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Abstract: This paper is devoted to study Lagrange interpolation based on nodal systems constituted by the roots of para-orthogonal polynomials with respect to analytic weights on the unit circle. The presented results address, in addition to algorithmic and convergence questions for continuous and discontinuous functions, a detailed study of the GibbsWilbraham phenomenon.
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# GIBBS-WILBRAHAM PHENOMENON ON LAGRANGE INTERPOLATION BASED ON ANALYTIC WEIGHTS ON THE UNIT CIRCLE 

E. BERRIOCHOA, A. CACHAFEIRO, J. M. GARCÍA AMOR


#### Abstract

This paper is devoted to study Lagrange interpolation based on nodal systems constituted by the roots of para-orthogonal polynomials with respect to analytic weights on the unit circle. The presented results address, in addition to algorithmic and convergence questions for continuous and discontinuous functions, a detailed study of the Gibbs-Wilbraham phenomenon.


Key words and phrases: Lagrange interpolation; analytic weights; para-orthogonal polynomials; unit circle; convergence; piecewise continuous functions; Gibbs-Wilbraham phenomenon.

## 1. Introduction

In the study of Fourier series related to piecewise continuous functions it appears a peculiar phenomenon called Gibbs or Gibbs-Wilbraham phenomenon, which states that in the proximity of the discontinuity points of the function the partial sums of the Fourier series have an oscillatory behavior. This fact has been discovered and investigated by several mathematicians and physicists, among which we highlight: H. Wilbraham, A.A. Michelson, A.E.H. Love and J.W. Gibbs (see [12]).

This phenomenon has also been studied in connection with Lagrange interpolation. Indeed the trigonometric interpolation polynomials related to periodic functions with jump discontinuities present, for growing order, a Gibbs phenomenon which differs in the overshot/undershot from the ones appeared for the partial sums of the Fourier series, (see [9], [10] and [11]). Another situation studied in connection with this phenomenon is the behavior of the Lagrange interpolants for piecewise continuous functions on the bounded interval by using Chebyshev nodes. The corresponding results that can be seen in [26], give information about the oscillatory behavior of Lagrange interpolants and can be adapted to interpolation for piecewise continuous functions on the unit circle by using equally spaced nodal points. For this last case there are few references and the results are very predictable. Indeed, they are essentially the same as in the papers of Helmberg and they can be seen in [20]. The novelty of this last work is the presentation of rigorous proofs of the results. Additionally we highlight the books [14] and [15] dedicated to this phenomenon as well as [21] and [6], which are relevant works on the Gibbs-Wilbraham phenomenon for spline approximation interpolation and generalizations. Other generalizations of splines can be seen in [8].

In the case of Hermite-Fejér interpolation, a seminal work in which piecewise continuous or piecewise smooth functions were considered is [19]. Two contributions more recently in this direction are [1] and [2]. In [1] it is described the behavior of the Hermite-Fejér and Hermite interpolation polynomials related to piecewise continuous functions or piecewise smooth functions on the unit circle by considering as nodal systems those constituted by the $n$ roots of complex unimodular numbers. Hence the nodal points are uniformly distributed on the circle. The main result given in this paper

[^0]is concerning the analysis of the Gibbs phenomenon near the discontinuities of the interpolated function and the obtention of the amplitude of the corresponding Gibbs height. In [2] the same type of problems were studied but considering as nodal points the zeros of the para-orthogonal polynomials with respect to measures in the Baxter class and such that the sequence of the first derivative of the reciprocal of the orthogonal polynomials is uniformly bounded on the circle. In this paper we have obtained that the Hermite-Fejér interpolants uniformly converge to the piecewise continuous function far away from the discontinuity points and we have described the oscillatory behavior of the interpolants near the discontinuities, where a Gibbs-Wilbraham phenomenon appears.

Now, in the present paper we develop one of the lines mentioned in the final considerations of [2] as possible future lines of research. Thus, we analyze the Lagrange interpolation process for piecewise continuous functions with suitable properties and by using some general nodal systems based on analytic measures, which constitutes a novel approach to the Lagrange interpolation theory. Indeed the nodal systems are the zeros of the para-orthogonal polynomials with respect to analytic weights, whose basic properties have been studied in [4]. In our case a Gibbs phenomenon appears whose study and knowledge allow us to use this interpolatory process by controlling the error.

The organization of the paper is the following. Section 2 is devoted to introduce the background and some auxiliary results, which play a fundamental role in the development of the main theorems. In order to make more readable the paper we give the proofs of the two lemmas in the Appendix at the end of the paper. In Section 3 we present the Lagrange interpolation problem on the circle and some results about convergence in case of continuous functions with appropriate modulus of continuity, as well as a result concerning the rate of convergence in case of smooth functions. Section 4 contains the main results given in Theorems 5 and 6 . In the last one we describe the oscillatory behavior of the interpolation polynomials near the discontinuity points of the piecewise continuous function. We provide detail proof of the results and the asymptotic amplitude of the Gibbs height. In Section 5 we have done several numerical experiments in order to present in a graphical way the main results of this article and finally, we summarize the conclusions and future goals.

## 2. Background and auxiliary results

Throughout all the paper we consider the following situation. Let $\nu$ be an analytic weight on the unit circle $\mathbb{T}:=\{z \in \mathbb{C}:|z|=1\}$, that is, it is associated with an analytic function on an open annulus with radii $1 / r$ and $r$, with $r<1$, which is positive on $\mathbb{T}$. Let $\left\{\phi_{n}(z)\right\}_{n \geq 0}$ be the monic orthogonal polynomial sequence related to $\nu$ and $\left\{\phi_{n}^{*}(z)\right\}_{n \geq 0}$ the sequence of reversed polynomials, defined by $\phi_{n}^{*}(z)=z^{n} \overline{\phi_{n}}\left(\frac{1}{z}\right)$, (see [24]). We also consider the para-orthogonal polynomials $W_{n}(z, \tau)$ defined by $W_{n}(z, \tau)=\phi_{n}(z)+\tau \phi_{n}^{*}(z)$, where $\tau$ is a unimodular complex number, ([16]) and we denote, as usual, by $\Pi(z)$ the normalized Szegő function related to $\nu$ (see [24] for details).

We recall some well known results related to the asymptotic behavior of $\phi_{n}(z), \phi_{n}^{*}(z)$ and $W_{n}(z, \tau)$. A more detailed introduction and proofs of these results can be seen in [4], where they were included to make the paper self contained.
(i)

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \phi_{n}^{*}(z)=\Pi(z) \text { and } \phi_{n}^{*}(z)=\Pi(z)+\mathcal{O}\left(r^{n}\right), \text { both uniformly on } \mathbb{T} . \\
& \lim _{n \rightarrow \infty} \frac{\phi_{n}}{z^{n}}=\bar{\Pi}\left(\frac{1}{z}\right) \text { and } \phi_{n}(z)=z^{n} \bar{\Pi}\left(\frac{1}{z}\right)+\mathcal{O}\left(r^{n}\right), \text { both uniformly on } \mathbb{T} .
\end{aligned}
$$

(ii)
$\lim _{n \rightarrow \infty}\left(\phi_{n}^{*}\right)^{\prime}(z)=\Pi^{\prime}(z)$ and $\left(\phi_{n}^{*}\right)^{\prime}(z)=\Pi^{\prime}(z)+\mathcal{O}\left(n^{2} r^{n}\right)$, both uniformly on $\mathbb{T}$.
$\lim _{n \rightarrow \infty} \frac{\phi_{n}^{\prime}(z)}{n z^{n-1}}=\bar{\Pi}\left(\frac{1}{z}\right)$ and $\phi_{n}^{\prime}(z)=n z^{n-1} \bar{\Pi}\left(\frac{1}{z}\right)+\mathcal{O}(1)$, both uniformly on $\mathbb{T}$.
(iii) There exist positive real numbers $A$ and $B$ such that for $z \in \mathbb{T}$ and for all $n$ we have

$$
\begin{equation*}
\left|W_{n}(z, \tau)\right| \leq A, B \leq \frac{\left|W_{n}^{\prime}(z, \tau)\right|}{n} \leq A, W_{n}^{\prime}(z, \tau)=n z^{n-1} \bar{\Pi}\left(\frac{1}{z}\right)+\mathcal{O}(1) . \tag{1}
\end{equation*}
$$

(iv) The roots of $W_{n}(z, \tau)$ are simple and they belong to $\mathbb{T}$, (see [16]). Moreover, under our conditions the roots are distributed in the following way. If $\alpha_{j}=e^{i \theta_{j}}$ and $\alpha_{j+1}=e^{i \theta_{j+1}}$ are two consecutive zeros of $W_{n}(z, \tau)$ then

$$
\begin{equation*}
\left|\theta_{j}-\theta_{j+1}\right|=\frac{2 \pi}{n}+\mathcal{O}\left(\frac{1}{n^{2}}\right) \tag{2}
\end{equation*}
$$

In this paper we will use the roots of $W_{n}(z, \tau)$ as nodal system on $\mathbb{T}$ to interpolate functions in the Lagrange sense. For convenience in the sequel we will work with the para-orthogonal polynomials $W_{2 n}(z, \tau)$ and for simplicity we will write $W_{2 n}(z)$. The odd case corresponding to take $W_{2 n+1}(z, \tau)$ could be done in a similar way. Moreover, if we denote by $\left\{\alpha_{j}\right\}_{j=0}^{2 n-1}$ the zeros of $W_{2 n}(z)$ we will assume that they are numbered in the clockwise sense beginning in $\alpha_{0}$. Besides, if $z$ and $w$ belong to $\mathbb{T}$ we will write $\widetilde{z-w}$ to refer to the lenght of the shortest arc joining both points.

