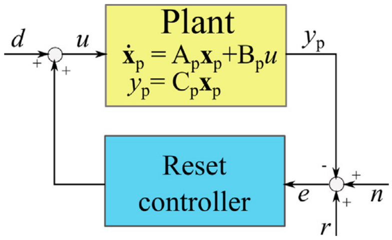


Fig. 1. A reset control system, with an LTI continuous-time plant and a feedback reset controller. The feedback loop is perturbed by errors both in the measurement (noise n) and the actuator (disturbance d); r is a reference signal.

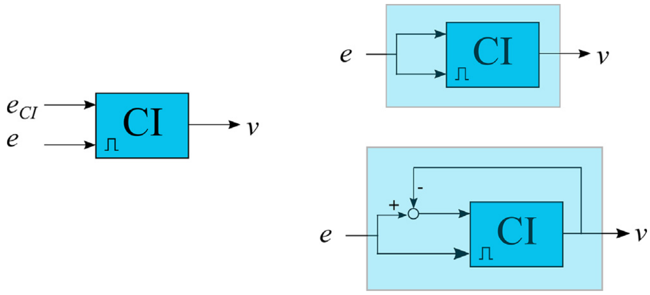


Fig. 2. (*left*) The block CI represents the Clegg integrator with two inputs and a zero-crossing detection; (*right*) The Clegg integrator with one input (*top*), and a first order reset element (FORE) (*bottom*), built upon the block CI.

2. From the Clegg integrator to reset controllers

In this work, the main focus is on reset control systems, in which a continuous-time plant is controlled by a reset controller with plant output feedback (see Fig. 1). This feedback control system, that uses plant output measurement, will be modeled in the HI framework by using (1). More specifically, the plant is LTI, single-input single-output, and described by the differential equation:

$$P : \begin{cases} \dot{\mathbf{x}}_p = A_p\mathbf{x}_p + B_p u \\ y_p = C_p\mathbf{x}_p \end{cases} \tag{2}$$

where $\mathbf{x}_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}$ the control input, $y_p \in \mathbb{R}$ is the plant output, and A_p, B_p and C_p are matrices of appropriate dimensions. The reset controller, with continuous state $\mathbf{x}_r \in \mathbb{R}^{n_r}$, will be endowed with a zero-crossing detection mechanism based on a discrete state $q \in \{1, -1\}$, being finally $(\mathbf{x}_r, q) \in \mathcal{O}^{n_r} := \mathbb{R}^{n_r} \times \{1, -1\}$ the controller state. In the following, the proposed reset controller will be analyzed in detail; since the controller setup will be based on a modification of the Clegg integrator [2], this is first described.

2.1. A Clegg integrator wih a zero-crossing detection mechanism

A basic and well-known reset controller is the Clegg integrator [2,3], which will be adapted in this work by attaching a zero-crossing detection procedure based on the discrete state $q \in \{1, -1\}$, and also adding an extra input. Besides the trigger input $e \in \mathbb{R}$ (usually the error signal in the case of an output feedback control system), the input signal $e_{CI} \in \mathbb{R}$ is proposed[1] (Fig. 2). Using (1), the result is a new model of the Clegg integrator in the HI framework. It is given by:

$$\text{CI} : \begin{cases} \dot{x}_r = e_{CI} & , (x_r, q, e_{CI}, e) \in \mathcal{C} \\ \begin{pmatrix} x_r^+ \\ q^+ \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_r \\ q \end{pmatrix} & , (x_r, q, e_{CI}, e) \in \mathcal{D} \end{cases} \tag{3}$$

where $(x_r, q) \in \mathcal{O}$ is the CI state, $(e_{CI}, e) \in \mathbb{R}^2$ is its input, and $v = x_r$ is its output, and the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are given by

$$\mathcal{C} = \{(x_r, q, e_{CI}, e) \in \mathcal{O} \times \mathbb{R}^2 : qe \leq 0\} \tag{4}$$

[1] It is worth noting that the original Clegg integrator is recovered from CI by removing the discrete state q and performing $e_{CI} = e$.

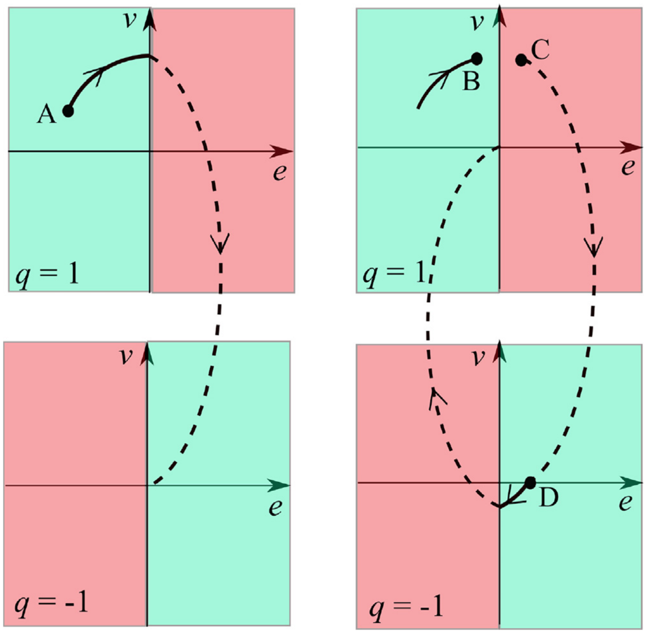


Fig. 3. Zero-crossing detection mechanism: (*left*) from the initial point A (*up*), the system flows until a zero-crossing is detected ($qe = 0$), jumping to $(e, v, q) = (0, 0, -1)$ (*bottom*); (*right*) a perturbation of the error signal at point B makes the system jump from B to C (*up*), then a zero-crossing is also detected ($qe > 0$) and the system jumps to D (*bottom*); from D the system flows again and finally jumps to $(0, 0, 1)$ (*up*).

and

$$\mathcal{D} = \{(x_r, q, e_{CI}, e) \in \mathcal{O} \times \mathbb{R}^2 : qe \geq 0\}, \tag{5}$$

respectively. Because the CI discrete state q is constant when flowing, its flow equation is not explicitly shown. When (x_r, q, e_{CI}, e) goes from $\mathcal{C}$ to $\mathcal{D}$ either crossing or jumping through their boundary, a jump of the CI state may be performed. This ensures the detection of a zero-crossing even if the signal e has jump discontinuities, such as due to noise measurement n (see Fig. 3), which is a clear advantage over previous reset controllers that have a zero-crossing resetting law [1,8,32], that only detects zero-crossings when the signal error is continuous.

By using CI as a building block, the two input signals, e_{CI} and e, can be used to model more complex reset controllers. For example, for the FORE of Fig. 2, $e_{CI} = e - v$. This capability will be fully exploited by higher-order reset controllers in the next Section.

2.2. A reset controller with a zero-crossing resetting law

In this work, a reset controller inspired in [1,8,32] is proposed. Here, the new CI is used as a building block, and thus the reset controller has also attached a zero-crossing detection mechanism. It has a state $(\mathbf{x}_r, q) \in \mathcal{O}^{n_r}$ and a scalar input e. Here $\mathbf{x}_r$ will be referred to as the *continuous* state and q as the *discrete* state. Using again (1), it is given by

$$R : \begin{cases} \dot{\mathbf{x}}_r = A_r \mathbf{x}_r + B_r e & , (\mathbf{x}_r, q, e) \in \mathcal{C} \\ \begin{pmatrix} \mathbf{x}_r^+ \\ q^+ \end{pmatrix} = \begin{pmatrix} A_\rho & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x}_r \\ q \end{pmatrix} & , (\mathbf{x}_r, q, e) \in \mathcal{D} \end{cases} \tag{6}$$

where the output and control signal is the scalar $v = C_r \mathbf{x_r} + D_r e$, and now the flow and jump sets, $\mathcal{C}$ and $\mathcal{D}$, are

$$\mathcal{C} = \{(\mathbf{x}_r, q, e) \in \mathcal{O}^{n_r} \times \mathbb{R} : qe \leq 0\} \tag{7}$$

and

$$\mathcal{D} = \{(\mathbf{x}_r, q, e) \in \mathcal{O}^{n_r} \times \mathbb{R} : qe \geq 0\}, \tag{8}$$

respectively. Here A_r, B_r, C_r, and D_r are real matrices with appropriate dimensions. When a zero-crossing is detected, some of the controller states are set to zero by using the matrix A_ρ (by convention, the last n_ρ states of $\mathbf{x}_r$ are set to zero, while the first $n_{\bar{\rho}} = n_r - n_\rho$ states are kept unchanged). It is given by

$$A_\rho = \begin{pmatrix} I_{n_{\bar{\rho}} \times n_{\bar{\rho}}} & 0_{n_{\bar{\rho}} \times n_\rho} \\ 0_{n_\rho \times n_{\bar{\rho}}} & 0_{n_\rho \times n_\rho} \end{pmatrix}, \tag{9}$$

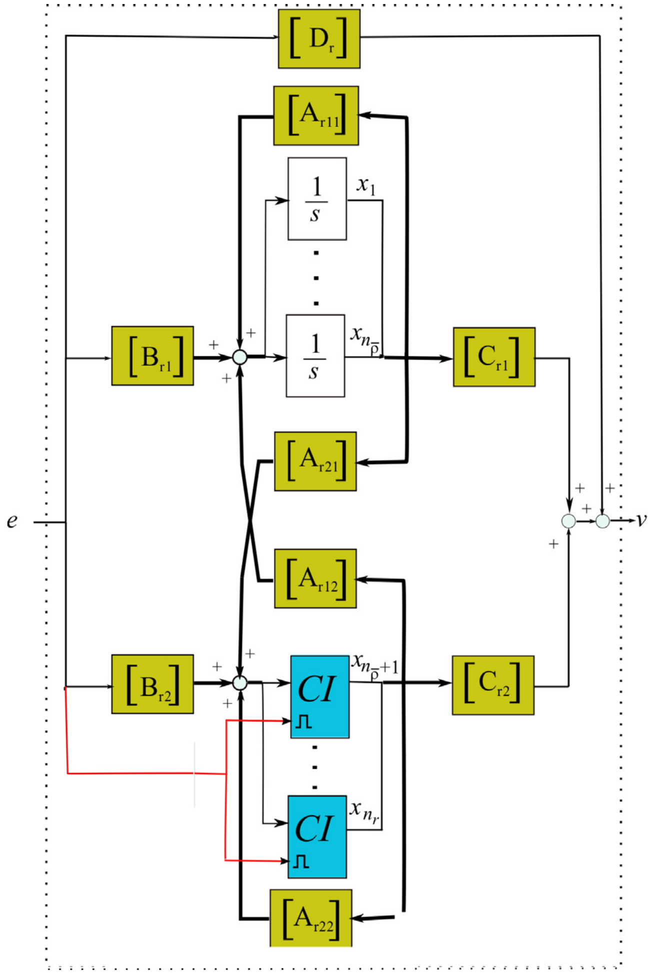


Fig. 4. A block diagram of the reset controller with a zero-crossing resetting law, given by (6), with state $(\mathbf{x}_r, q)$. Here $\mathbf{x}_r = (x_1, \ldots, x_{n_{\bar{\rho}}}, x_{n_{\bar{\rho}}+1}, \ldots, x_{n_r})$ and q is the discrete state that is common for all the CI blocks. For a block [M] the output vector is the matrix multiplication of M with the input vector (thin lines correspond to scalar signals, while thick lines correspond to vector signals).

In the case of a *full reset* controller $n_\rho = n_r$, while if $n_\rho < n_r$ then R is a *partial reset* controller. In addition, A_r, B_r, and C_r are partitioned into blocks with appropriate block dimensions:

$$A_r = \begin{pmatrix} A_{r_{11}} & A_{r_{12}} \\ A_{r_{21}} & A_{r_{22}} \end{pmatrix}, B_r = \begin{pmatrix} B_{r_1} \\ B_{r_2} \end{pmatrix}, C_r = \begin{pmatrix} C_{r_1} & C_{r_2} \end{pmatrix} \tag{10}$$

For a block diagram representation of the reset controller R, besides integrator blocks it is sufficient to use the modified Clegg integrator CI given by (3) as a basic block. A block diagram of R that allows a direct implementation is given in Fig. 4. It uses as basic blocks $n_{\bar{\rho}}$ $(= n_r - n_\rho)$ integrators and n_ρ (two-inputs) CIs; note that since the reset controller R is, in general, partially reset at the reset instants, the last n_ρ components of the controller state $\mathbf{x}_r$ that may be reset are attached to CI blocks, and the first $n_{\bar{\rho}}$ components of $\mathbf{x}_r$ are attached to integrators. In this case, the input e_{CI_k} to the kth CI block, $k = n_{\bar{\rho}} + 1, \ldots, n_r$, is simply obtained by (Fig. 4)

$$\mathbf{e}_{CI} = A_{r_{21}}\mathbf{x}_{\bar{\rho}} + A_{r_{22}}\mathbf{x}_\rho + B_{r_2} e$$

where $\mathbf{e}_{CI} = (e_{CI_{n_{\bar{\rho}}+1}}, \ldots, e_{CI_{n_r}})$, $\mathbf{x}_{\bar{\rho}} = (x_1, \ldots, x_{n_{\bar{\rho}}})$, and $\mathbf{x}_\rho = (x_{n_{\bar{\rho}}+1}, \ldots, x_{n_r})$.

3. The closed-loop reset control system

Once the plant and the reset controller are defined, the feedback control system $\mathcal{H}^{cl}_{\mathbf{w}}$ is obtained as a hybrid control system (see Fig. 1), with the plant output y_p serving as the feedback signal. Its state is $(\mathbf{x}_p, \mathbf{x}_r, q) \in \mathcal{O}^n$, and is given by

$$\mathcal{H}_{\mathbf{w}}^{cl}:\begin{cases}\begin{pmatrix}\dot{\mathbf{x}}_p\\ \dot{\mathbf{x}}_r\end{pmatrix}=\begin{pmatrix}A_p-B_pD_rC_p & B_pC_r\\ -B_rC_p & A_r\end{pmatrix}\begin{pmatrix}\mathbf{x}_p\\ \mathbf{x}_r\end{pmatrix}+\begin{pmatrix}B_pD_r & B_p\\ B_r & O\end{pmatrix}\mathbf{w} & ,(\mathbf{x},q,\mathbf{w})\in\mathcal{C}_{\mathbf{w}}^{cl}\\ \\ \begin{pmatrix}\mathbf{x}_r^+\\ q^+\end{pmatrix}=\begin{pmatrix}A_\rho & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{x}_r\\ q\end{pmatrix} & ,(\mathbf{x},q,\mathbf{w})\in\mathcal{D}_{\mathbf{w}}^{cl}\end{cases}\tag{11}$$

where the exogenous input is $\mathbf{w}=(w_1,w_2)=(r+n,d)$, and the flow and jump sets are given by

$$\mathcal{C}_{\mathbf{w}}^{cl}=\{(\mathbf{x}_p,\mathbf{x}_r,q,\mathbf{w})\in\mathcal{O}^{n_p+n_r}\times\mathbb{R}^2:q(w_1-C_p\mathbf{x}_p)\le 0\}\tag{12}$$

and

$$\mathcal{D}_{\mathbf{w}}^{cl}=\{(\mathbf{x}_p,\mathbf{x}_r,q,\mathbf{w})\in\mathcal{O}^{n_p+n_r}\times\mathbb{R}^2:q(w_1-C_p\mathbf{x}_p)\ge 0\},\tag{13}$$

respectively.

3.1. Closed-loop solutions and properties

Following Section 1.1, solutions to $\mathcal{H}_{\mathbf{w}}^{cl}$ are defined as pairs $(\mathbf{x},\mathbf{w})$ satisfying (11), where $\mathbf{x}$ is a hybrid arc and $\mathbf{w}$ is a hybrid input. Since $\mathcal{H}_{\mathbf{w}}^{cl}$ satisfies the hybrid basic conditions (the flow and jump maps are continuous, and the flow and the jump sets are closed), then it directly follows that for the case of no exogenous inputs, that is for $\mathbf{w}=\mathbf{0}$, the system $\mathcal{H}_{\mathbf{w}}^{cl}$ is well posed and the property of asymptotic stability is robust (see [21] for precise definitions and results).

Now, a key question is to analyze if $\mathcal{H}_{\mathbf{w}}^{cl}$ has also good properties for relevant sets of exogenous inputs. To begin, the problem of modeling relevant exogenous inputs arises; because the plant is originally continuous-time, it is natural to assume that exogenous signals only depend on time t and not on j, and thus hybrid inputs $\mathbf{w}$ can be considered as given by setting $\mathbf{w}(t,j)=\mathbf{w}'(t)$ for all $(t,j)\in E$, for some continuous time signal $\mathbf{w}'$ and any arbitrary time domain E. Moreover, since exogenous inputs should be relevant in control practice, it will be assumed that $\mathbf{w}'$ is generated by an exosystem Σ with state $\mathbf{x}_w$, that is

$$\Sigma:\begin{cases}\dot{\mathbf{x}}_w=A_w\mathbf{x}_w\\ \mathbf{w}=\begin{pmatrix}C_{w_1}\\ C_{w_2}\end{pmatrix}\mathbf{x}_w\end{cases}\tag{14}$$

This exosystem will be able to generate signals like steps, ramps, sinusoids, and, in general, Bohl functions. Finally, it is embedded in the reset control system (11) resulting in the (autonomous) reset control system $\mathcal{H}^{cl}$ with state $(\mathbf{x},q)=(\mathbf{x}_w,\mathbf{x}_p,\mathbf{x}_r,q)$:

$$\mathcal{H}^{cl}:\begin{cases}\dot{\mathbf{x}}=A\mathbf{x} & ,(\mathbf{x},q)\in\mathcal{C}^{cl}\\ \\ \begin{pmatrix}\mathbf{x}^+\\ q^+\end{pmatrix}=\begin{pmatrix}A_R & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ q\end{pmatrix} & ,(\mathbf{x},q)\in\mathcal{D}^{cl}\end{cases}\tag{15}$$

where the matrices A and A_R are given by

$$A=\begin{pmatrix}A_w & O & O\\ B_p(D_rC_{w_1}+C_{w_2}) & A_p-B_pD_rC_p & B_pC_r\\ B_rC_{w_1} & -B_rC_p & A_r\end{pmatrix},\quad A_R=\begin{pmatrix}I & O & O\\ O & I & O\\ O & O & A_\rho\end{pmatrix},\tag{16}$$

and the flow and jump sets are given by

$$\mathcal{C}^{cl}=\{(\mathbf{x},q)\in\mathcal{O}^n:qC\mathbf{x}\le 0\}\tag{17}$$

and

$$\mathcal{D}^{cl}=\{(\mathbf{x},q)\in\mathcal{O}^n:qC\mathbf{x}\ge 0\},\tag{18}$$

respectively, where $n=n_w+n_p+n_r$, and C is given by

$$C=\begin{pmatrix}C_{w_1} & -C_p & O\end{pmatrix}\tag{19}$$

The next proposition analyzes the existence of solutions to $\mathcal{H}^{cl}$ and some properties that will be useful in control practice. For definitions of well-posedness and Zeno solutions, see [21]. $\mathcal{H}^{cl}$ is *flow persistent* if for any $\xi\in\mathcal{O}^n$ there exist a solution ϕ to $\mathcal{H}^{cl}$ with $\phi(0,0)=\xi$, such that dom ϕ is unbounded in the t-direction, that is the set $\{t\in\mathbb{R}_{\ge 0}:\exists j\in\mathbb{N},(t,j)\in \text{dom }\phi\}$ is not upper-bounded.

Proposition 3.1. *Consider the reset control system $\mathcal{H}^{cl}$ as given by (15)–(19), and a point $\xi = (\mathbf{x}_0, q_0) \in \mathcal{O}^n$, then:*

1. *(Well-posedness) $\mathcal{H}^{cl}$ is well-posed.*
2. *(Existence of solutions) There exist nontrivial solutions ϕ to $\mathcal{H}^{cl}$ with $\phi(0,0) = \xi$, and if $\phi \in \mathcal{S}_{\mathcal{H}^{cl}}(\xi)$, i.e., if ϕ is a maximal solution, then it is complete. That is, $\mathcal{H}^{cl}$ is forward complete from $\mathcal{O}^n$.*
3. *(Flow persistence) $\mathcal{H}^{cl}$ is flow persistent.*

Proof.

1. It directly follows since $\mathcal{H}^{cl}$ satisfies the basic hybrid conditions. Note that $\mathcal{C}^{cl}$ and $\mathcal{D}^{cl}$ are closed, and the functions $f, g : \mathcal{O}^n \to \mathcal{O}^n$, defining the flowing and jumping dynamics, respectively, and given by $f((\mathbf{x}, q)) = (A\mathbf{x}, 0)$ and $g((\mathbf{x}, q)) = (A_R\mathbf{x}, -q)$, are continuous.
2. If $\xi \in \mathcal{C}^{cl} \setminus \mathcal{D}^{cl}$ then there exists a nontrivial solution ϕ with $\phi(t, 0) = (e^{At}\mathbf{x}_0, q_0) \in \mathcal{C}^{cl}$ for $t \in [0, \epsilon]$ and $\epsilon = \min\{t \in \mathbb{R}^+ : Ce^{At}\mathbf{x}_0 = 0\}$; in the case that $Ce^{At}\mathbf{x}_0 \neq 0$ for any $t > 0$ then $\phi(t, 0) = (e^{At}\mathbf{x}_0, q_0) \in \mathcal{C}^{cl}$ for $t \in [0, \infty)$. Alternatively, if $\xi \in \mathcal{D}^{cl}$ then there exists nontrivial solutions ϕ with $\phi(0, 1) = (A_R\mathbf{x}_0, -q_0)$ and $\phi(0, 0) = \xi$.
   Moreover, for $(\mathbf{x}, q) \in \mathcal{D}^{cl}$ it is true that $qC\mathbf{x} \geq 0$, and since $CA_R = C$ then after a jump $q^+C\mathbf{x}^+ = -qCA_R\mathbf{x} = -qC\mathbf{x} \leq 0$, that is $(\mathbf{x}^+, q^+) \in \mathcal{C}^{cl}$ and thus $g(\mathcal{D}^{cl}) \subset \mathcal{C}^{cl}$; since, in addition, any solution to the flow equation $\dot{\mathbf{x}} = A\mathbf{x}$ defined on an interval, open to the right, can be trivially extended to an interval including the right endpoint, it is concluded that any maximal solution is complete (by direct application of Prop. 2.10 [21]).
3. Firstly, it is shown that there always exists a solution $\phi \in \mathcal{S}_{\mathcal{H}^{cl}}(\xi)$ whose domain satisfies $[0, \varepsilon] \times \{0\} \subset \operatorname{dom} \phi$ or $[0, 0] \times \{0\} \cup [0, \varepsilon] \times \{1\} \subset \operatorname{dom} \phi$, for some $\varepsilon > 0$ (that is, the solution flows during some interval $[0, \varepsilon]$ after at most one jump). This is based on the fact that $q_0Ce^{At}\mathbf{x}_0$, as a mapping from $t \in \mathbb{R}_{\geq 0}$ to $\mathbb{R}$, is a real analytic function and thus it only has isolated zeros. As a consequence, there are only three possible cases: (i) there exist $\varepsilon > 0$ such that $q_0Ce^{At}\mathbf{x}_0 < 0$ for any $t \in (0, \varepsilon]$, (ii) there exist $\varepsilon > 0$ such that $q_0Ce^{At}\mathbf{x}_0 > 0$ for any $t \in (0, \varepsilon]$, (iii) $q_0Ce^{At}\mathbf{x}_0 = 0$ for any $t \geq 0$. In case (i), a solution ϕ directly exists such that $[0, \varepsilon] \times \{0\} \subset \operatorname{dom} \phi$; in case (ii), after an initial jump $q_0^+C\mathbf{x}_0^+ = -q_0CA_R\mathbf{x}_0 = -q_0C\mathbf{x}_0 < 0$ and thus a solution exists that flow in $(0, \varepsilon]$ after an initial jump, that is $[0, 0] \times \{0\} \cup [0, \varepsilon] \times \{1\} \subset \operatorname{dom} \phi$; finally, in case (iii), a solution ϕ with $\operatorname{dom} \phi = [0, \infty) \times \{0\}$ do exist.
   From the above property, it follows that, besides solutions that always flow after a finite number of jumps, there always exist solutions $\phi \in \mathcal{S}_{\mathcal{H}^{cl}}(\xi)$ with an infinite number of jumps at instants t_k, $k = 0, 1, \ldots$, where $t_0 = 0$ and $t_k = \varepsilon_1 + \cdots + \varepsilon_k$ for $k = 1, 2, \ldots$, and where $\varepsilon_k > 0$ for $k \geq 1$. Thus the only obstacle for $\mathcal{H}^{cl}$ to be flow persistent is that for any of these flowing-first solutions the corresponding sequence $\{t_k\}_{k=0}^{\infty}$ would converge to some finite value, resulting in Zeno solutions. But this is not possible since $\{t_k\}_{k=0}^{\infty}$ are not Cauchy sequences; this follows from [32]-Prop. 2.4. □

3.2. Analysis of defective solutions and time-regularized reset control system

In a reset control system formulated as (15)–(19), in which the hybrid dynamics is due to the controller (the plant is an LTI continuous-time system), it is important to analyze how the hybrid time domains of solutions, and the non-deterministic behavior of the system are related to the final operation in control practice. For example, for a hybrid time domain that consists of the union of intervals $I^j \times j = [t_j, t_{j+1}] \times j$, with $0 = t_0 < t_1 < t_2 = t_3 = t_4 = t_5 < t_6 < \cdots$, the solution flows in the time interval $[t_0, t_1]$, jumps at t_1, flows in $[t_1, t_2]$, then it performs three consecutive jumps, keeps flowing in $[t_5, t_6] = [t_2, t_6]$, performs another jump at t_6, etc. From a practical point of view, for solutions to be implemented in a controller, it is necessary to assume that the controller is able to perform a finite number of consecutive jumps instantaneously. In addition, it is compulsory that from any point, there always exist solutions that are unbounded in the t-direction. Otherwise, the control system would only present Zeno solutions (genuinely or eventually discrete) that are simply unimplementable in practice and are considered defective solutions.