Next we present some auxiliary results in relation with $W_{2 n}(z)$ of interest in the sequel. Notice that throughout all the paper we will use the Landau notation: $\mathcal{O}\left(a_{n}\right)$ and $o\left(a_{n}\right)$, (see [24]) and we assume $\left\{\alpha_{j}\right\}_{j=0}^{2 n-1}$ are the zeros of $W_{2 n}(z)$. In order to simplify the notation we will use the same notation $\mathcal{O}$ of Landau for different constants. Moreover, notice that the asymptotic properties that we obtain hold uniformly on $\mathbb{T}$ and $n$, that is, the positive constants are unique for all the circumference and for all $n$.

To make the paper more readable, we present the proofs of the following lemmas in a final Appendix at the end of the work.

Lemma 1. Let $\left\{\alpha_{j}\right\}_{j=0}^{2 n-1}$ be the nodal system determined by $W_{2 n}(z)$. If $z \in \mathbb{T}$ and $\alpha_{0}$ and $\alpha_{2 n-1}$ are the two nodal points closest to $z$ then the following properties hold:
(i) There exists a positive constant $C$ such that

$$
\begin{equation*}
\left|\frac{W_{2 n}(z)}{z-\alpha_{0}}\right|,\left|\frac{W_{2 n}(z)}{z-\alpha_{2 n-1}}\right| \leq 2 n C . \tag{3}
\end{equation*}
$$

(ii) For some positive constant $D$ it holds

$$
\begin{equation*}
\left|\frac{W_{2 n}(z)}{z-\alpha_{j}}\right|,\left|\frac{W_{2 n}(z)}{z-\alpha_{2 n-1-j}}\right| \leq \frac{2 n D}{j} \text { for } j>0 . \tag{4}
\end{equation*}
$$

(iii) If $j \geq \sqrt{n}$ then

$$
\frac{z-\alpha_{j}}{z-\alpha_{j+1}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \text { and } \frac{z-\alpha_{2 n-j}}{z-\alpha_{2 n-j-1}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) .
$$

In what follows we use the following notation to represent any point $z \in \mathbb{T}$ by using the first nodal point $\alpha_{0}$, that is, $z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}$, with $d$ a real number, $d \in[-n, n]$. Moreover we will write arcs to refer to those determined by two any consecutive roots of $W_{2 n}(z)$.

Lemma 2. Under our conditions the following properties are satisfied
(i) For each pair of consecutive nodes $\alpha_{j}$ and $\alpha_{j+1}$ it holds

$$
\begin{equation*}
\frac{\alpha_{j+1}}{\alpha_{j}}=1+\mathcal{O}\left(\frac{1}{n}\right) \text { and }\left(\frac{\alpha_{j+1}}{\alpha_{j}}\right)^{n}=-1+\mathcal{O}\left(\frac{1}{n}\right) . \tag{5}
\end{equation*}
$$

(ii) If $j<\sqrt{n}$ then
(a)

$$
\begin{equation*}
\left(\frac{\alpha_{j}}{\alpha_{0}}\right)^{n}=(-1)^{j}\left(1+\mathcal{O}\left(\frac{j}{n}\right)\right) \text { and }\left(\frac{\alpha_{j}}{\alpha_{0}}\right)^{2 n}=1+\mathcal{O}\left(\frac{j}{n}\right) . \tag{6}
\end{equation*}
$$

(b)

$$
\begin{equation*}
\left(\frac{\alpha_{2 n-1-j}}{\alpha_{0}}\right)^{n}=(-1)^{j+1}\left(1+\mathcal{O}\left(\frac{j+1}{n}\right)\right) \tag{7}
\end{equation*}
$$

(iii) (a) If $\alpha_{j}$ and $\alpha_{j+1}$ are two consecutive nodes then

$$
\frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)}=1+\mathcal{O}\left(\frac{1}{n}\right) .
$$

(b) If $j<\sqrt{n}$

$$
\begin{equation*}
\frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{j}\right) \alpha_{j}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \text { and } \frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{2 n-1-j}\right) \alpha_{2 n-1-j}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) . \tag{8}
\end{equation*}
$$

(iv) If $z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}$ and the distance over the circumference between $\alpha_{0}$ and $z$ is less than the length of $\sqrt{n} \operatorname{arcs},(-\sqrt{n} \leq d \leq \sqrt{n})$, then

$$
\begin{equation*}
\frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}=-2 i \sin (\pi d)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) . \tag{9}
\end{equation*}
$$

(v) (a) If $z=\alpha_{0} e^{i \frac{2 \pi}{2 n} d}$ with $\frac{1}{2} \leq d \leq \sqrt{n}$ and $j \leq \sqrt{n}$ then

$$
\begin{equation*}
\frac{1}{\left(\frac{z}{\alpha_{j}}-1\right) 2 n}=-\frac{i}{2 \pi(j+d)}+\mathcal{O}\left(\frac{1}{n}\right) . \tag{10}
\end{equation*}
$$

(b) If $z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}$, with $-\frac{1}{2} \leq d \leq \sqrt{n}$ and $j \leq \sqrt{n}$ then

$$
\begin{equation*}
\frac{1}{\left(\frac{z}{\alpha_{2 n-1-j}}-1\right) 2 n}=\frac{i}{2 \pi(j+d+1)}+\mathcal{O}\left(\frac{1}{n}\right) \tag{11}
\end{equation*}
$$

(vi) If $\alpha_{j}$ and $\alpha_{j+1}$ are consecutive nodes and their angular distance to $z$ is at least $\sqrt{n}$ arcs then

$$
\frac{\alpha_{j+1}^{n} W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}{\alpha_{j}^{n} W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}=-1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) .
$$

## 3. Lagrange interpolation problem. Convergence

In the sequel we study the Lagrange interpolation process on the unit circle taking as nodal system the para-orthogonal polynomials $W_{2 n}(z)$ considered before, that is, the nodal points are the zeros $\left\{\alpha_{j}\right\}_{j=0}^{2 n-1}$ of $W_{2 n}(z)$. As it is well known we construct the interpolation polynomials in the space of Laurent polynomials $\Lambda[z]$ by using the appropriate subspaces $\Lambda_{p, q}[z]=\operatorname{span}\left\{z^{k}: p \leq k \leq q\right\}$ with $p$ and $q$ integer numbers such that $p \leq q$. Due to the density of $\Lambda[z]$ in the space of continuous functions, the natural choices are $\Lambda_{-n, n-1}[z]$ or $\Lambda_{-(n-1), n}[z]$ when we use $2 n$ nodes. Thus, if we take $2 n$ nodes we recall that the Laurent polynomial of Lagrange interpolation related to a function $F$ defined in $\mathbb{T}$ and with nodal system $\left\{\alpha_{j}\right\}_{j=0}^{2 n-1}, \mathcal{L}_{-n, n-1}(F, z) \in \Lambda_{-n, n-1}[z]$, is characterized by satisfying

$$
\mathcal{L}_{-n, n-1}\left(F, \alpha_{j}\right)=F\left(\alpha_{j}\right) \text { for each } j=0, \cdots, 2 n-1 .
$$

In the odd case, that is, using $2 n+1$ nodes the natural choice is the space $\Lambda_{-n, n}[z]$.
This polynomial can be expressed in terms of the so called fundamental polynomials of Lagrange interpolation as follows

$$
\begin{array}{r}
\mathcal{L}_{-n, n-1}(F, z)=\frac{1}{z^{n}} \sum_{j=0}^{2 n-1} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} F\left(\alpha_{j}\right) \\
=\frac{W_{2 n}(z)}{z^{n}} \sum_{j=0}^{2 n-1} \frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} F\left(\alpha_{j}\right), \tag{12}
\end{array}
$$

but for our numerical purposes it is more convenient to use the barycentric expression given by

$$
\mathcal{L}_{-n, n-1}(F, z)=\frac{\sum_{j=0}^{2 n-1} \frac{w_{j}}{z-\alpha_{j}} F\left(\alpha_{j}\right)}{\sum_{j=0}^{2 n-1} \frac{w_{j}}{z-\alpha_{j}}}
$$

where $w_{j}=\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)}$.
One of the advantages of the last expression is its numerical stability (see [13]).