It has been introduced a property, *flow persistence*, that is useful to analyze whether a reset control system may be effectively used in control practice regarding the existence of non-defective solutions. It is worth noting that if a control system is flow persistent, there is always a solution that is unbounded in the t-direction; on the other hand, if it is not flow persistent, there may exist points from which all solutions are bounded in the t-direction. Thus, while flow persistence is a necessary property in control practice, it is less clear whether it is a sufficient property; that is, for a given initial point, is the existence of Zeno solutions besides solutions that are unbounded in the t-direction a problem?. Note that there are always infinite Zeno solutions starting at the points $(\mathbf{0}, 1), (\mathbf{0}, -1) \in \mathcal{C}^{cl} \cap \mathcal{D}^{cl}$, besides an infinite number of solutions unbounded in the t-direction.

Another important aspect regarding the final implementation of the hybrid controller is also related to that non-deterministic behavior. In principle, the above formulation allows a (finite or infinite) number of different solutions from some initial points. In practice, it is clear that any realistic controller implementation entails a decision such that a solution is selected within all the existing solutions. At this point, a possible answer to the above question is that there is no problem once it is assumed that the controller is able to choose only those solutions that are unbounded in the t-direction. However, this type of implementation would require some procedure to properly select the implementable solutions.

A more simple and common approach to implementing the non-deterministic behavior is to assume that it is irrelevant which solution the controller selects, and thus the reset control system would correctly perform for any chosen solution. This is the approach to be followed in this work, and thus it is necessary to remove all the defective solutions. A standard way to avoid the existence of defective solutions in hybrid systems is by means of *time regularization* [10,26]. For the reset control system $\mathcal{H}^{cl}$ in (15)–(19), this is simply achieved by introducing a timer $\tau \in [0,\infty)$ that prevents the system from performing two o more consecutive jumps, simply by initializing τ to 0 after a jump, and avoiding performing a new jump until $\tau \geq \tau_m$, where $\tau_m > 0$ is a design parameter (usually referred to as the *minimum dwell-time*).

A time-regularized reset control system $\mathcal{H}^{cl}_{\tau_m}$, with state $(\mathbf{x}, q, \tau) \in \mathcal{O}^n \times [0,\infty)$, is given by:

$$\mathcal{H}^{cl}_{\tau_m} : \begin{cases} \dot{\tau} = 1, \quad \dot{\mathbf{x}} = A\mathbf{x} & , (\mathbf{x}, q, \tau) \in \mathcal{C}^{cl}_{\tau_m} \\ \tau^+ = 0, \quad \begin{pmatrix} \mathbf{x}^+ \\ q^+ \end{pmatrix} = \begin{pmatrix} A_R & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ q \end{pmatrix} & , (\mathbf{x}, q, \tau) \in \mathcal{D}^{cl}_{\tau_m} \end{cases} \tag{20}$$

where

$$\mathcal{C}^{cl}_{\tau_m} = \left(\mathcal{C}^{cl} \times [0,\infty)\right) \cup \left(\mathcal{D}^{cl} \times [0,\tau_m]\right) \tag{21}$$

and

$$\mathcal{D}^{cl}_{\tau_m} = \mathcal{D}^{cl} \times [\tau_m, \infty). \tag{22}$$

where $\mathcal{C}^{cl}$ and $\mathcal{D}^{cl}$ are defined in (17)–(18).

Moreover, it is considered that the reset control system will always perform a jump at the initial instant. By simplicity, this is not explicitly described in (20)–(22), it is modeled by assuming that $\mathcal{H}^{cl}_{\tau_m}$ is initially at a state $((\mathbf{z}, \mathbf{0}), q, 0)$, where $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$ and $q \in \{1, -1\}$. That is, the last n_ρ continuous states of the reset controller and the timer are always initially at rest.

The following property of $\mathcal{H}^{cl}_{\tau_m}$ easily follows since it always jumps from $\mathcal{D}^{cl}_{\tau_m}$ to the interior of $\mathcal{C}^{cl}_{\tau_m}$, and then it flows for at least a time $\tau_m > 0$. Note that from (21) flow is possible for values $(\mathbf{x}, q) \in \mathcal{D}^{cl}$ when the timer τ takes values $\tau \leq \tau_m$.

Corollary 3.2. *For any $\tau_m > 0$, $\mathcal{H}^{cl}_{\tau_m}$, as given by (20)–(22), is flow persistent and does not have Zeno solutions.*

In principle, an election of a small value of the minimum dwell-time τ_m is all that is needed to prevent the existence of defective solutions. Note, however, that this does not avoid the existence of multiple solutions for some initial conditions, this is, for example, the case of points $(\mathbf{0}, 1, 0)$ and $(\mathbf{0}, -1, 0)$. This non-deterministic behavior will be explored in the next example.

Example 3.3. Consider the reset control system of Fig. 5, composed of a Horowitz reset controller R and a first-order plant. Its flow persistence for an exogenous input $w = r$ corresponding to a step reference (no disturbances present) will be analyzed, as will the influence of τ_m on the reset control system solutions. For some $\tau_m > 0$, the time regularized reset control system is given by (20), with state $(\mathbf{x}, q, \tau)$ being $\mathbf{x} = (x_w, x_p, x_{r_1}, x_{r_2})$, and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 4 & -4 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \tag{23}$$

Here the reset controller output is $v = x_{r_1} + x_{r_2}$ and the error signal is $e = x_w - x_p$. The sets $\mathcal{C}^{cl}$ and $\mathcal{D}^{cl}$ are depicted in Fig. 5 as green and red regions, respectively, in the $e - v$ planes corresponding to $q = 1$ and $q = -1$. Note that the flow and jump sets are given by (21)–(22), and thus flowing is, in principle, possible in the set $\mathcal{D}^{cl}$. It is chosen an initial point A corresponding to a unit step reference, with the controller and plant at rest; then the reset control system flows from A to B. Because the reset control system is flow persistent, there exist a solution that is unbounded in the t-direction; this solution is shown in Fig. 5 where jumps from B to C, from D to E, ... are visible (it is the unique solution for the initial point A). In this case, the solution is not flowing in the set $\mathcal{D}^{cl}$ since it always flows during a time larger than τ_m before jumps are enabled. This is true as far as $\tau_m < \tau_m^\star \approx 1.4416$; in fact, without time regularization, the reset interval sequence is periodic with a fundamental period $\tau_m^\star$ after the second jump. This simple structure of the reset instants sequence is common in low-order reset control systems, and in these cases, time regularization would not be necessary for most initial points. However, as discussed further below, there may be initial points where time regularization must be used to remove defective solutions.

This is the case of any initial point with $\mathbf{x}_0 = (1, 1, 1, 0)$, corresponding again to a unit step reference but now the controller output is initially $v = 1$ (with the CI initially at rest), it can be easily checked that the initial point belongs to $\mathcal{C}^{cl}_{\tau_m} \cap \mathcal{D}^{cl}_{\tau_m}$; moreover, since $A\mathbf{x}_0 = \mathbf{0}$ and $A_R\mathbf{x}_0 = \mathbf{x}_0$ it directly follows that there exists an infinite number of solutions having one of the following hybrid time domains: $[0,\infty) \times \{0\}$, $[0, t_1) \times \{0\} \cup [t_1, \infty) \times \{1\}$, $[0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup [t_2, \infty) \times \{2\}, \ldots$, where $t_1 \in [0,\infty)$ and $t_{j+1} \in [t_j + \tau_m, \infty)$ for $j = 1, 2, \ldots$. That is, there is an only-flowing solution and an infinite number of solutions that jump a finite or infinite number of times, so all the solutions are unbounded in the t-direction as far as $\tau_m > 0$. Finally, note that any solution ϕ satisfies $\phi(t, j) = \xi$ for any $(t, j) \in \text{dom}\,\phi$; informally, all the solutions produce the same values of controller output and error, so it makes no difference which solution is chosen by the controller.

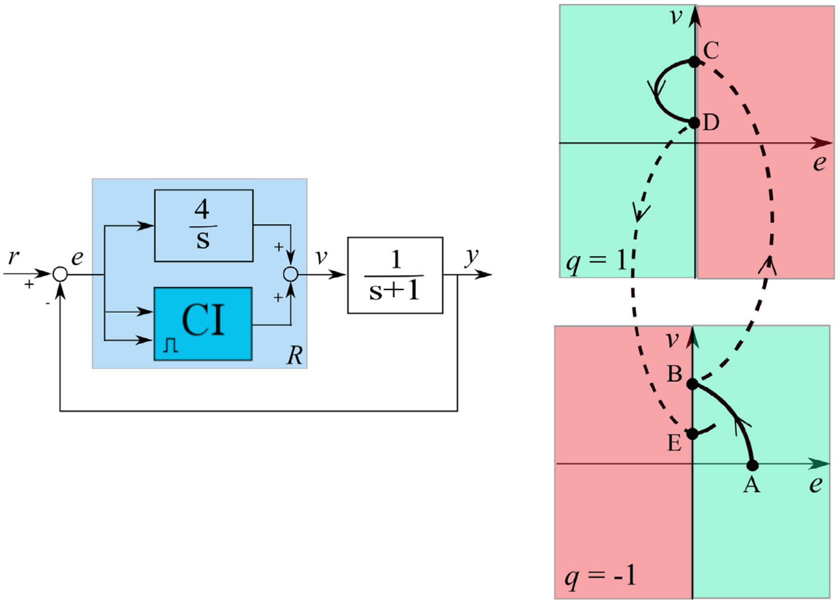


Fig. 5. Flow persistence: (*left*) Reset control system with a Horowitz reset controller (taken from [3]); (*right*) Simulation for a unit step reference, with the reset control system initially at point A: $(x_w, x_p, x_{r_1}, x_{r_2}) = (1, 0, 0, 0)$, $q = -1$ and with $\tau = 0$ (solid lines correspond to flows, an dotted lines to jumps from B to C, from D to E, ...). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3.3. Reset controllers with a sector resetting law

Although this work is focused on reset control systems with a zero-crossing resetting law, it is instructive to analyze other resetting laws that have been developed in the literature. The sector resetting law was introduced in [33], and has been the main approach within the framework of hybrid inclusions, followed and also extended in several works [13,34,35].

A basic reset controller with a sector resetting law, and with a state $\mathbf{x}_r$ and input e, is given by

$$R : \begin{cases} \dot{\mathbf{x}}_r = A_r\mathbf{x}_r + B_r e & , (\mathbf{x}_r, e) \in \mathcal{C} \\ \mathbf{x}_r^+ = A_\rho \mathbf{x}_r & , (\mathbf{x}_r, e) \in \mathcal{D} \end{cases} \tag{24}$$

where $\mathcal{C} = \{(\mathbf{x}_r, e) \in \mathbb{R}^{n_r+1} : ev \geq 0\}$ and $\mathcal{D} = \{(\mathbf{x}_r, e) \in \mathbb{R}^{n_r+1} : ev \leq 0\}$, being $v = C_r\mathbf{x}_r$ the controller output. Thus, the basic jump set $\mathcal{D}$ is a sector in the $e - v$ plane, consisting of its second and fourth quadrants. In combination with the plant (2) and the exosystems (14), and also including time-regularization, the resulting reset control system is given by

$$\begin{cases} \dot{\tau} = 1, & \dot{\mathbf{x}} = A\mathbf{x} & , (\mathbf{x}, \tau) \in \mathcal{C}^{cl}_{\tau_m} \\ \tau^+ = 0, & \mathbf{x}^+ = A_R\mathbf{x} & , (\mathbf{x}, \tau) \in \mathcal{D}^{cl}_{\tau_m} \end{cases} \tag{25}$$

where the closed-loop state is now $(\mathbf{x}, \tau) = (\mathbf{x}_w, \mathbf{x}_p, \mathbf{x}_r, \tau)$, and the flow and jump sets are also given by (21) and (22), respectively. But now the sets $\mathcal{C}^{cl}$ and $\mathcal{D}^{cl}$ defined by the sector resetting law, are given by

$$\mathcal{C}^{cl} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T M\mathbf{x} \geq 0\} \tag{26}$$

and

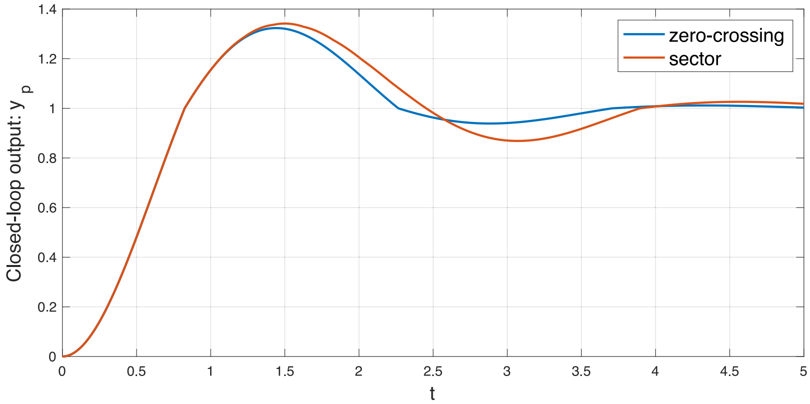
$$\mathcal{D}^{cl} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T M\mathbf{x} \leq 0\} \tag{27}$$

respectively, where

$$M = \begin{pmatrix} O & O & C_{w_1}^T C_r \\ O & O & -C_p^T C_r \\ C_r^T C_{w_1} & -C_r^T C_p & O \end{pmatrix} \tag{28}$$

It should be noted that time-regularization may cause solutions to flow in the jump set $\mathcal{D}^{cl}$, or in the $e - v$ plane to flow in the sector $\mathcal{D}$. It easily follows that this reset control system is flow persistent and does not have defective solutions.