### 3.1. Convergence of the Laurent polynomial of Lagrange interpolation related to contin-

 uous functions. First we establish the Lebesgue constant for the interpolatory Lagrange process in the next result.Theorem 1. There exists a positive constant $L>0$ such that for every function $F$ bounded on $\mathbb{T}$ it holds that

$$
\left|\mathcal{L}_{-n, n-1}(F, z)\right| \leq L\|F\|_{\infty} \log n
$$

for every $z \in \mathbb{T}$, where $\left\|\|_{\infty}\right.$ denotes the supremum norm on $\mathbb{T}$.
Proof. The result is evident if $z$ is a nodal point. So let us assume that $z$ is an arbitrary point of $\mathbb{T}$ and it is not a nodal point. If we assume that the nodes are ordered in such a way that $\alpha_{0}$ and $\alpha_{2 n-1}$ are the two nodal points closest to $z$, then by applying (12), (4), (3) and (1) we get

$$
\begin{array}{r}
\left|\mathcal{L}_{-n, n-1}(F, z)\right|=\left|\frac{1}{z^{n}} \sum_{j=0}^{2 n-1} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} F\left(\alpha_{j}\right)\right| \leq \sum_{j=0}^{2 n-1}\left|\frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} F\left(\alpha_{j}\right)\right| \\
\leq \sum_{j=0}^{n-1}\left|\frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} F\left(\alpha_{j}\right)\right|+\sum_{j=n}^{2 n-1}\left|\frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} F\left(\alpha_{j}\right)\right| \\
\leq 2\|F\|_{\infty}\left(\frac{2 n C}{2 n B}+\sum_{j=1}^{n-1} \frac{2 n D}{2 n B} \frac{1}{j}\right) \leq 2\|F\|_{\infty} E\left(1+\sum_{j=1}^{n-1} \frac{1}{j}\right)=2\|F\|_{\infty} E\left(1+H_{n-1,1}\right),
\end{array}
$$

where $H_{n-1,1}$ is the harmonic number defined by $\sum_{j=1}^{n-1} \frac{1}{j}$.
Therefore, if we take into account that $\lim _{n \rightarrow \infty} \frac{1+H_{n-1,1}}{\log n}=1$, we obtain the result.
Note 1. The previous theorem states that the norm of the operator $\mathcal{L}_{-n, n-1}$ is, at most, $L \log n$. In this sense the growth with $n$ of the norm of the images is quite slow and in fact it will be bounded for some types of functions.

As a consequence of the preceding result we can obtain the uniform convergence of the interpolatory process when we deal with functions having convenient modulus of continuity. To obtain this result first we recall an adaptation of Jackson's theorem about approximation of continuous functions on $\mathbb{T}$ by Laurent polynomials given in [3].

Theorem 2. Let $F$ be a continuous function on $\mathbb{T}$ with modulus of continuity $w(F, \delta)$. Then there exists a positive constant $M$ such that for each natural number $n$ there exists a Laurent polynomial $p_{-n, n-1}(z)$ satisfying

$$
\left|F(z)-p_{-n, n-1}(z)\right|<M w\left(F, \frac{2 \pi}{n}\right)
$$

Proof. It can be seen in [3].
Theorem 3. Let $F$ be a function continuous on $\mathbb{T}$ with modulus of continuity $w(F, \delta)=o\left(|\log \delta|^{-1}\right)$. Then $\mathcal{L}_{-n, n-1}(F, z)$ converges uniformly to $F$ on $\mathbb{T}$.

Proof. From the preceding result it is clear that for each $n$ there exists a Laurent polynomial $p_{-n, n-1}(z)$ such that $\left|F(z)-p_{-n, n-1}(z)\right|<M o\left(\left|\log \left(\frac{2 \pi}{n}\right)\right|^{-1}\right)=M \frac{o(1)}{\log n-\log 2 \pi}$, that is, given $\varepsilon>0$ then for $n$ large enough it holds that $\left|F(z)-p_{-n, n-1}(z)\right|<\frac{\epsilon}{\log n}$ and therefore

$$
\left|\mathcal{L}_{-n, n-1}\left(F-p_{-n, n-1}, z\right)\right|<L\left\|F-p_{-n, n-1}\right\|_{\infty} \log n \leq L \frac{\epsilon}{\log n} \log n=L \epsilon
$$

Hence, if we use Theorem 1, we have

$$
\begin{aligned}
\left|F(z)-\mathcal{L}_{-n, n-1}(F, z)\right| & =\left|F(z)-p_{-n, n-1}(z)+p_{-n, n-1}(z)-\mathcal{L}_{-n, n-1}(F, z)\right| \\
\leq & \left|F(z)-p_{-n, n-1}(z)\right|+\left|p_{-n, n-1}(z)-\mathcal{L}_{-n, n-1}(F, z)\right| \\
& =\left|F(z)-p_{-n, n-1}(z)\right|+\left|\mathcal{L}_{-n, n-1}\left(p_{-n, n-1}-F, z\right)\right|
\end{aligned}
$$

and thus the result is proved.

Note 2. Notice that the preceding theorem is as good as the well known classical results on the bounded interval. Indeed it is also possible to construct functions for which the Laurent polynomial interpolatory process diverges. We recall that our interpolation process is closely related to the algebraic interpolation on the bounded interval through the Szegö transformation (see [24]). Therefore every example of a function for which the interpolants on the bounded interval with Chebyshev nodes diverge, will give an example of function for which the Laurent interpolants with equally spaced nodes diverge. These examples can be seen in the papers of Grünwald and Marcinkiewicz, (see [7] and [18]) and the interesting tribute [25].
3.2. Rate of convergence for smooth functions. We want to point out that the results given in this subsection could be very useful for some applications of the interpolatory process, like the numerical gaussian integration.

Theorem 4. Let $F(z)$ be a function defined on $\mathbb{T}$.
(i) If $F(z)=\sum_{-\infty}^{\infty} A_{k} z^{k}$ with $\left|A_{k}\right| \leq K \frac{1}{|k|^{c}}$ for $k \neq 0$, with $c>1$ then $\mathcal{L}_{-n, n-1}(F, z)$ uniformly converges to $F$ on $\mathbb{T}$ and the rate of convergence is $\mathcal{O}\left(\frac{\log n}{n^{c-1}}\right)$.
(ii) If $F(z)$ is an analytic function in an open annulus containing $\mathbb{T}$, then $\mathcal{L}_{-n, n-1}(F, z)$ uniformly converges to $F$ on $\mathbb{T}$. Besides, the rate of convergence is geometric.
Proof. (i) If we write $F(z)=F_{1, n}(z)+F_{2, n}(z)$ with $F_{1, n}(z)=\sum_{k=-n}^{n-1} A_{k} z^{k}$ and $F_{2, n}(z)=\sum_{-\infty}^{-n-1} A_{k} z^{k}+$ $\sum_{k=n}^{\infty} A_{k} z^{k}$ then it holds that $\mathcal{L}_{-n, n-1}\left(F_{1, n}, z\right)=F_{1, n}(z)$ and $F_{2, n}(z)$ is such that

$$
\left|F_{2, n}(z)\right| \leq \sum_{-\infty}^{-n-1}\left|A_{k}\right|+\sum_{k=n}^{\infty}\left|A_{k}\right| \leq 2 \sum_{k=n}^{\infty} \frac{K}{k^{c}}=2 K\left(H(c)-H_{n-1, c}\right) \leq \frac{2 K}{(c-1)} \frac{1}{(n-1)^{c-1}}
$$

where $H(c)$ denotes the sum of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k^{c}}$ and $H_{n-1, c}$ is its $(n-1)$-partial sum. By applying Theorem 1 we have that $\left|\mathcal{L}_{-n, n-1}\left(F_{2, n}, z\right)\right| \leq L \frac{2}{(c-1)} \frac{1}{(n-1)^{c-1}} \log n$ and therefore the error of interpolation is

$$
\left|F(z)-\mathcal{L}_{-n, n-1}(F, z)\right|=\left|F_{2, n}(z)-\mathcal{L}_{-n, n-1}\left(F_{2, n}, z\right)\right| \leq K_{1} \frac{\log n}{n^{c-1}}
$$

Hence the rate of convergence is $\mathcal{O}\left(\frac{\log n}{n^{c-1}}\right)$.
(ii) It is obtained in a similar way. Indeed if $F(z)=\sum_{-\infty}^{\infty} A_{k} z^{k}$ with $\left|A_{k}\right| \leq \operatorname{Pr} r^{|k|}$ for some $P>0$ and $0<r<1$, by considering the decomposition of $F$ given before we get that $\mathcal{L}_{-n, n-1}\left(F_{1, n}, z\right)=F_{1, n}(z)$ and

$$
\left|F_{2, n}(z)\right| \leq \sum_{-\infty}^{-n-1}\left|A_{k}\right|+\sum_{k=n}^{\infty}\left|A_{k}\right| \leq 2 P \sum_{k=n}^{\infty} r^{k}=Q r^{n}
$$

Then $\left|\mathcal{L}_{-n, n-1}\left(F_{2, n}, z\right)\right| \leq L Q r^{n} \log n$ and the error of interpolation is given by

$$
\left|F(z)-\mathcal{L}_{-n, n-1}(F, z)\right|=\left|F_{2, n}(z)-\mathcal{L}_{-n, n-1}\left(F_{2, n}, z\right)\right| \leq \operatorname{Tr}^{n}(1+\log n) \leq S r_{1}^{n}
$$

for some positive constants $T, S$ and $r_{1}$ with $r<r_{1}<1$.
4. Lagrange interpolation for piecewise continuous functions. Gibbs-Wilbraham PHENOMENON

In this section our aim is to study the behavior of the Lagrange interpolants related to piecewise continuous functions with appropriate modulus of continuity. Without loss of generality we consider two functions $f$ and $g$ continuous on $\mathbb{T}$, both with good modulus of continuity. If $A$ is an arc contained in $\mathbb{T}$ with extreme points $a_{1}$ and $a_{2}$ and length $\ell_{A}$, we consider the piecewise continuous function $F=f \chi_{A}+g \chi_{\mathbb{T} \backslash A}$, where $\chi_{A}$ is defined by $\chi_{A}(z)=1$ if $z \in A$ and $\chi_{A}(z)=0$ if $z \notin A$.

In order to do our analysis first we rewrite the interpolants in a convenient form as follows. Thus, if we assume that $F(z)=f(z)$ then

$$
\begin{array}{r}
\mathcal{L}_{-n, n-1}(F, z)-F(z)=\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} F\left(\alpha_{j}\right)-f(z) \\
=\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right)+\sum_{\substack{j=0 \\
\alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-f(z) \\
=\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right)-\sum_{\substack{j=0 \\
\alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right) \\
+\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-f(z) \\
=\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right)-f(z)+\sum_{\substack{j=0 \\
\alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right) \\
=\mathcal{L}_{-n, n-1}(f, z)-f(z)+\sum_{\substack{j=0 \\
\alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right) . \tag{13}
\end{array}
$$

For the convergence we are going to consider the following regions in $\mathbb{T}$, like the next Figure 1 shows. In this particular case we have taken $a_{1}=1, a_{2}=i$ and $A$ the shortest arc between $a_{2}$ and $a_{1}$.
(1) Since the extremal points of the arc play the same role, we study the behavior near $a_{2}$ and for simplicity we will assume that $\alpha_{0}$ is the nodal point in $A$ which is closer to $a_{2}$. Notice that $\alpha_{0}$, which depends on $n$, may be equal or different to $a_{2}$, which is a difference with respect to classical and recent references. Moreover, we denote the points of $\mathbb{T}$ refering to $\alpha_{0}$. An alternative point of view is to consider that the arc A ends (for the interpolatory process) in $\alpha_{0}$. This last interpretation connects better with other references.
(2) We denote by $I_{n}$ the region of A that is far from $\mathbb{T} \backslash A$ more than $\sqrt{n} \operatorname{arcs}$ determined by two any consecutive roots of $W_{2 n}(z)$, that is,

$$
I_{n}=\left\{z \in A: z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}, d \in\left(\sqrt{n}, \ell_{A} \frac{2 \pi}{2 n}-\sqrt{n}\right]\right\}
$$



Figure 1. Regions considered in $\mathbb{T}$.

We also consider the region $J_{n}$ of $\mathbb{T} \backslash A$ that is far from $A$ more than $\sqrt{n}$ arcs, which can be defined in a similar way. Since the behavior is similar in both situations we only study the first case in next Theorem 5.
(3) We consider the regions of $A, K_{n}$ and $K_{n}^{\prime}$, which are less than $\sqrt{n} \operatorname{arcs}$ from the extreme points of $A$. The same is considered for $\mathbb{T} \backslash A$. Since in these regions, $L_{n}$ and $L_{n}^{\prime}$, the behavior is similar, we only study the behavior of $K_{n}$ and $L_{n}$ in Theorem 6 .

Theorem 5. Let $F=f \chi_{A}+g \chi_{\mathbb{T} \backslash A}$, with $f$ and $g$ continuous functions on $\mathbb{T}$ with modulus of continuity $o\left(|\log \delta|^{-1}\right)$. Then

$$
\mathcal{L}_{-n, n-1}(F, z) \text { uniformly converges to } F(z) \text { on } I_{n} \subset A \text {. }
$$

Proof. By applying Theorem 3 we have that $\mathcal{L}_{-n, n-1}(f, z)$ uniformly converges to $f(z)$ on $\mathbb{T}$. Thus, in order to obtain the result, we have to prove that the last sum in (13)

$$
\sum_{\substack{j=0 \\ \alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right)
$$

converges to 0 uniformly on $I_{n}$.
In this last expression there are nodes $\alpha_{j}$ attained from $z$ turning in the clockwise sense less that $\pi$. The corresponding set of indices is denoted $N_{1}$. In the same way there are also nodes attained from $z$ turning in the counterclockwise sense less that $\pi$ and their indices are $N_{2}$. Obviously it suffices to study only one of these sums. So let us study

$$
\sum_{j \in N_{1}} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right) .
$$

We gather in pairs the nodes with indices in $N_{1}$, beginning with the node which is nearest to $z$. Then we have

$$
\begin{aligned}
\left.\leq\left|\frac{W_{2 n}(z)}{z^{n}}\right| \sum_{j \in N_{1}} \right\rvert\,( & \left.\left(\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right)+\frac{\left.\sum_{j \in N_{1}} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j+1}\right)\right) \right\rvert\,\left(z-\alpha_{j+1}\right)}{}\left(f\left(\alpha_{j+1}\right)-g\left(\alpha_{j+1}\right)\right)\right) \right\rvert\, \\
& \leq \underbrace{\left|\frac{W_{2 n}(z)}{z^{n}}\right| \sum_{j \in N_{1}}\left|\left(\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}+\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\right)\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right)\right|}_{*} \\
& +\underbrace{\left|\frac{W_{2 n}(z)}{z^{n}}\right| \sum_{j \in N_{1}}\left|\left(\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\left(f\left(\alpha_{j+1}\right)-f\left(\alpha_{j}\right)+g\left(\alpha_{j}\right)-g\left(\alpha_{j+1}\right)\right)\right)\right|}_{* *} .
\end{aligned}
$$

Clearly by applying (1) we get that $*$ satisfies

$$
\begin{align*}
& * \leq M \sum_{j \in N_{1}}\left|\left(\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}+\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\right)\right| \\
= & M \sum_{j \in N_{1}}\left|\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}\right|\left|1+\frac{\alpha_{j+1}^{n}}{\alpha_{j}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\right| . \tag{14}
\end{align*}
$$

From (1) we have that $\frac{1}{\left|W_{2 n}^{\prime}\left(\alpha_{j}\right)\right|} \leq \frac{1}{2 n B}$ and from the proof of (ii) in Lemma 1 we get that $\frac{1}{\left|z-\alpha_{j}\right|} \leq \frac{2 n E}{j}$. Hence it holds

$$
\left|\frac{\alpha_{j}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}\right| \leq \frac{E}{B} \frac{1}{j}
$$

Besides, if we apply (vi) in Lemma 2 we have

$$
\frac{\alpha_{j+1}^{n}}{\alpha_{j}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}=-1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)
$$

Then

$$
* \leq Q \sum_{j \in N_{1}} \frac{\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)}{j}
$$

which goes to 0 as $n$ goes to infinity.
To study $* *$ we take into account the following facts related to the modulus of continuity of $f-g$. Indeed since $\left|\alpha_{j}-\alpha_{j+1}\right| \leq \frac{3 \pi}{n}$ then

$$
\left|f\left(\alpha_{j+1}\right)-g\left(\alpha_{j+1}\right)-f\left(\alpha_{j}\right)+g\left(\alpha_{j}\right)\right| \leq w\left(f-g, \frac{3 \pi}{n}\right) \leq o\left(\left|\log \left(\frac{3 \pi}{n}\right)\right|^{-1}\right) \approx o\left(\frac{1}{\log n}\right)
$$

and

$$
\begin{array}{r}
* * \leq\left|\frac{W_{2 n}(z)}{z^{n}}\right| \sum_{j \in N_{1}}\left|\left(\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\left(f\left(\alpha_{j+1}\right)-f\left(\alpha_{j}\right)+g\left(\alpha_{j}\right)-g\left(\alpha_{j+1}\right)\right)\right)\right| \\
\leq A \sum_{j \in N_{1}}\left(\left|\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\right| \frac{\varepsilon}{\log n}\right)=A \frac{\epsilon}{\log n} \sum_{j \in N_{1}}\left|\frac{\alpha_{j+1}^{n}}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}\right| \\
<\frac{Q \varepsilon}{\log n} \sum_{j \in N_{1}} \frac{1}{j+1}, \tag{15}
\end{array}
$$

which goes to 0 when $n$ is large enough.

Note 3. Notice that as a consequence of (14) and (15) we can state that, under our hypotheses, the nodes which are at least $\sqrt{n}$ arcs far from a point do not significantly contribute to the calculus of the Lagrange interpolation polynomial at this point, that is, it can be bounded for any arbitrary $\epsilon>0$.

The second theorem of the present section is devoted to study the behavior of the interpolation polynomial related to $F$ near the discontinuity points. Since the behavior is similar in both extreme points of the arc $A$, we only study the point $a_{2}$.