Example 3.4. A detailed comparison between the proposed zero-crossing resetting law and the sector resetting law is performed for a case. Consider the reset control system of Fig. 5 with both a zero-crossing resetting law (Example 3.3) and



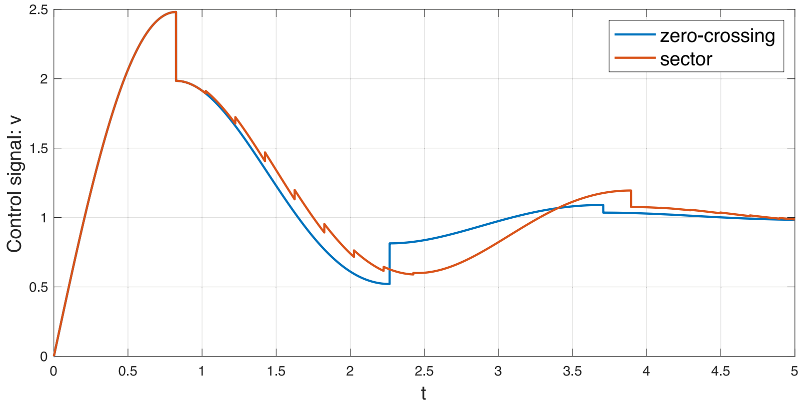


Fig. 6. Zero-crossing and sector resetting laws for Example 3.3: (*top*) Closed-loop outputs, (*botton*) Control signals.

a sector resetting law. For the sector resetting law, the state is $(\mathbf{x}, \tau) = (x_w, x_p, x_{r_1}, x_{r_2}, \tau)$, and the reset control system is given by (25)–(28), with A and A_R given by (23), and

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{29}$$

Figs. 6–7 show a simulation of both reset control systems for $\tau_m = 0.2$. Fig. 6 shows the step responses, including closed-loop outputs and control signals. Note that there is an important difference in how both resetting laws perform jumps, especially in the case in which $e < 0$ and $v > 0$ and thus a jump is enabled. In this case, which corresponds, for example, to the first jump in Fig. 6, the zero-crossing resetting law performs a jump to its flow set and then the solution flows until the next jump at $t \approx 2.25$ s, while the sector resetting law performs a jump to its jump set. This produces chattering behavior in the sector resetting law; and in fact, the obtained solution would be defective if time-regularization had not been used. On the other hand, strictly speaking, time-regularization is not necessary for this solution of the zero-crossing resetting law, and the same solution is obtained with or without time-regularization (as far as $\tau_m < \tau_m^\star$ -see Example 3.3). The control signals and their components are best analyzed in Fig. 7: note that after the first jump, in contrast with the zero-crossing resetting law, the sector resetting law periodically reset its x_{r_2} state every $\tau_m = 0.2$ s, until $t \approx 2.45$ s. This is the cause of its chattering, and of its bigger overshoot and undershoot in the step response. The undershoot is worsened by the fact that when the error signal changes its sign at $t \approx 2.45$, $x_{r_2} \approx 0$ due to a recent reset. This example shows that the responses of both resetting laws may be very different in general, and in this case, it is clear that the response of the zero-crossing resetting law is qualitatively better in terms of tracking error and control signal chattering. Some quantitative performance figures are shown in Table 1. Note that there is a significant reduction of 15% in the IAE and of 53% in the undershoot percentage when the zero-crossing resetting law is compared with the sector resetting law, and with slightly better figures in the overshoot percentage and control signal energy. For values of $\tau_m < 0.2$ the figures for the sector resetting law are even worse.

Table 1
(Example 3.4) Performance figures for the reset control system of Fig. 6 (Horowitz reset controller) with the sector resetting law and the proposed zero-crossing resetting law. IAE stands for the integral of the tracking error absolute value, and $\|v - v_{ss}\|^2$ is a measure of the control signal energy, defined as $\|v - v_{ss}\|^2 = \sum_{j=0}^{\infty} \left(\int_{t_j}^{t_{j+1}} |v(t,j) - v_{ss}|^2 dt \right)$, where $v_{ss} = 1$ is its the steady-state value.

Resetting law	IAE	% overshoot	% undershoot	$\|v - v_{ss}\|$
Zero-crossing	0.85	32.34	6.20	1.11
Sector	1.00	34.42	13.20	1.15

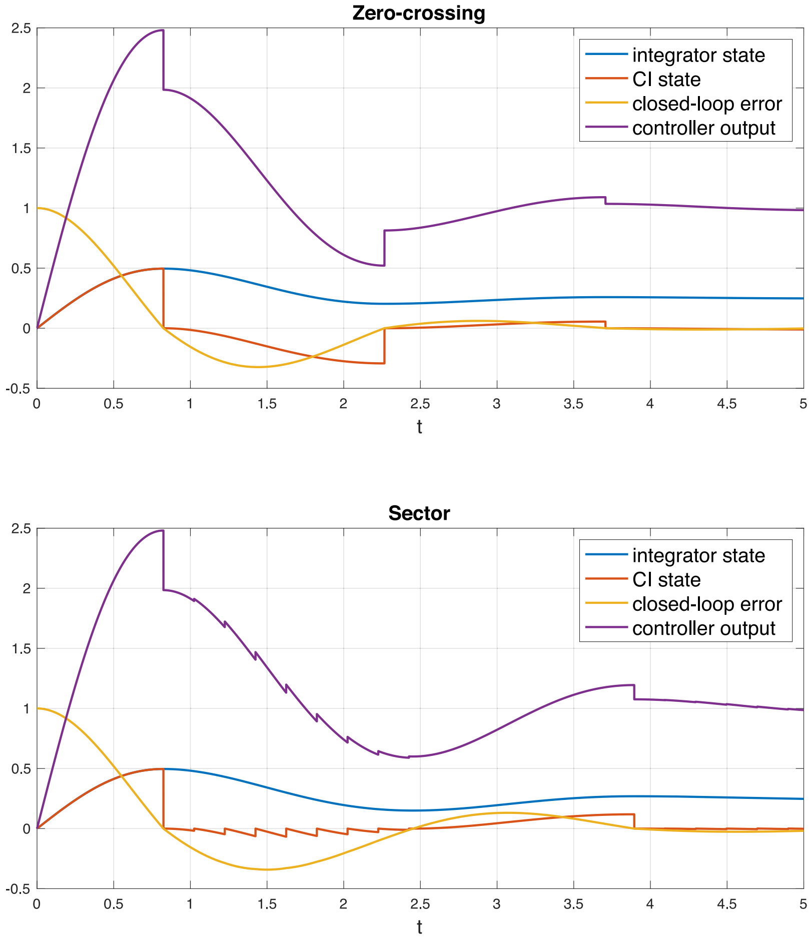


Fig. 7. Zero-crossing and sector resetting laws for Example 3.3.

4. Stability analysis

The stability of the time-regularized reset control system $\mathcal{H}^{cl}_{\tau_m}$ with no exogenous inputs is analyzed in the following. Reset times-dependent stability criteria will be developed, inspired from previous results developed in [5,32]. The basic idea of this approach is to analyze stability by using a discrete-time system (a Poincaré-like map), that represents the sampling of $\mathcal{H}^{cl}_{\tau_m}$ at the after-reset instants.

As it is usual in hybrid dynamical systems, stability is referred to sets instead of to a single point. The following stability definitions are based on [20], note that they are applicable to continuous-time or discrete-time systems as particular cases of hybrid systems. Consider a generic hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. For a set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $\phi \in \mathbb{R}^n$, the notation $\|\phi\|_{\mathcal{A}} = \min\{\|\phi - \psi\| : \psi \in \mathcal{A}\}$ indicates the distance of ϕ to $\mathcal{A}$. The set $\mathcal{A}$ is *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exist $\delta > 0$ such that $\|\phi(0,0)\|_{\mathcal{A}} \leq \delta$ implies $\|\phi(t,j)\|_{\mathcal{A}} \leq \varepsilon$ for all solutions ϕ to $\mathcal{H}$ and all $(t,j) \in \text{dom } x$. The set $\mathcal{A}$ is *attractive* if there exists a ball $\mathbb{B} \subset \mathbb{R}^n$ centered at the origin such that for any $\xi \in \mathbb{B}$ solutions ϕ to $\mathcal{H}^{cl}$ with $\phi(0,0) = \xi$ converge to a set $\mathcal{A}$, that is $\|\phi(t,j)\|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, where $(t,j) \in \text{dom } \phi$. The set $\mathcal{A}$ is *asymptotically stable* if it is stable and attractive, and its *basin of attraction* is $\mathbb{B}$. If the basin of attraction is $\mathbb{R}^n$ then $\mathcal{A}$ is *globally asymptotically stable*.

In the case of the reset control $\mathcal{H}^{cl}_{\tau_m}$ given by (20), stability will be referred to the set

$$\mathcal{A}_{\mathbf{0}} = \{\mathbf{0}\} \times \{1, -1\} \times [0, \infty) \tag{30}$$

Define for simplicity of notation the matrix J as

$$J = \begin{pmatrix} I_{(n-n_\rho)\times(n-n_\rho)} \\ O_{n_\rho \times (n-n_\rho)} \end{pmatrix} \tag{31}$$

such that for any $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$, $J\mathbf{z} = \begin{pmatrix} \mathbf{z} \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^n$. It easily follows that $A_R = JJ^\top$, $J = A_R J$ and $J^\top = J^\top A_R$. Now, the mapping $I : \mathbb{R}^{n-n_\rho} \to \mathbb{R}_{\geq 0}$, is defined as

$$I(\mathbf{z}) = \min\{\tau \geq \tau_m : Ce^{A\tau} J\mathbf{z} \geq 0\} \tag{32}$$

Moreover, the following standing assumption will operate in the rest of this work. It is assumed that there exists an upper bound of I in the cases in which the flow dynamics is unstable. This is justified to avoid cases in which the closed-loop base system is unstable and it does jump a finite number of times. In these cases, the reset controller would be unable to stabilize the flow dynamics.

Assumption A. Either the state matrix A in (20) is Hurwitz or the mapping I is upper bounded (that is there exists $\tau_M > 0$, that will be referred to as the *reset intervals upper bound*, such that $I(\mathbf{z}) \leq \tau_M$ for any $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$).

Here, a Poincaré-*like* map that gives the evolution from one after-jump state to the next one with a sign change is postulated. By definition, the discrete-time system $\mathcal{DH}^{cl}_{\tau_m}$, with state $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$, is given by

$$\mathbf{z}^+ = g(\mathbf{z}) = -J^T e^{A \cdot I(\mathbf{z})} J\mathbf{z} \tag{33}$$

It is worth noting that the reset matrix A_R is somehow embedded in J, and that the sign change allows one to obtain the successive reset intervals starting at $q = 1$ by using the mapping I, which does not explicitly depend on q. Also, to obtain a simplified dynamic discrete system $\mathcal{DH}^{cl}_{\tau_m}$ in which the state $\mathbf{z}$ only consists of the first $n - n_\rho$ values of $\mathbf{x} = J\mathbf{z}$. Some homogeneity properties of the maps I and g easily follow: for any $\lambda > 0$ it is true that

$$\begin{array}{ll} (i) & I(\lambda \mathbf{z}) = I(\mathbf{z}) \\ (ii) & g(\lambda \mathbf{z}) = \lambda g(\mathbf{z}) \end{array} \tag{34}$$

Proposition 4.1. *The set $\mathcal{A}_{\mathbf{0}}$ is (globally) asymptotically stable for the reset control system $\mathcal{H}^{cl}_{\tau_m}$ if and only if the origin $\{\mathbf{0}\}$ is (globally) asymptotically stable for the discrete-time system $\mathcal{DH}^{cl}_{\tau_m}$.*

Proof. It is an adaptation of [5]- Proposition 3.1 to the hybrid formalism adopted in this work. Consider solutions $\mathbf{z}$ to $\mathcal{DH}^{cl}_{\tau_m}$, given by (33), with $\mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{R}^{n-n_\rho}$. First, it is important to point out that only solutions $\phi = (\mathbf{x}, q, \tau)$ to $\mathcal{H}^{cl}_{\tau_m}$, as given by (20), with $\phi(0, 0) = \xi = (J\mathbf{z}_0, 1, 0)$ need to be considered. On one hand, it is assumed that $\mathcal{H}^{cl}_{\tau_m}$ is initially at a state $(J\mathbf{z}_0, q, 0)$, with $\mathbf{z}_0 \in \mathbb{R}^{n-n_\rho}$ and $q \in \{-1, 1\}$ (see paragraph after (22)); and, in addition, it easily follows that a solution ϕ^- to (20) with $\phi^-(0, 0) = (-J\mathbf{z}_0, -1, 0)$ is the additive inverse of the solution ϕ^+ with $\phi^+(0, 0) = (J\mathbf{z}_0, 1, 0)$, that is dom $\phi^- =$ dom ϕ^+ and $\phi^-(t, j) = -\phi^+(t, j)$ for any $(t, j) \in$ dom ϕ^+.

(*only if*) From (20)–(22) and (33) it follows that the considered solutions ϕ to $\mathcal{H}^{cl}_{\tau_m}$, with dom $\phi = [0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup \cdots$, satisfy

$$\begin{array}{l} \phi(0, 0) = (J\mathbf{z}_0, 1, 0) \\ \phi(t_1, 1) = (-J\mathbf{z}(1), -1, 0) \\ \phi(t_2, 2) = (J\mathbf{z}(2), 1, 0) \\ \cdots \end{array} \tag{35}$$

and thus $\|\phi(t_j, j)\|_{\mathcal{A}_{\mathbf{0}}} = \|J\mathbf{z}(j)\| = \|\mathbf{z}(j)\|$ for any $j = 0, 1, 2, \ldots$. Now, by contradiction, if $\{\mathbf{0}\}$ is not stable for $\mathcal{DH}^{cl}_{\tau_m}$ then there exists $\varepsilon > 0$ such that for any solution $\mathbf{z}$ to $\mathcal{DH}^{cl}_{\tau_m}$ there exist some j with $\|\mathbf{z}(j)\| > \varepsilon$, and thus $\|\phi(t_j, j)\|_{\mathcal{A}_{\mathbf{0}}} > \varepsilon$. As a result, $\mathcal{A}_{\mathbf{0}}$ is not stable for $\mathcal{H}^{cl}_{\tau_m}$. On the other hand, if $\{\mathbf{0}\}$ is not attractive for $\mathcal{DH}^{cl}_{\tau_m}$ then there will exist a sequence of values $\|\phi(t_j, j)\|_{\mathcal{A}_{\mathbf{0}}} = \|\mathbf{z}(j)\|$, for $j = 1, 2, \ldots$ which does not converge to zero, and thus $\mathcal{A}_{\mathbf{0}}$ will not be attractive for $\mathcal{H}^{cl}_{\tau_m}$.

(*if*) From the flow equation in (20), for a solution $\phi = (\mathbf{x}, q, \tau)$ with dom $\phi = [0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup \cdots$, it directly follows that

$$\mathbf{x}(t, j) = e^{A(t-t_j)} \mathbf{x}(t_j, j) \tag{36}$$

for any $(t, j) \in$ dom ϕ, and thus there exist numbers $M > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|\mathbf{x}(t, j)\| \leq \|\mathbf{x}(t_j, j)\| M e^{\alpha(t-t_j)} = \|(-1)^{j+1} J\mathbf{z}(j)\| M e^{\alpha(t-t_j)} = \|\mathbf{z}(j)\| M e^{\alpha(t-t_j)} \tag{37}$$

and thus

$$\|\phi(t,j)\|_{\mathcal{A}_\mathbf{0}} \leq \|\mathbf{z}(j)\| M e^{\alpha(t-t_j)} \tag{38}$$

Now, if $\{\mathbf{0}\}$ is stable for $\mathcal{DH}^{cl}_{\tau_m}$ then it is true that there exists $\gamma > 0$ such that $\|\mathbf{z}(j)\| \leq \gamma\|\mathbf{z}_0\| = \gamma\|\xi\|_{\mathcal{A}_\mathbf{0}}$ and thus (38) results in (redefining $M\gamma$ as γ)

$$\|\phi(t,j)\|_{\mathcal{A}_\mathbf{0}} \leq \gamma e^{\alpha(t-t_j)}\|\xi\|_{\mathcal{A}_\mathbf{0}} \tag{39}$$

Now, from Assumption A it follows that either $\alpha < 0$ (when A is a Hurwitz matrix) or $t - t_j < \tau_M$ for any $(t,j) \in \text{dom}\,\phi$, and thus stability of the set $\mathcal{A}_\mathbf{0}$ for $\mathcal{H}^{cl}_{\tau_m}$ directly follows. Asymptotic stability is obtained from the fact that for the discrete-time system $\mathcal{DH}^{cl}_{\tau_m}$ it is true that $\|\mathbf{z}(j)\| < \gamma\lambda^j\|\mathbf{z}_0\|$, for $j = 1, 2, \ldots$, and for any $\mathbf{z}_0 \in \mathbb{B}$ and $0 \leq \lambda < 1$ (here $\mathbb{B} \subset \mathbb{R}^{n_p+n_{\hat{\rho}}}$ is a ball centered at the origin). Substituting in (38) and using again the fact that $\|\xi\|_{\mathcal{A}_\mathbf{0}} = \|\mathbf{z}_0\|$, it results that

$$\|\phi(t,j)\|_{\mathcal{A}_\mathbf{0}} \leq \gamma\lambda^j e^{\alpha(t-t_j)}\|\xi\|_{\mathcal{A}_\mathbf{0}} \tag{40}$$

and thus, since either $\alpha < 0$ or $t - t_j < \tau_M$, it directly follows that $\|\phi(t,j)\|_{\mathcal{A}_\mathbf{0}} \to 0$ as $t + j \to \infty$, where $(t,j) \in \text{dom}\,\phi$, and $\xi \in (\mathbb{B} \times \{\mathbf{0}\}) \times \{1\} \times \{0\}$. As a result, global asymptotic stability of $\mathcal{A}_\mathbf{0}$ for $\mathcal{H}^{cl}_{\tau_m}$ comes from the global asymptotic stability of $\{\mathbf{0}\}$ for $\mathcal{DH}^{cl}_{\tau_m}$. $\square$

4.1. Stability based on periods of the reset interval sequences

It is well known that reset interval sequences have a particularly simple structure in some relevant cases in control practice [32]. For $n_r = n_\rho$ (full reset) and a second-order plant, $n_p = 2$, reset interval sequences are periodic with some fundamental period $\Delta > 0$, after the second reset (this is also the case of Example 3.3 with no exogenous inputs as long as τ_m is small enough). Note that in these cases, stability can be easily checked by determining whether $A_R e^{A\Delta}$ is a Schur matrix.

This section will look at how periodic reset interval patterns can be used to obtain stability criteria for the reset control system $\mathcal{H}^{cl}_{\tau_m}$. It is defined the *angle mapping* $\Pi_g : S^{n-n_\rho-1} \to S^{n-n_\rho-1}$ as

$$\mathbf{s}^+ = \Pi_g(\mathbf{s}) = \frac{g(\mathbf{s})}{\|g(\mathbf{s})\|} \tag{41}$$

with g given by (33). Here, $\mathbf{s}$ is the projection of $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$ on the unit $(n - n_\rho - 1)$-sphere $S^{n-n_\rho-1}$, and thus the mapping Π_g will produce orbits of those projections. A natural form of analyzing periodic interval sequences of the reset control system $\mathcal{H}^{cl}_{\tau_m}$ is by analyzing periodic points of Π_g. These points will define the periodic structure of the reset interval sequences, allowing the development of an asymptotic stability criterion.

Some definitions about stability of periodic points follow (see, for example, [36] and [28] for technical details). Firstly, $\Pi_g^k(\mathbf{s})$ is defined to be the result of applying k times Π_g to the point $\mathbf{s}$. The *orbit* of $\mathbf{s}$ under Π_g is the set of points $\{\mathbf{s} = \Pi_g^0(\mathbf{s}), \Pi_g(\mathbf{s}), \Pi_g^2(\mathbf{s}), \ldots, \}$. $\mathbf{p}$ is a *periodic-k point* if $\Pi_g^k(\mathbf{p}) = \mathbf{p}$ and if k is the smaller such positive integer; and the orbit of $\mathbf{p}$ with k points, that is $\{\mathbf{p}, \Pi_g(\mathbf{p}), \ldots, \Pi_g^{k-1}(\mathbf{p})\}$, is called a *periodic-k orbit*. For $k = 1$, $\mathbf{p}$ is referred to as a *fixed point*. Assume that Π_g is differentiable in a neighborhood U of a fixed point $\mathbf{p}$ and let $\mathbf{D}\Pi_g(\mathbf{p})$ be the Jacobian matrix of Π_g at $\mathbf{p}$; the fixed point $\mathbf{p}$ is called a *sink* if $\mathbf{D}\Pi_g(\mathbf{p})$ is a Schur matrix, and a *source* if all eigenvalues of $\mathbf{D}\Pi_g(\mathbf{p})$ have a magnitude greater than 1. The stable manifold of $\mathbf{p}$, denoted as $\mathcal{S}(\mathbf{p})$, is the set of points $\mathbf{s} \in S^{n-n_\rho-1}$ such that $\|\Pi_g^k(\mathbf{s}) - \mathbf{p}\| \to 0$ as $k \to \infty$. Analogously, for a periodic-k point $\mathbf{p}$, its periodic-k orbit is a *sink* (*source*) if $\mathbf{p}$ is a sink (source) for the map Π_g^k.

Proposition 4.2. *Assume that the angle map Π_g has a periodic-k point $\mathbf{p}$, being Π_g^k differentiable in a neighborhood U of $\mathbf{p}$, and that its periodic-k orbit is a sink with stable manifold $\mathcal{S}(\mathbf{p}) = S^{n-n_\rho-1}$. Then, $\mathbf{p}$ is an eigenvector of the matrix*

$$M_\mathbf{p} := J^\top e^{A\cdot I(\Pi_g^{k-1}(\mathbf{p}))} \cdots A_R e^{A\cdot I(\Pi_g(\mathbf{p}))} A_R e^{A\cdot I(\mathbf{p})} J \tag{42}$$

corresponding to a real eigenvalue $\lambda_\mathbf{p}$, and the set $\mathcal{A}_\mathbf{0}$ is asymptotically stable for the reset control system $\mathcal{H}^{cl}_{\tau_m}$ if and only if $|\lambda_\mathbf{p}| < 1$. Moreover, the basin of attraction of $\mathcal{A}_\mathbf{0}$ is $\mathbb{R}^n \times \{1, -1\} \times [0, \infty)$ (stability is global).

Proof. Consider the discrete system $\mathcal{DH}^{cl}_{\tau_m}$ as given by (33), $\mathbf{z}_0 \in \mathbb{R}^{n-n_\rho}$, $\mathbf{z}_j = g^j(\mathbf{z}_0)$, and also $\mathbf{s}_0 = \frac{\mathbf{z}_0}{\|\mathbf{z}_0\|}$, and $\mathbf{s}_j = \Pi_g^j(\mathbf{s}_0)$, for $j = 1, 2, \ldots$. From the homogeneity property (34), and (41), it directly follows that

$$I(\mathbf{z}_j) = I(\mathbf{s}_j) \tag{43}$$

for $j = 0, 1, 2, \ldots$. The proof is particularized for the case in which $\mathbf{p}$ is a fixed point of the angle map Π_g (that is $k = 1$); for the cases $k = 2, 3, \ldots$, the proof is similar, using Π_g^k instead of Π_g in the following reasoning. Now, define the matrix functions $M, \delta M : \mathbb{R}^{n-n_\rho} \to \mathbb{R}^{(n-n_\rho)\times(n-n_\rho)}$, such that $M(\mathbf{z}) = J^\top e^{AI(\mathbf{z})}J$, and thus $M(\mathbf{p}) = M_\mathbf{p}$ as given by (42), and $\delta M(\mathbf{z}) = M(\mathbf{z}) - M_\mathbf{p}$.

Firstly, it will be shown that $\mathbf{p}$ is an eigenvector of $M_{\mathbf{p}}$. Since $\mathbf{p}$ is a sink with stable manifold the whole sphere, then it is true that $\|\mathbf{s}_j - \mathbf{p}\| \to 0$ as $j \to \infty$, for any $\mathbf{s}_0 \in \mathbb{S}^{n-n_\rho}$. Moreover, since the mapping I is continuous at $\mathbf{p}$ (otherwise Π_g would not be differentiable), and thus the mapping M is also continuous at $\mathbf{p}$, it also follows that $\|I(\mathbf{s}_j) - I(\mathbf{p})\| \to 0$ and finally

$$\|M(\mathbf{s}_j) - M_{\mathbf{p}}\| \to 0 \tag{44}$$

as $j \to \infty$, for any $\mathbf{s}_0 \in \mathbb{S}^{n-n_\rho}$. Then, by directly using (33) and (41), two points $\mathbf{s}_j$ and $\mathbf{s}_{j+1} = \Pi_g(\mathbf{s}_j)$ are related by

$$\mathbf{s}_{j+1} = -\frac{M(\mathbf{s}_j)\mathbf{s}_j}{\|M(\mathbf{s}_j)\mathbf{s}_j\|} \tag{45}$$

or equivalently

$$M(\mathbf{s}_j)\mathbf{s}_j = -\|M(\mathbf{s}_j)\mathbf{s}_j\|\mathbf{s}_{j+1} \tag{46}$$

and finally for $j \to \infty$ it results that

$$M_{\mathbf{p}}\mathbf{p} = -\|M_{\mathbf{p}}\mathbf{p}\|\mathbf{p} \tag{47}$$

that is $\mathbf{p}$ is an eigenvector of $M_{\mathbf{p}}$ with eigenvalue $\lambda_{\mathbf{p}} := -\|M_{\mathbf{p}}\mathbf{p}\|$, a (non-positive) real number.