We recall that the special function HurwitzLerchPhi $[-1, s, d]$ (see [5]) is defined by

$$
\text { HurwitzLerchPhi }[-1, s, d]=\sum_{j=0}^{\infty} \frac{(-1)^{j}}{(j+d)^{s}}
$$

For our purposes we use that corresponding to take $s=1$, that is, HurwitzLerchPhi $[-1,1, d]$ and for simplicity, we denote it by $\eta(d)$.
Theorem 6. Let $F=f \chi_{A}+g \chi_{\mathbb{T} \backslash A}$, with $f$ and $g$ continuous functions on $\mathbb{T}$ with modulus of continuity $o\left(|\log \delta|^{-1}\right)$ and assume that $\ell_{1}=\lim _{z \rightarrow a_{2}} f(z) \neq \lim _{z \rightarrow a_{2}} g(z)=\ell_{2}$. Then
(i) If $z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}$, with $d \in\left[-\frac{1}{2}, \sqrt{n}\right)$ and $n$ is large enough then

$$
\mathcal{L}_{-n, n-1}(F, z) \text { behaves like }\left(\ell_{1}-\ell_{2}\right) \eta(1+d) \frac{\sin \pi d}{\pi}+\ell_{1}
$$

that is, for an arbitrary $\varepsilon>0$ and $n$ large enough the distance between $\mathcal{L}_{-n, n-1}(F, z)$ and $\left(\ell_{1}-\ell_{2}\right) \eta(1+d) \frac{\sin \pi d}{\pi}+\ell_{1}$ is less than $\varepsilon$.
(ii) If $z=\alpha_{0} e^{-i \frac{2 \pi}{2 n} d}$, with $d \in\left(-\sqrt{n},-\frac{1}{2}\right]$ and $n$ is large enough then

$$
\mathcal{L}_{-n, n-1}(F, z) \text { behaves like }\left(\ell_{1}-\ell_{2}\right) \eta(|d|) \frac{\sin \pi|d|}{\pi}+\ell_{2} .
$$

Proof. (i) If we compute the difference $\mathcal{L}_{-n, n-1}(F, z)-f(z)$ by applying (13), we get

$$
=\mathcal{L}_{-n, n-1}(f, z)-f(z)+\sum_{\substack{j=0 \\ \alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{\mathcal{L}_{-n, n-1}(F, z)-f(z)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right) .
$$

From Theorem 3 we know that $\mathcal{L}_{-n, n-1}(f, z)$ uniformly converges to $f(z)$ on $\mathbb{T}$. Hence, to solve our problem, we have to study the behavior of the last sum in (16) for $n$ large enough. If we take into
account Note 3, we have to study the behavior of this sum considering only the nodes which are less than $\sqrt{n} \operatorname{arcs}$ from $a_{2}$. Thus we consider

$$
\begin{array}{r}
\sum_{j=2 n-\sqrt{n}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right) \\
=\frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}} \times \sum_{j=2 n-\sqrt{n}}^{2 n-1} \frac{\alpha_{j}^{n}}{\alpha_{0}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{j}\right) \alpha_{j}} \frac{1}{\left(\frac{z}{\alpha_{j}}-1\right) 2 n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right),
\end{array}
$$

that can be rewritten like

$$
\begin{array}{r}
\frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}} \times \\
\sum_{j=0}^{\sqrt{n}-1} \frac{\alpha_{2 n-1-j}^{n}}{\alpha_{0}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{2 n-1-j}\right) \alpha_{2 n-1-j}} \frac{1}{\left(\frac{z}{\alpha_{2 n-1-j}}-1\right) 2 n}\left(g\left(\alpha_{2 n-1-j}\right)-f\left(\alpha_{2 n-1-j}\right)\right) . \tag{17}
\end{array}
$$

If we define $h(z)=g(z)-f(z)-\ell_{2}+\ell_{1}$ and we take into account that $\lim _{z \rightarrow a_{2}} h(z)=0$, then given $\varepsilon>0$ there exists a neighborhood of $a_{2}, \mathcal{E}_{a_{2}}$, such that $\left|h\left(\alpha_{2 n-1-j}\right)\right|<\varepsilon$ for those $\alpha_{2 n-1-j} \in \mathcal{E}_{a_{2}}$. Hence, by applying (9), (7), (8) and (11) of Lemma 2 in (17) we obtain

$$
\begin{array}{r}
\sum_{j=2 n-\sqrt{n}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(g\left(\alpha_{j}\right)-f\left(\alpha_{j}\right)\right)=\left(-2 i \sin (\pi d)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right) \times \\
\sum_{j=0}^{\sqrt{n}-1}(-1)^{j+1}\left(1+\mathcal{O}\left(\frac{j+1}{n}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{i}{2 \pi(j+d+1)}+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(h\left(\alpha_{2 n-1-j}\right)+\ell_{2}-\ell_{1}\right) .
\end{array}
$$

After doing some calculus we get that the right hand side of the preceding equality is

$$
\begin{aligned}
& \underbrace{-2 i \sin (\pi d) \sum_{j=0}^{\sqrt{n}-1}(-1)^{j+1}\left[\frac{i}{2 \pi(j+d+1)}+\frac{i}{2 \pi(j+d+1)} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right]\left(h\left(\alpha_{2 n-1-j}\right)+\ell_{2}-\ell_{1}\right)}_{(*)} \\
& +\underbrace{\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \sum_{j=0}^{\sqrt{n}-1}(-1)^{j+1}\left[\frac{i}{2 \pi(j+d+1)}+\frac{i}{2 \pi(j+d+1)} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right]\left(h\left(\alpha_{2 n-1-j}\right)+\ell_{2}-\ell_{1}\right)}_{(* *)} .
\end{aligned}
$$

It is immediate to see that for $n$ large enough $\left({ }^{*}\right)$ behaves like

$$
-\frac{\sin (\pi d)}{\pi} \eta(1+d)\left(\ell_{2}-\ell_{1}\right)
$$

and $\left({ }^{* *}\right)$ goes to zero when $n$ tends to $\infty$. Thus, it follows that for $n$ large enough (i) holds.
(ii) Proceeding like in (13) we compute the difference $\mathcal{L}_{-n, n-1}(F, z)-g(z)$. Thus we have

$$
\begin{array}{r}
\mathcal{L}_{-n, n-1}(F, z)-g(z)=\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} F\left(\alpha_{j}\right)-g(z) \\
=\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right)+\sum_{\substack{j=0 \\
\alpha_{j} \in \mathbb{T} \backslash A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-g(z) .
\end{array}
$$

Hence

$$
\begin{array}{r}
\mathcal{L}_{-n, n-1}(F, z)-g(z)=\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} f\left(\alpha_{j}\right) \\
+\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-g(z) \\
=\sum_{j=0}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n} g\left(\alpha_{j}\right)-g(z)+\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right) \\
=\mathcal{L}_{-n, n-1}(g, z)-g(z)+\sum_{\substack{j=0 \\
\alpha_{j} \in A}}^{2 n-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right) . \tag{18}
\end{array}
$$

From Theorem 3 we know that $\mathcal{L}_{-n, n-1}(g, z)$ uniformly converges to $g(z)$ on $\mathbb{T}$. Hence, to solve our problem, we have to study the behavior of the last sum in (18) for $n$ large enough. If we take into account Note 3, we have to study the behavior of this sum considering only the nodes which are less than $\sqrt{n} \operatorname{arcs}$ from $a_{2}$. Thus we consider

$$
\begin{array}{r}
\sum_{j=0}^{\sqrt{n}-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right) \\
=\frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}} \times \sum_{j=0}^{\sqrt{n}-1} \frac{\alpha_{j}^{n}}{\alpha_{0}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{j}\right) \alpha_{j}} \frac{1}{\left(\frac{z}{\alpha_{j}}-1\right) 2 n}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right) . \tag{19}
\end{array}
$$

If we define $h(z)=g(z)-f(z)-\ell_{2}+\ell_{1}$ and we take into account that $\lim _{z \rightarrow a_{2}} h(z)=0$, then given $\varepsilon>0$ there exists a neighborhood of $a_{2}, \mathcal{E}_{a_{2}}$, such that $\left|h\left(\alpha_{j}\right)\right|<\varepsilon$ for those $\alpha_{j} \in \mathcal{E}_{a_{2}}$. Hence, by applying (9), (6), (8) and (10) of Lemma 2 in (19) we obtain

$$
\begin{array}{r}
\sum_{j=0}^{\sqrt{n}-1} \frac{1}{z^{n}} \frac{W_{2 n}(z)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)} \alpha_{j}^{n}\left(f\left(\alpha_{j}\right)-g\left(\alpha_{j}\right)\right)=\left(-2 i \sin (\pi d)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right) \times \\
\sum_{j=0}^{\sqrt{n}-1}(-1)^{j}\left(1+\mathcal{O}\left(\frac{j}{n}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)\left(-\frac{i}{2 \pi(j-d)}+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(-h\left(\alpha_{j}\right)+\ell_{1}-\ell_{2}\right) .
\end{array}
$$

After doing some calculus we get that the right hand side of the preceding equality is

$$
\begin{aligned}
& \underbrace{-2 i \sin (\pi d) \sum_{j=0}^{\sqrt{n}-1}(-1)^{j}\left[-\frac{i}{2 \pi(j-d)}-\frac{i}{2 \pi(j-d)} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right]\left(-h\left(\alpha_{j}\right)+\ell_{1}-\ell_{2}\right)}_{(*)} \\
& +\underbrace{\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \sum_{j=0}^{\sqrt{n}-1}(-1)^{j}\left[-\frac{i}{2 \pi(j-d)}-\frac{i}{2 \pi(j-d)} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right]\left(-h\left(\alpha_{j}\right)+\ell_{1}-\ell_{2}\right)}_{(* *)} .
\end{aligned}
$$

It is immediate to see that for $n$ greater enough (*) behaves like

$$
\frac{\sin (\pi d)}{\pi} \eta(|d|)\left(\ell_{2}-\ell_{1}\right)=\frac{\sin (\pi|d|)}{\pi} \eta(|d|)\left(\ell_{1}-\ell_{2}\right)
$$

and $(* *)$ goes to zero when $n$ tends to $\infty$. Thus, it follows the result.

The behavior of the interpolants near to $a_{2}$ is plotted in the next Figure 2 for the case $\ell_{1}=1$, $\ell_{2}=0$ and $d \in[-7,7]$. It is given by $1+\frac{\sin \pi d}{\pi} \eta(1+d)$ and $\frac{\sin \pi|d|}{\pi} \eta(|d|)$.