Secondly, consider the (unique) orthogonal decomposition of $\mathbf{z}_j$ as

$$\mathbf{z}_j = \alpha_j\mathbf{p} + \alpha_j\delta_j\mathbf{p}^{\perp} \tag{48}$$

where $\mathbf{p}^{\perp}$ is a vector perpendicular to $\mathbf{p}$, and α_j and δ_j are real numbers, for any $j = 1, 2, \ldots$. Since it is true that $\|\mathbf{s}_j - \mathbf{p}\| \to 0$ as $j \to \infty$, then it directly follows that $\delta_j \to 0$ as $j \to \infty$. In addition, consider two points $\mathbf{z}_j$ and $\mathbf{z}_{j+1} = g(\mathbf{z}_j)$, for $j = 1, 2, \ldots$. It results that

$$\frac{\|\mathbf{z}_{j+1}\|}{\|\mathbf{z}_j\|} = \frac{\|-M(\mathbf{z}_j)\mathbf{z}_j\|}{\|\mathbf{z}_j\|} = \frac{\|(M_{\mathbf{p}} + \delta M_j)(\alpha_j\mathbf{p} + \alpha_j\delta_j\mathbf{p}^{\perp})\|}{\|\alpha_j\mathbf{p} + \alpha_j\delta_j\mathbf{p}^{\perp}\|} \to \frac{\|M_{\mathbf{p}}\mathbf{p}\|}{\|\mathbf{p}\|} = |\lambda_{\mathbf{p}}| \tag{49}$$

for $j \to \infty$, where by definition $\delta M_j = \delta M(\mathbf{z}_j)$, and it has been used the fact that $\|\delta M_j\| \to 0$ as $j \to \infty$ (it easily follows from (44)).

(*if*) If $|\lambda_{\mathbf{p}}| < 1$ then from (49) if follows that for some real number λ such that $0 \le |\lambda_{\mathbf{p}}| < \lambda < 1$, there exist an integer N such that $\|\mathbf{z}_{j+1}\| = \|M(\mathbf{z}_j)\mathbf{z}_j\| < \lambda\|\mathbf{z}_j\|$, for $j \ge N$, and this is true for any point $\mathbf{z}_0$. Since, from Assumption A, there exists $M_{\max} > 0$ such that $\|M(\mathbf{z})\| = \|J^T e^{AI(\mathbf{z})} J\| \le M_{\max}$ for any $\mathbf{z} \in \mathbb{R}^{n-n_\rho}$, then it results that

$$\|\mathbf{z}_j\| \le \|M(\mathbf{z}_{j-1})\|\|M(\mathbf{z}_{j-2})\|\cdots\|M(\mathbf{z}_N)\|\|M(\mathbf{z}_{N-1})\|\cdots\|M(\mathbf{z}_0)\|\|\mathbf{z}_0\| \le \lambda^{j-N}M_{\max}^N\|\mathbf{z}_0\| \tag{50}$$

that is

$$\|\mathbf{z}_j\| \le \gamma\lambda^j\|\mathbf{z}_0\| \tag{51}$$

for $j \ge N$, where $\gamma = (M_{\max}/\lambda)^N$ is a constant. It is also true that $\|\mathbf{z}_j\| \le (M_{\max})^j\|\mathbf{z}_0\|$, for $j < N$. It easily follows that the origin is globally asymptotically stable for $\mathcal{DH}_{\tau m}^{cl}$ and Proposition 4.1 certifies global asymptotic stability of $\mathcal{A}_{\mathbf{0}}$ for the reset control system $\mathcal{H}_{\tau m}^{cl}$.

(*only if*) Consider an initial condition $\mathbf{z}_0 = \alpha_0\mathbf{p}$, where α_0 is a non-zero arbitrary number. On the other hand, $M(c\mathbf{p}) = M(\mathbf{p})$ for any constant c. Thus, $\mathbf{z}_1 = -M(\mathbf{z}_0)\mathbf{z}_0 = -M(\alpha_0\mathbf{p})\alpha_0\mathbf{p} = -\alpha_0 M(\mathbf{p})\mathbf{p} = -\alpha_0\lambda_{\mathbf{p}}\mathbf{p}$, and it is obtained that $\mathbf{z}_j = (-1)^j\alpha_0\lambda_{\mathbf{p}}^j\mathbf{p}$ for $j = 1, 2, \ldots$. Stability of $\mathcal{A}_{\mathbf{0}}$ for $\mathcal{H}_{\tau m}^{cl}$ implies stability of the origin for $\mathcal{DH}_{\tau m}^{cl}$, and this implies that $|\lambda_{\mathbf{p}}| < 1$. □

Note that, in general, the angle map Π_g may exhibit several periodic-k points, not necessarily sinks, with different integer values of $k \ge 1$, each with its own stable manifold. For example, in the case in which the point is a source, the stable manifold is the point itself. The following result follows using similar arguments to the proof of Proposition 4.2

Corollary 4.3. *Assume that the angle map Π_g has a finite number of periodic points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, with periods $k_1, k_2, \ldots, k_n$, respectively, and that their stable manifolds satisfy*

$$\bigcup_{i=1}^{n} \mathcal{S}(\mathbf{p}_i) = S^{n-n_\rho-1}. \tag{52}$$

Then, for each $i = 1, 2, \ldots, n$, the point $\mathbf{p}_i$ is an eigenvector of the matrix

$$M_{\mathbf{p}_i} = A_R e^{A\cdot I(\Pi_g^{k_i-1}(\mathbf{p}_i))} \cdots A_R e^{A\cdot I(\Pi_g(\mathbf{p}_i))} A_R e^{A\cdot I(\mathbf{p}_i)} \tag{53}$$

corresponding to a real eigenvalue $\lambda_{\mathbf{p}_i}$, and the set $\mathcal{A}_{\mathbf{0}}$ is asymptotically stable for the reset control system $\mathcal{H}_{\tau m}^{cl}$ if and only if $|\lambda_{\mathbf{p}_i}| < 1$ for any $i = 1, 2, \ldots, n$. Moreover, the basin of attraction of $\mathcal{A}_{\mathbf{0}}$ is $\mathbb{R}^n \times \{1, -1\} \times [0, \infty)$ (stability is global).

Note that the angle map Π_g defines a (nonlinear) discrete dynamic system given by (40), and its periodic points are simply the fixed points of the maps Π_g (angle map), Π_g^2, Π_g^3, etc. Thus, for application of Proposition 4.2 or Corollary 4.3, first it is needed to compute the fixed points of those maps. This is a basic computational issue in the field of discrete dynamical systems; see for example [30,36]. Notice that the angle map Π_g is defined on an $m-$dimensional sphere, where $m = n - n_\rho - 1$. For small values of m like $m = 1$ (the case of Section 4.2) or $m = 2$ (the case of Section 4.3) fixed points can be obtained without much computational effort. Note that these cases correspond to relevant practical cases, for example, $m = 1$ may correspond to a case with a second order plant and a full reset controller of arbitrary order. In general, for larger values of m the computation of fixed points is more involved. However, note that although the computational burden is a drawback, this is the price paid for having a *necessary and sufficient* stability condition. In the following, several cases are analyzed in detail.

4.2. A case study with the Horowitz reset controller

For the Example 3.3 without exogenous inputs, the reset control system $\mathcal{H}^{cl}_{\tau_m}$ is given by the matrices A, A_R and C obtained by removing the first column and the first row of the matrices in (23), that is

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -4 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \tag{54}$$

and with a minimum dwell-time τ_m. Here, stability of the set $\mathcal{A}_{\mathbf{0}} = \{(0, 0, 0)\} \times \{1, -1\} \times [0, \infty)$ for the reset control system $\mathcal{H}^{cl}_{\tau_m}$ is analyzed. Firstly, Assumption A must be checked. It easily follows since although A has eigenvalues $\lambda_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{19}}{2}$ and $\lambda_3 = 0$, and thus A is not a Hurwitz matrix, the reset intervals upper bound is $\tau_M = \tau_m + \frac{2\pi}{\sqrt{19}}$ (details are omitted by brevity). Now, for application of Proposition 4.1/Cor. 4.2, firstly note that $m = n - n_\rho - 1 = 1$, and thus Π_g maps values of $\mathbf{s}$ in the unit circle S^1. By simplicity, using the parametrization $\mathbf{s} = (\cos(\theta), \sin(\theta))$, for $\theta \in (-\pi, \pi]$, it is possible to redefine the angle map (with some abuse of notation) as $\Pi_g : (-\pi, \pi] \to (-\pi, \pi]$, and $\theta^+ = \Pi_g(\theta)$, with $\mathbf{s} = (\cos(\theta), \sin(\theta))$ and $\mathbf{s}^+ = (\cos(\theta^+), \sin(\theta^+))$ in (40), and from where periodic points at $\theta = p_i$, $i = 1, \ldots, n$ can be easily obtained.

Moreover, for the computation of the matrices M_{p_i}, $i = 1, \ldots, n$, the evaluation of I at the periodic points of Π_g at S^1 may also be simplified in an analogous manner, by redefining the mapping as $I : (-\pi, \pi] \to \mathbb{R}_{\geq 0}$ obtaining $I(\theta)$ with $\mathbf{z} = (\cos(\theta), \sin(\theta))$ in (31). Fig. 8 depicts the graphs of Π_g and I for two different values of τ_m.

For $\tau_m = 0.25$, Π_g has a unique periodic point, it is a fixed point at $p = \pi/2$, which is a sink with a basin of attraction $\mathcal{S}(\pi/2) = (-\pi, \pi]$ (corresponding to the unit circle). Direct application of Proposition 4.2 results in the reset control system $\mathcal{H}^{cl}_{0.25}$ being globally asymptotically stable, since $I(p) = I(\pi/2) = 2\pi/\sqrt{19}$ and

$$M_p = J^\top e^{A \cdot I(p)} J = \begin{pmatrix} -0.4863 & 0 \\ 0 & -0.1891 \end{pmatrix} \tag{55}$$

has an eigenvector $(\cos(p), \sin(p)) = (0, 1)$ with corresponding eigenvalue -0.1891.

For $\tau_m = 1.35$, the mapping Π_g exhibits a more complex structure (Fig. 8). Π_g has three fixed points: $p_1 \approx 0.5322$, $p_2 \approx 1.4246$, and $p_3 = \pi/2$. Both p_1 and p_3 are sinks, while p_2 is a source. Their basins of attraction are $\mathcal{S}(p_1) = [d, p_2)$, $\mathcal{S}(p_2) = \{p_2\}$ and $\mathcal{S}(p_3) = (-\pi, d) \cup (p_2, \pi]$, where both Π_g and I have a jump discontinuity at $\theta = d$ (see Fig. 8-right). Finally, $I(p_1) = I(p_2) = \tau_m = 1.35$ and $I(p_3) = 2\pi/\sqrt{19}$. As a result, since $\mathcal{S}(p_1) \cup \mathcal{S}(p_2) \cup \mathcal{S}(p_3) = (-\pi, \pi]$, and the three matrices

$$M_{p_1} = J^\top e^{A \cdot I(p_1)} J = \begin{pmatrix} -0.5222 & 0.0463 \\ -0.1850 & -0.1808 \end{pmatrix}, \tag{56}$$

$M_{p_2} = M_{p_1}$, and $M_{p_3} = A_R e^{A \cdot I(p_3)} = M_p$ (as given in (55)) are Schur matrices (and thus all eigenvalues are strictly inside the unit circle), it also follows that $\mathcal{H}^{cl}_{1.35}$ is globally asymptotically stable.

As expected, τ_m has a strong influence on the periodic point patterns. Although, in principle, a small value τ_m is all that is required to avoid defective solutions in control practice, and following the above analysis, it is not difficult to show that $\mathcal{H}^{cl}_{\tau_m}$ is globally asymptotically stable for any $\tau_m < 1.35$, it is illustrative to analyze how the periodic point patterns change for increasing values of it. Although an exhaustive analysis is out of scope of this work and it will be given elsewhere, there are several bifurcation points delimiting zones with one sink, with two sinks plus a source, with periodic-2 sinks, with periodic-3 sinks, etc. To conclude the example, a case with periodic-3 sinks is analyzed in the following.

For $\tau_m = 2$, there are three periodic-3 points (Fig. 9). They are $p_1 \approx -1.5494$, $p_2 \approx -0.2162$, and $p_3 = \pi/2$. For p_1, its periodic-3 orbit $\{p_1, \Pi_g(p_1), \Pi_g^2(p_1)\} = \{p_1, p_3, p_2\}$ is a sink and, in addition, $I(p_1) \approx 2.9042$, and $I(p_3) = I(p_2) = \tau_m = 2$, and

$$M_{p_1} = J^\top e^{A \cdot I(p_2)} A_R e^{A \cdot I(p_3)} A_R e^{A \cdot I(p_1)} J = 10^{-2} \cdot \begin{pmatrix} -2.2711 & 0.0337 \\ 0.0011 & -3.8597 \end{pmatrix}, \tag{57}$$

is a Schur matrix. For p_2 and p_3, its periodic-3 orbits are $\{p_2, p_1, p_3\}$ and $\{p_3, p_2, p_1\}$, respectively. They are also sinks, and the matrices $M_{p_2} = J^\top e^{A \cdot I(p_3)} A_R e^{A \cdot I(p_1)} A_R e^{A \cdot I(p_2)} J$, and $M(p_3) = J^\top e^{A \cdot I(p_1)} A_R e^{A \cdot I(p_2)} A_R e^{A \cdot I(p_3)} J$ can be easily checked to be Schur matrices. Finally, applying Corollary 4.3, and using the fact that the union of the three basins of attraction is $(-\pi, \pi]$, it results that $\mathcal{H}^{cl}_2$ is globally asymptotically stable.