Figure 2. Approximation of $\mathcal{L}_{-n, n-1}\left(F, \alpha_{0} e^{-i \frac{2 \pi}{2 n} d}\right)$ near to $d=0$.

## 5. Numerical experiments

We have done several numerical experiments in order to present in a graphical way the different results of the article. In all of them we use a barycentric formula of the second type to compute the values of the interpolation polynomials.
Example 1. Our first numerical experiment is designed to show in a graphical way the central result of the article, that is, the existence of a Gibbs-Wilbraham phenomenon near the discontinuities of a piecewise continuous function. Moreover, the phenomenon can be described in the terms given on statements of Theorem 6 .

To see this behavior we consider a discontinuous test function $F$ given by $F(z)=\Re(z)$ on the shortest arc between $-i$ and 1 and defined by 0 elsewhere. In this case $a_{1}=-i, a_{2}=1, \ell_{1}=1$ and $\ell_{2}=0$. We also consider the interpolation polynomials $\mathcal{L}_{-n, n-1}(F, z)$ and $\mathcal{M}_{-n, n-1}(F, z)$ related to the nodal systems determined by the para-orthogonal polynomials $W_{2 n}(z, \tau)$ with respect to the Bernstein-Szegő measure, with normalized Szegő function $\left(1+\frac{z}{2}\right)\left(1+i \frac{z}{4}\right)\left(1-i \frac{z}{4}\right)$,


Figure 3. $F(z), \Re\left(\mathcal{L}_{-20,19}(F, z)\right), \Re\left(\mathcal{M}_{-20,19}(F, z)\right)$ and $1+\eta(1+d) \frac{\sin \pi d}{\pi},(d \in$ $\left.\left[-\frac{1}{2}, 7\right]\right), \eta(|d|) \frac{\sin \pi|d|}{\pi},\left(d \in\left[-7,-\frac{1}{2}\right)\right)$ for $z=e^{-i \frac{\pi d}{20}}$.
taking $\tau=-1,1$ respectively. We have represented three times the same graphic changing only the number of nodes, (taking $n=20$ in Figure 3, $n=360$ and $n=500$ in Figure 4; (i) and (ii), respectively). The illustrations always gather the plots of the test function $F(z)$ (green), the real part of the interpolation polynomials $\Re\left(\mathcal{L}_{-n, n-1}(F, z)\right)$ (red) and $\Re\left(\mathcal{M}_{-n, n-1}(F, z)\right)$ (blue) and the approximations given by $\ell_{1}+\left(\ell_{1}-\ell_{2}\right) \frac{\sin \pi d}{\pi} \eta(1+d)=1+\frac{\sin \pi d}{\pi} \eta(1+d), d \in\left[-\frac{1}{2}, 7\right]$ and $\ell_{2}+\left(\ell_{1}-\ell_{2}\right) \frac{\sin \pi|d|}{\pi} \eta(|d|)=\frac{\sin \pi|d|}{\pi} \eta(|d|), d \in\left[-7,-\frac{1}{2}\right)$ (black), for $z=e^{-i \frac{\pi d}{n}}$. Notice that we do not consider the imaginary part of the interpolants due to the results given in Theorem 6.

To make the paper more readable we summarize this information in the following table.

| Function | Description | Colour |
| :---: | :---: | :---: |
| F(z) | Test Function | Green |
| $\Re(z)$ on the shortest arc between $-i$ and 1 and 0 elsewhere |  |  |
| $\mathcal{L}_{-n, n-1}(F, z)$ | Interpolation polynomial | Red |
| based on the nodal polynomial $W_{2 n}=\phi_{2 n}-\phi_{2 n}^{*}$ |  |  |
| $\mathcal{M}_{-n, n-1}(F, z)$ | Interpolation polynomial | Blue |
| based on the nodal polynomial $W_{2 n}=\phi_{2 n}+\phi_{2 n}^{*}$ |  |  |
| $1+\frac{\sin \pi d}{\pi} \eta(1+d)$ |  |  |
| $\frac{\sin \pi\|d\|}{\pi} \eta(\|d\|)$ | Approximation of $\mathcal{L}_{-n, n-1}(F, z)$ stated in Theorem 6 | Black |

In this way, we can see the two interpolation processes based in different nodal systems in Figure 3 and it can be appreciated a sharp difference with the prediction or approximation given by the theorem. Moreover, we can point out some difference between the shapes of $\Re\left(\mathcal{L}_{-n, n-1}(F, z)\right)-F(z)$ and $\Re\left(\mathcal{M}_{-n, n-1}(F, z)\right)-F(z)$.

When $n$ grows the situation must change. Indeed the interpolation polynomials must have the same shape near the discontinuity and this shape is described by $1+\frac{\sin \pi d}{\pi} \eta(1+d)$ on the right. This evolution can be observed in the first graphic of Figure 4.

Finally, the second graphic of Figure 4 shows that $1+\frac{\sin \pi d}{\pi} \eta(1+d)$ and $\Re\left(\mathcal{L}_{-n, n-1}(F, z)\right)$ are indistinguishable close to 1 and that $\Re\left(\mathcal{M}_{-n, n-1}(F, z)\right)$ shares their shape. A similar comment can be done on the left hand side.


Figure 4. (i) $F(z), \Re\left(\mathcal{L}_{-360,359}(F, z)\right), \Re\left(\mathcal{M}_{-360,359}(F, z)\right)$ and $1+\eta(1+d) \frac{\sin \pi d}{\pi}$ for $z=e^{-i \frac{\pi d}{360}}, d \in[0,7]$, (ii) $F(z), \Re\left(\mathcal{L}_{-500,499}(F, z)\right), \Re\left(\mathcal{M}_{-500,499}(F, z)\right)$ and $1+\eta(1+$ $d) \frac{\sin \pi d}{\pi}$ for $z=e^{-i \frac{\pi d}{500}}, d \in[0,7]$.

Example 2. Our second numerical experiment is designed to present in a graphical way the main results of the article, that is the global behavior of the Lagrange interpolation polynomial for a piecewise continuous function. That behavior includes uniform convergence far away the discontinuities and the existence of a Gibbs-Wilbraham phenomenon near the discontinuities with a well determined shape described on statements of Theorem 6. Moreover, all these behaviors are the same for nodal systems based in different measures of analytical weights.

We have chosen a more variable test function for this example. Indeed we use for the experiment the function $F(z)$ given by $2+(1-\Re(z)) \sin \frac{1}{1-\Re(z)}$ on the shortest arc between $-i$ and $1,-2+(1-$ $\Re(z)) \sin \frac{1}{1-\Re(z)}$ on the shortest arc between 1 and $i$ and 0 elsewhere. So $1, i$ and $-i$ are discontinuity points. We use two different nodal systems with 200 points based on two different analytic measures. This choice allows to illustrate the oscillatory behavior of the Gibbs-Wilbraham phenomena. The first measure is just the Bernstein-Szegő measure used in Example 1. The second one is a RogersSzegő measure (RS). We recall that the RS measures (or wrapped Gaussian measures) are one of the classical examples of measures on the unit circle. These type of measures are analytical weights on the unit circle. A complete description of the RS measures which depend on a parameter $q$ can be obtained in [23]. In particular for our example we use $q=0.05$ and we take $\tau=-1$ for the para-orthogonal polynomials. In this situation 1 is a nodal point for both systems.

Figure 5 presents the plots of $F(z)$ (green), $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)$ (red) and $\Re\left(\mathcal{M}_{-100,99}(F, z)\right)$ (blue), for $z=e^{-i \frac{\pi d}{n}}$ on the interval $d \in[-100,100]$. With arrows we have pointed out the areas of uniform convergence and with ellipses the areas where the Gibbs-Wilbraham phenomena appears.

Figure 6 presents the plots of $F(z)$ (green), $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)$ (red) and $\Re\left(\mathcal{M}_{-100,99}(F, z)\right)$ (blue), for $z=e^{-i \frac{\pi d}{n}}$ on the interval $d \in[-7,7]$, that is close to 1 .

Finally we have plotted in Figure 7 the difference between $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)-\Re\left(\mathcal{M}_{-100,99}(F, z)\right)$ (black) and $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)-F(z)$ (red) on the interval $d \in[-7,7]$, that is close to 1 .


Figure 5. $F(z), \Re\left(\mathcal{L}_{-100,99}(F, z)\right)$ and $\Re\left(\mathcal{M}_{-100,99}(F, z)\right), z=e^{-i \frac{\pi d}{n}}, d \in[-100,100]$.


Figure 6. $F(z), \Re\left(\mathcal{L}_{-100,99}(F, z)\right)$ and $\Re\left(\mathcal{M}_{-100,99}(F, z)\right), z=e^{-i \frac{\pi d}{n}}, d \in[-7,7]$.


Figure 7. $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)-\Re\left(\mathcal{M}_{-100,99}(F, z)\right)$ and $\Re\left(\mathcal{L}_{-100,99}(F, z)\right)-F(z)$.