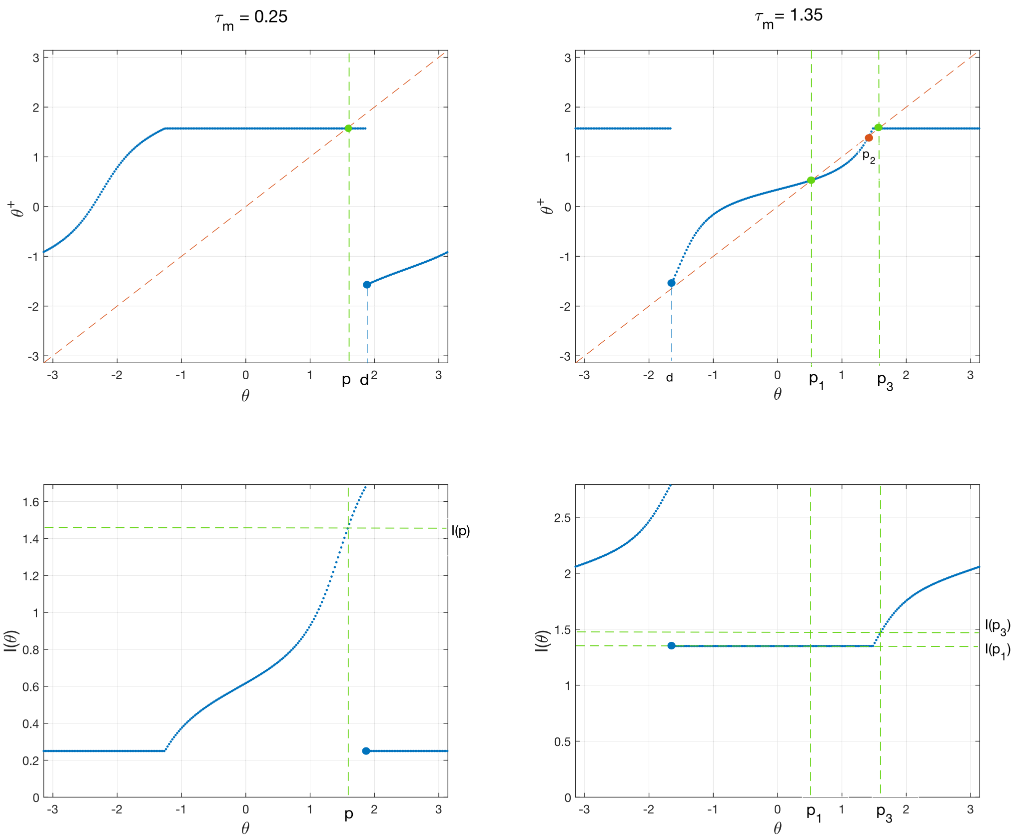


Fig. 8. Graphs of the maps Π_g (*top*) and I (*bottom*) for the reset control system of Example 3.3: (*left*) $\tau_m = 0.25$ and $\tau_M \approx 1.69$; Π_g has only a periodic point (a fixed point) at $p = \frac{\pi}{2}$, and a jump discontinuity at $d = \pi + \frac{1}{2}(1 - \frac{\sqrt{19}}{\tan(\frac{\sqrt{19}}{8})})$; (*right*) $\tau_m = 1.35$ and $\tau_M \approx 2.79$; Π_g has three fixed points at $p_1 \approx 0.5322$, $p_2 \approx 1.4246$, and $p_3 = \pi/2$, and a jump discontinuity at $d = -\pi + \frac{1}{2}(1 - \frac{\sqrt{19}}{\tan(\frac{\sqrt{19}}{2}\cdot 1.35)})$.

4.3. A case with chaotic sequences of reset intervals

Obviously, Proposition 4.2–Corollary 4.3 are useful in practice for those cases in which periodic orbits of Π_g can be found with a reasonable effort, like in the cases analyzed above. Although this is the case in many practical cases, even some low-order reset control systems may exhibit extraordinarily complex periodic point patterns that make their application elusive, motivating the investigation of alternative stability criteria. The following describes a reset control system consisting of a FORE and a third-order plant that produces chaotic sequences of reset intervals.

Consider the time-regularized reset control system $\mathcal{H}^{\mathrm{cl}}_{0.1}$, as given by (20) with $\tau_m = 0.1$ and

$$A = \begin{pmatrix} 0 & 0 & 3.5 & 5 \\ 1 & 0 & -4.3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}, \tag{58}$$

This reset control system, with state $(\mathbf{x}, q, \tau) = (\mathbf{x}_p, x_r, q, \tau)$, is the feedback composition of a FORE with state (x_r, q) and a third order plant with state $\mathbf{x}_p = (x_1, x_2, x_3)$. Here, the mapping $\Pi_g : S^2 \to S^2$ has an invariant set $\mathcal{T} = \{(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0) \in S^2 : t \in [-3, 4]\}$, that is $\Pi_g(\mathcal{T}) \subset \mathcal{T}$.

Again, the maps Π_g and I are particularized to $\mathcal{T}$ and reparameterized (with some abuse of notation) as $\Pi_g : [-3, 4] \to [-3, 4]$, and $I : [-3, 4] \to \mathbb{R}_{\geq 0}$ (their graphs are shown Fig. 10-left), being a point of the circle $\mathcal{T}$ represented by t instead of $\mathbf{s} = (\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0)$. In this case, since the mapping Π_g is continuous on the interval $[-3, 4]$ and has periodic-3 points (see Fig. 10), then it turns out that Π_g has periodic-k points for any $k = 1, 2, 3, \ldots$, according to Sharkovskii theorem [36,37]. For example, Fig. 10 shows the graphs of Π_g, Π_g^2, and Π_g^3, explicitly marking 1 periodic-1 point (a fixed point), 2 periodic-2 points, and 6 periodic-3 points. Note that all the periodic points are sources, and in fact this is the case for any periodic point since, as it is well known, period 3 implies chaos [27]. As a result, any initial point of $\mathcal{H}^{\mathrm{cl}}_{0.1}$, with

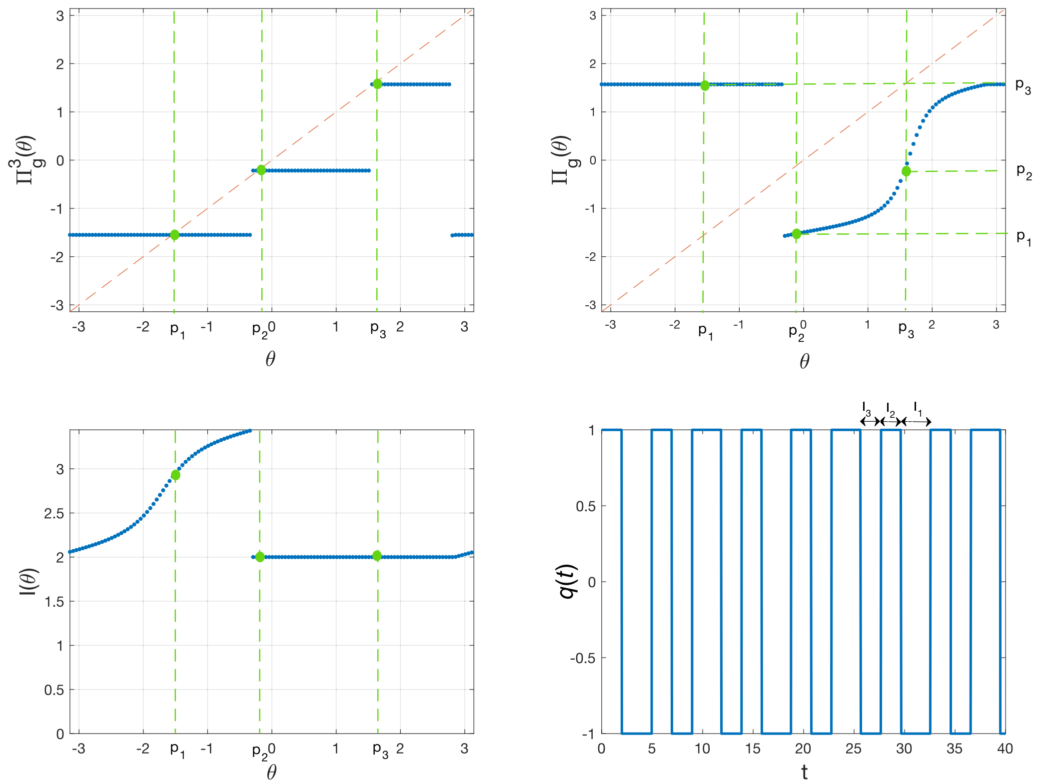


Fig. 9. Reset control system of Example 3.3 for $\tau_m = 2$: (*top*) Graph of the mapping Π_g^3 (*left*), showing the periodic-3 points p_1, p_2, and p_3, and graph of Π_g (*right*); (*bottom*) graph of the map I (*left*), and plot of $q(t) \in \{1, -1\}$ showing the periodicity of the reset intervals (*rigth*), where $I_3 = I(p_3) = 2$, $I_2 = I(p_2) = 2$, and $I_1 = I(p_1) \approx 2.9042$.

$\xi \in (\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0, 0) \times \{1\} \times \{0\}$, and $t \in [-3, 4]$, will produce solutions with chaotic sequences of reset intervals, making elusive a direct application of Proposition 4.2–Corollary 4.3.

4.4. Reset-times dependent stability conditions: Minimum dwell-time

An alternative approach to stability analysis of $\mathcal{H}_{\tau_m}^{cl}$ will be based on the use of Lyapunov functions, with an explicit consideration of the reset interval sequences $\tau_\phi = \{\tau_1, \tau_2, \ldots\}$ corresponding to solutions $\phi = (\mathbf{x}, q, \tau)$ to $\mathcal{H}_{\tau_m}^{cl}$. This approach is based on the reset-times dependent stability criteria early developed in [5,32]. The set of all possible reset interval sequences is defined as

$$S_{\mathcal{H}_{\tau_m}^{cl}} = \{\tau_\phi = \{\tau_1, \tau_2, \ldots\} \subset \mathbb{R}_{\geq 0} : \tau_i = \tau(t_i, i), (t_i, i) \in \text{ dom } \phi, \phi \text{ is a solution to } \mathcal{H}_{\tau_m}^{cl}\} \tag{59}$$

When A is a Hurwitz matrix, one strategy is to embed the set of reset interval sequences in a larger set S_{τ_m} characterized by the minimum dwell-time τ_m associated to $\mathcal{H}_{\tau_m}^{cl}$. It is defined as

$$S_{\tau_m} = \{\{\tau_1, \tau_2, \ldots\} \subset \mathbb{R}_{\geq 0} : \tau_i \geq \tau_m, i = 1, 2, \ldots\} \tag{60}$$

and then stability conditions are considered for any possible reset interval sequence in S_{τ_m}. This approach will allow a direct application of computationally efficient methods imported from the impulsive systems literature. And although, in principle, results may be conservative due to the fact that $S_{\mathcal{H}_{\tau_m}^{cl}}$ is a meager set compared to S_{τ_m}, since it is clear that $S_{\tau_m} \supset S_{\mathcal{H}_{\tau_m}^{cl}}$, in practice they may permit the computation of a first value of τ_m for which stability is guaranteed.

Proposition 4.4. *Consider the reset control system $\mathcal{H}_{\tau_m}^{cl}$ with a Hurwitz matrix A. The set $\mathcal{A}_0$ is globally asymptotically stable for $\mathcal{H}_{\tau_m}^{cl}$ if there exist a sequence of positive definite matrices $\{P_1, P_2, \ldots\}$, such that*

$$\begin{aligned} &\eta I \leq P_k \leq \rho I \\ &e^{A^T \tau_k} A_R P_{k+1} A_R e^{A\tau_k} - P_k \leq -\varepsilon I \end{aligned} \tag{61}$$

holds for $k = 1, 2, \ldots$, for some positive constants η, ρ, and ε, and any $\{\tau_1, \tau_2, \ldots\} \in S_{\tau_m}$.

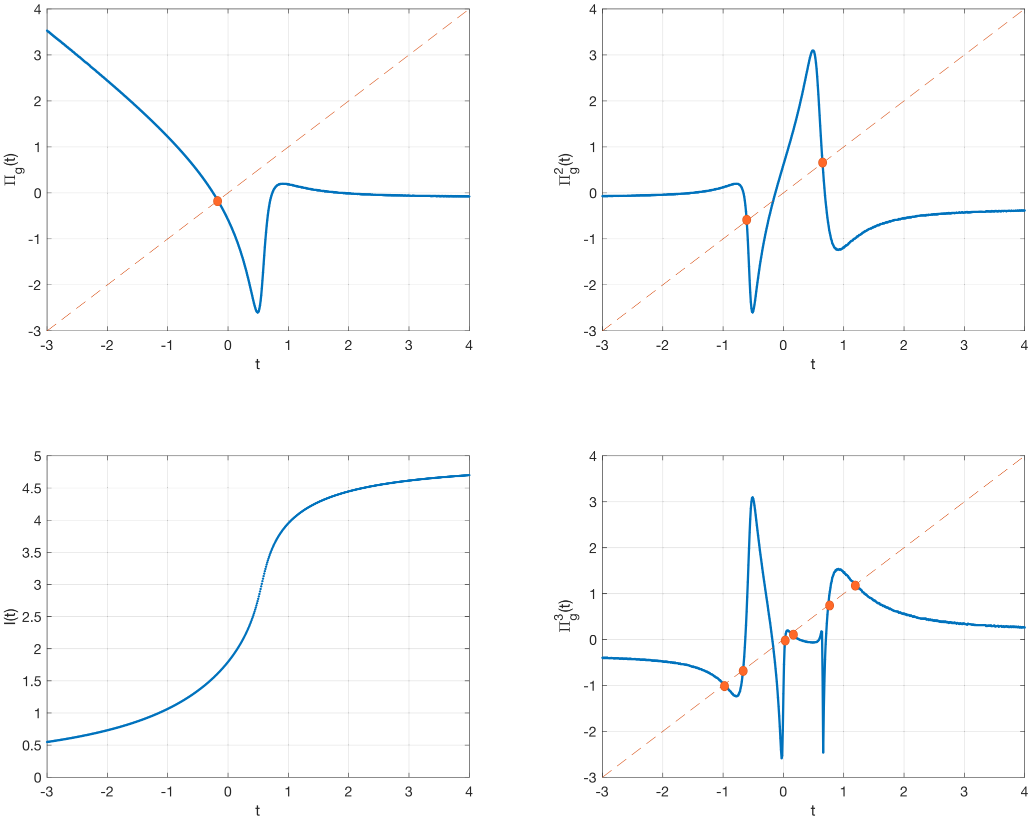


Fig. 10. An example of reset control system with a chaotic sequence of reset intervals: graphs of the mapps Π_g, Π_g^2, and Π_g^3 (with marks at the periodic points), and I.