## 6. Conclusions

In this article we use nodal systems which are useful for Lagrange interpolation process for continuous or piecewise continuous functions. Indeed we have valid algorithms without stability problems. The approximation obtained for smooth continuous functions is fast and this is an important reason for using methods based on Lagrange interpolation. When we use Lagrange interpolation for piecewise continuous functions we obtain an accurate description of where and when the Gibbs phenomenon appears. Following the ideas of Krylov in [17], who knows the error can correct it, that is, since we know the oscillations caused by the Gibbs phenomenon, we can use them to improve the convergence. As a consequence of our study we conclude that this phenomenon is more analogous to Fourier series than Hermite interpolation. Besides, we think that the techniques that we use in the
article could be used with more general nodal systems without the restriction of being related with para-orthogonal polynomials.

## 7. Appendix

In this section we give the proofs of Lemmas 1 and 2.
Proof of Lemma 1. (i) Taking into account that $W_{2 n}\left(\alpha_{0}\right)=0$ and $\frac{d}{d \theta}\left(W_{2 n}\left(e^{i \theta}\right)\right)=W_{2 n}^{\prime}(z) e^{i \theta} i$ for $z=e^{i \theta}$, then

$$
\begin{aligned}
\left|W_{2 n}(z)\right|=\left|W_{2 n}(z)-W_{2 n}\left(\alpha_{0}\right)\right| & =\left|W_{2 n}\left(e^{i \theta}\right)-W_{2 n}\left(e^{i \theta_{0}}\right)\right| \leq \max _{\theta \in[0,2 \pi]}\left|\frac{d}{d \theta}\left(W_{2 n}\left(e^{i \theta}\right)\right)\right|\left|\theta-\theta_{0}\right| \\
& \leq \max _{z \in \mathbb{T}}\left|W_{2 n}^{\prime}(z)\right|\left|z-\alpha_{0}\right| \frac{\pi}{2} \leq 2 n A\left|z-\alpha_{0}\right| \frac{\pi}{2}=2 n C\left|z-\alpha_{0}\right| .
\end{aligned}
$$

The second inequality can be obtained in the same way.
(ii) As a consequence of (2) the angular distance between two consecutive nodes is $\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{4 n^{2}}\right)$ and therefore the angular distance $\widetilde{z-\alpha_{j}} \geq j\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{4 n^{2}}\right)\right)$.
Since $\frac{2}{\pi}\left(\widetilde{z-\alpha_{j}}\right) \leq\left|z-\alpha_{j}\right|$ then

$$
\frac{1}{\left|z-\alpha_{j}\right|} \leq \frac{\pi}{2} \frac{1}{\widetilde{z-\alpha_{j}}} \leq \frac{\pi}{2} \frac{1}{j} \frac{1}{\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{4 n^{2}}\right)\right)}=\frac{2 n}{j} \frac{\pi}{2\left(2 \pi+\mathcal{O}\left(\frac{1}{2 n}\right)\right)} \leq \frac{2 n}{j} E .
$$

By applying that $W_{2 n}(z)$ is bounded on $\mathbb{T}$ we get (4).
(iii) Taking into account that

$$
\frac{\pi}{2}\left|z-\alpha_{j+1}\right| \geq \overline{z-\alpha_{j+1}} \geq j\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{4 n^{2}}\right)\right) \geq \sqrt{n}\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{4 n^{2}}\right)\right)
$$

and the fact that $\alpha_{j+1}-\alpha_{j}=\mathcal{O}\left(\frac{1}{n}\right)$ we have

$$
\frac{z-\alpha_{j}}{z-\alpha_{j+1}}=\frac{z-\alpha_{j+1}+\alpha_{j+1}-\alpha_{j}}{z-\alpha_{j+1}}=1+\frac{\alpha_{j+1}-\alpha_{j}}{z-\alpha_{j+1}}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)
$$

Proof of Lemma 2. (i) Since $\left|\theta_{j}-\theta_{j+1}\right|=\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{n^{2}}\right)$ then

$$
\frac{\alpha_{j+1}}{\alpha_{j}}=e^{i\left(\theta_{j+1}-\theta_{j}\right)}=e^{i\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\right)}=1+\mathcal{O}\left(\frac{1}{n}\right)
$$

which proves the first part of (5). In the same way

$$
\left(\frac{\alpha_{j+1}}{\alpha_{j}}\right)^{n}=\left(e^{i\left(\theta_{j+1}-\theta_{j}\right)}\right)^{n}=\left(e^{i\left(\frac{2 \pi}{2 n}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\right)}\right)^{n}=(-1) e^{i n \mathcal{O}\left(\frac{1}{n^{2}}\right)}=(-1)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),
$$

from which it follows the second part of (5). Notice that the asymptotic constants of (5) are independent of $n$ and $j$. Notice that the same result is true for the quotients $\frac{\alpha_{j}}{\alpha_{j+1}}$.
(ii) (a) By applying the second equality in (5) we have

$$
\begin{aligned}
\left(\frac{\alpha_{j}}{\alpha_{0}}\right)^{n}=\left(\frac{\alpha_{j}}{\alpha_{j-1}}\right)^{n}\left(\frac{\alpha_{j-1}}{\alpha_{j-2}}\right)^{n} \cdots\left(\frac{\alpha_{1}}{\alpha_{0}}\right)^{n} & =\underbrace{\left(-1+\mathcal{O}\left(\frac{1}{n}\right)\right) \cdots\left(-1+\mathcal{O}\left(\frac{1}{n}\right)\right)}_{j \text { factors }} \\
& =(-1)^{j} \underbrace{\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \cdots\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)}_{j \text { factors }}
\end{aligned}
$$

Due to Lemma 15.3 in [22] we have that

$$
\left\lvert\, \begin{array}{r}
|\underbrace{\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \cdots\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)}_{j \text { factors }}-1| \leq \underbrace{\left(1+\left|\mathcal{O}\left(\frac{1}{n}\right)\right|\right) \cdots\left(1+\left|\mathcal{O}\left(\frac{1}{n}\right)\right|\right)}_{j \text { factors }}-1 \\
\leq \exp \sum_{1}^{j}\left|\mathcal{O}\left(\frac{1}{n}\right)\right|-1 \leq \exp \left|\mathcal{O}\left(\frac{j}{n}\right)\right|-1 \leq\left|\mathcal{O}\left(\frac{j}{n}\right)\right| \tag{20}
\end{array}\right.
$$

which implies $\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \cdots\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)=1+\mathcal{O}\left(\frac{j}{n}\right)$, and thus the first part of (6) is proved. Although we maintain the same notation of $\mathcal{O}\left(\frac{j}{n}\right)$ in the last equality, the asymptotic constant is different from the one that appeared en (20). The second part is an immediate consequence of the first one.
(b) It can be obtained in a similar way.

In the sequel when we apply in an equality the identity $\frac{A+B}{C+D}=\frac{A}{C}+\frac{C B-A D}{C(C+D)}$, we will use the notation $\xlongequal{\circ}$ instead of $=$.
(iii) (a) Since $\frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)}=\frac{\alpha_{j} W_{2 n}^{\prime}\left(\alpha_{j}\right)}{\alpha_{j+1} W_{2 n}^{\prime}\left(\alpha_{j+1}\right)} \frac{\alpha_{j+1}}{\alpha_{j}}$, by using (1) and (5) we get

$$
\begin{array}{r}
\frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)}=\frac{2 n \alpha_{j}^{2 n} \overline{\Pi\left(\alpha_{j}\right)}+\mathcal{O}(1)}{2 n \alpha_{j+1}^{2 n} \overline{\Pi\left(\alpha_{j+1}\right)}+\mathcal{O}(1)} \frac{\alpha_{j+1}}{\alpha_{j}} \\
\doteq\left[\frac{\alpha_{j}^{2 n} \overline{\Pi\left(\alpha_{j}\right)}}{\alpha_{j+1}^{2 n} \overline{\Pi\left(\alpha_{j+1}\right)}}+\frac{\mathcal{O}(1) 2 n \alpha_{j+1}^{2 n} \overline{\Pi\left(\alpha_{j+1}\right)}-\mathcal{O}(1) 2 n \alpha_{j}^{2 n} \overline{\Pi\left(\alpha_{j}\right)}}{2 n \alpha_{j+1}^{2 n} \overline{\Pi\left(\alpha_{j+1}\right)}\left[2 n \alpha_{j+1}^{2 n} \overline{\Pi\left(\alpha_{j+1}\right)}+\mathcal{O}(1)\right]}\right]\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\
=\left[\left(\frac{\alpha_{j}}{\alpha_{j+1}}\right)^{2 n} \overline{\overline{\Pi\left(\alpha_{j}\right)}}\right. \\
\left.\overline{\overline{\Pi\left(\alpha_{j+1}\right)}}+\mathcal{O}\left(\frac{1}{n}\right)\right]\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\
=\left[\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \frac{\overline{\Pi\left(\alpha_{j}\right)}}{\overline{\Pi\left(\alpha_{j+1}\right)}}+\mathcal{O}\left(\frac{1}{n}\right)\right]\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)=\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) .
\end{array}
$$

(b) By using the first part of this item

$$
\begin{array}{r}
\frac{W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}{W_{2 n}^{\prime}\left(\alpha_{j}\right) \alpha_{j}}=\frac{\alpha_{0}}{\alpha_{j}} \frac{W_{2 n}^{\prime}\left(\alpha_{0}\right)}{W_{2 n}^{\prime}\left(\alpha_{1}\right)} \cdots \frac{W_{2 n}^{\prime}\left(\alpha_{j-1}\right)}{W_{2 n}^{\prime}\left(\alpha_{j}\right)} \\
=\frac{\alpha_{0}}{\alpha_{j}} \underbrace{\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \cdots\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)}_{j \text { factors }}=\frac{1}{\frac{\alpha_{j}}{\alpha_{0}}}\left(1+\mathcal{O}\left(\frac{j}{n}\right)\right) .
\end{array}
$$