Proof. Firstly, it is a standard result [30], Th. 23.3) that if (61) holds then the time-dependent quadratic Lyapunov function $V(\mathbf{z},k) = (J\mathbf{z})^T P_k (J\mathbf{z})$, certifies that every (time-varying) discrete-time system

$$\begin{cases} \mathbf{z}^+ = g(\mathbf{z}) = -J^T A_R e^{A\tau_k} J\mathbf{z} \\ k^+ = k+1 \end{cases} \tag{62}$$

with $\{\tau_1, \tau_2, \ldots\} \in S_{\tau_m}$, is globally asymptotically stable.

Now, since it is true that $I(\mathbf{z}) > \tau_m$ for any $\mathbf{z} \in \mathbb{R}^{n_p+n_{\hat{\rho}}}$ (it directly follows from its definition in (32)), and thus the solution $\mathbf{z}$ to $\mathcal{DH}^{cl}_{\tau_m}$ with $\mathbf{z}(0) = \mathbf{z}_0$ corresponds with the solution to a discrete-time system like (62) with $\{\tau_1, \tau_2, \tau_3, \ldots\} = \{I(\mathbf{z}_0), I(g(\mathbf{z}_0)), I(g^2(\mathbf{z}_0)), \ldots\} \in S_{\tau_m}$, then it follows from (61) that (see [30], Th. 23.3)

$$\|\mathbf{z}(k)\|^2 = \|g^k(\mathbf{z}_0)\|^2 \leq \frac{\rho}{\eta}\lambda^{2k}\|\mathbf{z}_0\|^2 \tag{63}$$

for $k \geq 0$, and for some $\lambda < 1$. Since the constants ρ, η, and λ do not depend on $\mathbf{z}_0$ then it directly follows that the origin $\{\mathbf{0}\}$ is globally asymptotically stable for the discrete time system $\mathcal{DH}^{cl}_{\tau_m}$. Application of Proposition 4.1 ends the proof. □

Proposition 4.4 gives a nice and simple connection between stability of the reset control system $\mathcal{H}^{cl}_{\tau_m}$ and stability of impulsive systems with impulses at fixed instants, since condition (61) can be easily linked with the stability of an impulsive system with impulses at instants $t_k = t_{k-1} + \tau_k$, $k = 1, 2, \ldots$. Moreover, the following Corollary directly follows.

Corollary 4.5. *Assume that (61) hold for $k = 1, 2, \ldots$, for η, ρ, $\varepsilon > 0$, and for any $\{\tau_1, \tau_2, \ldots\} \in S_{\tau_m^\star}$. Then the set $\mathcal{A}_0$ is globally asymptotically stable for $\mathcal{H}^{cl}_{\tau_m}$, for any $\tau_m \geq \tau_m^*$.*

A particular simple instance of (61) is obtained by considering a time independent Lyapunov function, that is $P_k = P$, for any $k = 1, 2, \ldots$. In this case, the procedure for solving (61) is reduced to searching for a matrix $P > 0$ such that

$$e^{A^T\tau} A_R P A_R e^{A\tau} - P \leq -\varepsilon I \tag{64}$$

for some $\varepsilon > 0$ and any $\tau \geq \tau_m^*$. This problem is well-known in the literature and there are a number of effective approaches for resolving it [5,25,32,38]. The next result is directly imported from [38].

Corollary 4.6. *Consider the reset control system $\mathcal{H}^{cl}_{\tau_m}$ given by (20). Assume that there exist a differentiable matrix function $R : [0, \tau_m^*] \to \mathbb{S}^n$, $R(0) > 0$, and $\varepsilon > 0$ such that*

$$\begin{aligned} &A^T R(0) + R(0)A < 0 \\ &A^T R(\theta) + R(\theta)A + \dot{R}(\theta) \le 0, \\ &A_R R(0) A_R - R(\tau_m^*) \le -\varepsilon I, \end{aligned} \tag{65}$$

hold for any $\theta \in [0, \tau_m^*]$. Then, the set $\mathcal{A}_\mathbf{0}$ is globally asymptotically stable for $\mathcal{H}^{cl}_{\tau_m}$, for any $\tau_m \ge \tau_m^*$.

This Corollary provides an efficient method for solving the $\mathcal{H}^{cl}_{\tau_m}$ stability problem, particularly when searching for the matrix function R in the set of matrix polynomials with a given degree d_R, that is when $R(\theta) = \sum_{i=0}^{d_R} R_i \theta^i$, $R_i > 0$. In this case, (65) may be inserted in sum-of-squares conditions, and the problem can be efficiently solved by using some sum-of-squares programming package (e.g. SOSTOOLS [29]).

Example 4.7. This is a classical example of a reset control system, considered in early works about reset control like [1]. It consists of a feedback interconnection between a FORE and a second order plant. Here, it is defined as a time-regularized reset control system $\mathcal{H}^{cl}_{\tau_m}$ with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -0.2 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix} \tag{66}$$

and a minimum dwell-time τ_m. Since A is a Hurwitz matrix, in fact it has two complex dominant eigenvalues at $-\frac{1}{10} \pm j\frac{3\sqrt{11}}{10}$, stability of the set $\mathcal{A}_\mathbf{0} = \{\mathbf{0}\} \times \{1, -1\} \times [0, \infty)$ will be investigated using Corollary 4.6. After working with SOSTOOLS, a value of $\tau_m^* = 0.6145$ is obtained, and thus global asymptotic stability is guaranteed for any $\tau_m \ge 0.6145$.

As it is expected, the result is somehow conservative. In this case, application of Proposition 4.2 to the values $0 < \tau_m \le 0.6145$ yields only one fixed point $\mathbf{p} = (-1, 0)$ of the mapping Π_g (and no other periodic points); and, in addition, $I(\mathbf{p}) = \frac{\pi}{\beta}$ where $\beta = 3\frac{\sqrt{11}}{10}$. Moreover, the corresponding matrix $M_\mathbf{p} = J^\top e^{A\frac{\pi}{\beta}} J$ is easily checked to be a Schur matrix. As a result, Proposition 4.2 certifies global asymptotic stability of $\mathcal{A}_\mathbf{0}$ for $\mathcal{H}^{cl}_{\tau_m}$ for $\tau_m \le 0.6145$. And a combination of both results certifies global asymptotic stability for any value of $\tau_m > 0$.

Interestingly, the application of Proposition 4.2 is less direct for $\tau_m > \frac{\pi}{\beta} \approx 3.1574$. For increasing values of τ_m they appear two fixed points (one sink and a source), periodic-2 points, etc. (details are omitted for brevity).

4.5. Reset-times dependent stability conditions: Ranged dwell-time

If A is not Hurwitz (but reset intervals are bounded by τ_M according to Assumption A) then the reset interval sequences set $S_{\mathcal{H}^{cl}_{\tau_m}}$, as defined in (59), is embedded in a larger set $S_{[\tau_m, \tau_M]}$ characterized by a ranged dwell-time. It is defined as

$$S_{[\tau_m, \tau_M]} = \{\{\tau_1, \tau_2, \ldots\} \subset \mathbb{R}_{\ge 0} : \tau_i \in [\tau_m, \tau_M]\} \tag{67}$$

and it is clear that $S_{\mathcal{H}^{cl}_{\tau_m}} \subset S_{[\tau_m, \tau_M]}$.

The next Proposition is a direct adaptation of Proposition 4.4 and Corollary 4.5 and easily follows.

Proposition 4.8. *Consider the reset control system $\mathcal{H}^{cl}_{\tau_m}$ with a reset intervals upper bound $\tau_M > 0$. The set $\mathcal{A}_0$ is globally asymptotically stable for $\mathcal{H}^{cl}_{\tau_m}$, with $[\tau_m, \tau_M] \subset [\tau_m^*, \tau_M^*]$, if there exist a sequence of positive definite matrices $\{P_1, P_2, \ldots\}$, such that*

$$\begin{aligned} &\eta I \le P_k \le \rho I \\ &e^{A^T \tau_k} A_R P_{k+1} A_R e^{A \tau_k} - P_k \le -\varepsilon I \end{aligned} \tag{68}$$

hold for $k = 1, 2, \ldots$, for some positive constants η, ρ, and ε, and any $\{\tau_1, \tau_2, \ldots\} \in S_{[\tau_m^*, \tau_M^*]}$.

Again, if it is considered a sequence of constant matrices $P_k = P > 0$ in Proposition 4.8, then some efficient methods in the literature can be applied. For example, in [32,39] a method for obtaining a set of intervals $[\tau_m, \tau_M]$ is developed. Also in [38] a method based on sum-of-squares conditions is given for this case of ranged dwell-time; the following Corollary is directly based on it.

Corollary 4.9. *Assume that there exist a differentiable matrix function $R : [0, \tau_M] \to \mathbb{S}^n$, $R(0) > 0$, and $\varepsilon > 0$ such that*

$$\begin{aligned} &A^T R(\theta) + R(\theta)A + \dot{R}(\theta) \le 0, \\ &A_R R(0) A_R - R(\tau) \le -\varepsilon I, \end{aligned} \tag{69}$$

hold for any $\theta \in [0, \tau_M^*]$ and any $\tau \in [\tau_m^*, \tau_M^*]$. Then the set $\mathcal{A}_\mathbf{0}$ is globally asymptotically stable for $\mathcal{H}^{cl}_{\tau_m}$, and for any $[\tau_m, \tau_M] \subset [\tau_m^*, \tau_M^*]$.

Example 4.10. Consider again the reset control system with the Horowitz reset controller (Fig. 6), described in Section 4.2. Note that the matrix A is not Hurwitz, since it has an eigenvalue in the closed right half plane, but the reset intervals upper bound is $\tau_M = \tau_m + \frac{2\pi}{\sqrt{19}}$. Using Corollary 4.9, it is discovered that (69) is feasible for $\tau_m^* = 0.1$ and $\tau_M^* = 37.5$ (the sum of squares tool SOSTOOLS, with a polynomial matrix function $R(\theta) = \sum_{i=0}^{6} R_i \theta^i$ of degree 6, has been used). As a result, $\mathcal{A}_\mathbf{0}$ is globally asymptotically stable for $\mathcal{H}^{cl}_{\tau_m}$, and for $[\tau_m, \tau_M] = [\tau_m, \tau_m + \frac{2\pi}{\sqrt{19}}] \subset [0.1, 37.5]$, that is for any τ_m such that $0.1 \leq \tau_m \leq 36.05$. Here, the result is somehow conservative as expected, note that in Section 4.2 stability is obtained for τ_m arbitrarily small.

5. Conclusions

A new model of the Clegg integrator with an attached error zero-crossing mechanism has been developed. This results in a reset controller model in the hybrid inclusions framework, equipping the resulting reset control system with good structural properties like robustness against measurement noise and robustness in stability. The manuscript has been focused on analysis of well-posedness and stability, adapting and extending previous work of the authors to the new reset model. More specifically, stability has been approached by analyzing the stability of a Poincaré-like map, following two paths: a test (necessary and sufficient conditions) based on the eigenvalues of matrices related with periods of reset interval sequences, and Lyapunov functions-based sufficient conditions. Both approaches have been analyzed in detail, including several examples.

On the one hand, although checking eigenvalues is a simple and efficient way to test stability, its applicability depends on the computation of a finite number of periodic points of a nonlinear map, being the computational burden increasing with the number of states that are unchanged at jumps. As an interesting result, it has been formally shown that this is not always possible since in some cases reset intervals may produce chaotic sequences. As an alternative, Lyapunov function-based results may be applied: different results have been obtained for the case in which the base control system is stable or unstable. Although they may be conservative, these (sufficient) conditions may be efficiently solved by using sum-of squares programming. In practical cases, a combination of both methods may be useful to analyze stability for different values of minimum dwell-time. As a final conclusion, it is believed that the manuscript gives a solid framework for reset control systems with a zero-crossing resetting law, that may serve as a basis for new theoretical and practical advances.

CRediT authorship contribution statement

Alfonso Baños: Conceptualization, Methodology, Investigation, Writing – review & editing. **Antonio Barreiro:** Conceptualization, Methodology, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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