Now by taking into account that if $j<\sqrt{n}$ then $\mathcal{O}\left(\frac{j}{n}\right)=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and by using the fact that $\frac{\alpha_{j}}{\alpha_{0}}=1+\mathcal{O}_{1}\left(\frac{j}{n}\right)$ we get

$$
\frac{\alpha_{0}}{\alpha_{j}}\left(1+\mathcal{O}\left(\frac{j}{n}\right)\right)=\frac{1+\mathcal{O}\left(\frac{j}{n}\right)}{1+\mathcal{O}_{1}\left(\frac{j}{n}\right)} \stackrel{=1+\frac{\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)}{1+\mathcal{O}\left(\frac{j}{n}\right)}, ., ~}{\text {, }}
$$

which proves (b).
(iv) By applying the properties given in (i) and (ii) at the beginning of section 2 related to the behavior of $\phi_{n}^{*}(z), \phi_{n}(z)$ and their derivatives, we have

$$
\frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}=\frac{\left(z^{2 n} \overline{\Pi(z)}+\tau \Pi(z)+\mathcal{O}\left(r^{2 n}\right)\right) \alpha_{0}^{n} 2 n}{z^{n}\left(2 n \alpha_{0}^{2 n-1} \overline{\Pi\left(\alpha_{0}\right)}+\mathcal{O}(1)\right) \alpha_{0}}
$$

From the fact that $W_{2 n}\left(\alpha_{0}\right)=0$ we obtain $\tau=\frac{-\phi_{2 n}\left(\alpha_{0}\right)}{\phi_{2 n}^{*}\left(\alpha_{0}\right)}$, that is,

$$
\tau=\frac{-\alpha_{0}^{2 n} \overline{\Pi\left(\alpha_{0}\right)}+\mathcal{O}\left(r^{2 n}\right)}{\Pi\left(\alpha_{0}\right)+\mathcal{O}\left(r^{2 n}\right)} \stackrel{-\alpha_{0}^{2 n} \overline{\Pi\left(\alpha_{0}\right)}}{\Pi\left(\alpha_{0}\right)}+\mathcal{O}\left(r^{2 n}\right)
$$

Therefore

$$
\begin{aligned}
& \frac{W_{2 n}(z) \alpha_{0}^{n} 2 n}{z^{n} W_{2 n}^{\prime}\left(\alpha_{0}\right) \alpha_{0}}=\frac{\left(z^{2 n} \overline{\Pi(z)}-\frac{\alpha_{0}^{2 n} \overline{\Pi\left(\alpha_{0}\right)}}{\Pi\left(\alpha_{0}\right)} \Pi(z)\right) \alpha_{0}^{n} 2 n+\mathcal{O}\left(2 n r^{2 n}\right)}{2 n z^{n} \alpha_{0}^{2 n} \overline{\Pi\left(\alpha_{0}\right)}+\mathcal{O}(1)} \\
& \stackrel{z^{2 n} \overline{\Pi(z)}-\frac{\alpha_{0}^{2 n} \overline{\Pi\left(\alpha_{0}\right)}}{\Pi \Pi\left(\alpha_{0}\right)}}{z^{n} \alpha_{0}^{n} \overline{\Pi\left(\alpha_{0}\right)}}+\frac{\mathcal{O}(1)}{2 n}=2 i \Im\left(\frac{z^{n} \overline{\Pi(z)}}{\alpha_{0}^{n} \overline{\Pi\left(\alpha_{0}\right)}}\right)+\frac{\mathcal{O}(1)}{2 n} .
\end{aligned}
$$

Now, if we use that $\frac{\overline{\Pi(z)}}{\overline{\Pi\left(\alpha_{0}\right)}}=1+\frac{\overline{\Pi(z)}-\overline{\bar{\Pi}\left(\alpha_{0}\right)}}{\overline{\Pi\left(\alpha_{0}\right)}}, \overline{\Pi\left(\alpha_{0}\right)} \neq 0$, and $\overline{\Pi(z)}$ is a lipschitz function then

$$
\left|\frac{\overline{\Pi(z)}-\overline{\Pi\left(\alpha_{0}\right)}}{\overline{\Pi\left(\alpha_{0}\right)}}\right| \leq K\left|z-\alpha_{0}\right|=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) .
$$

Hence

$$
\begin{aligned}
& 2 i \Im\left(\frac{z^{n} \overline{\Pi(z)}}{\alpha_{0}^{n} \overline{\Pi\left(\alpha_{0}\right)}}\right)+\frac{\mathcal{O}(1)}{2 n} \\
&=2 i \Im\left(\left(e^{-i \frac{\pi d}{n}}\right)^{n}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)\right)+\frac{\mathcal{O}(1)}{2 n}=-2 i \sin \pi d+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) .
\end{aligned}
$$

(v) (a) By taking into account that $\alpha_{j}=\alpha_{0} e^{-i\left(\frac{2 \pi}{2 n} j+\mathcal{O}\left(\frac{j}{n^{2}}\right)\right)}$ then

$$
\frac{1}{\left(\frac{z}{\alpha_{j}}-1\right) 2 n}=\frac{1}{\left(\frac{\alpha_{0} e^{\frac{i 2 \pi d}{2 n}}}{\alpha_{0} e^{-i\left(\frac{2 \pi}{2 n} j+\mathcal{O}\left(\frac{j}{n^{2}}\right)\right)}}-1\right) 2 n}=\frac{1}{\left(e^{i\left(\frac{2 \pi}{2 n}(d+j)+\mathcal{O}\left(\frac{j}{n^{2}}\right)\right)}-1\right) 2 n}
$$

If we take into account that for small values of $x$ it holds $e^{i x}-1=i x+\mathcal{O}\left(x^{2}\right)$, then

$$
\begin{array}{r}
\frac{1}{\left(\frac{z}{\alpha_{j}}-1\right) 2 n}=\frac{1}{\left(i\left(\frac{2 \pi}{2 n}(d+j)+\mathcal{O}\left(\frac{j}{n^{2}}\right)\right)+\mathcal{O}\left(\frac{2 \pi}{2 n}(d+j)+\mathcal{O}\left(\frac{j}{n^{2}}\right)\right)^{2}\right) 2 n} \\
=\frac{1}{i 2 \pi(d+j)+\mathcal{O}\left(\frac{j}{n}\right)+\mathcal{O}\left(\frac{(d+j)^{2}}{n^{2}}\right) 2 n}=\frac{1}{i 2 \pi(d+j)+\mathcal{O}\left(\frac{(d+j)^{2}}{n}\right)} \\
=\frac{1}{i 2 \pi(d+j)}\left(\frac{1}{1+\mathcal{O}\left(\frac{d+j}{n}\right)}\right)=\frac{1}{i 2 \pi(d+j)}\left(1+\mathcal{O}\left(\frac{d+j}{n}\right)\right)=\frac{1}{i 2 \pi(d+j)}+\mathcal{O}\left(\frac{1}{n}\right) .
\end{array}
$$

(b) It is obtained in a similar way.
(vi) It is an immediate consequence of the preceding items (i) and (iii) and Lemma 1. Indeed

$$
\begin{aligned}
& \frac{\alpha_{j+1}^{n} W_{2 n}^{\prime}\left(\alpha_{j}\right)\left(z-\alpha_{j}\right)}{\alpha_{j}^{n} W_{2 n}^{\prime}\left(\alpha_{j+1}\right)\left(z-\alpha_{j+1}\right)}=\frac{\alpha_{j+1}^{n}}{\alpha_{j}^{n}} \frac{W_{2 n}^{\prime}\left(\alpha_{j}\right)}{W_{2 n}^{\prime}\left(\alpha_{j+1}\right)} \frac{z-\alpha_{j}}{z-\alpha_{j+1}} \\
& =\left(-1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right),
\end{aligned}
$$

from which it follows the result.
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Editors J. Comput. Appl. Math.
Dear Editors:
We want to thank to the referee for their useful comments that have improved this new version of the paper.

In this version we have taken into account all the comments made by the referee. More precisely:
1.1. References.

We have added the books recommended by the referee, as well as the all the papers except which are left to the discretion of the authors because they are less related with the subject.

## 1.2.

The distribution of the nodal points is given in (2). With respect to the question if one of the ends of the $\operatorname{arc} \mathrm{A}$ is a point of interpolation, we have added the following explanation before Theorem 5:
"Notice that $\alpha_{0}$, which depends on n, may be equal or different to $a_{2}$, which is a difference with respect to classical and recent references. Moreover, we denote the points of $\mathbb{T}$ refering to $\alpha_{0}$. An alternative point of view is to consider that the arc A ends (for the interpolatory process) in $\alpha_{0}$. This last interpretation connects better with other references."
1.3.

We have corrected all the remarks related to the grammar or the notation made by the referee.
We have formatted the splitted equalities/inequalities with the corresponding symbols on the lefthand side.

We have cleared the meaning of reversed polynomials (Section 2) and the set In (page 8) given their definitions. We have also cleared the statements "behaves like" in Theorem 6 and "Following the ideas of Krylov" in section 6 by adding some short comments.

On behalf of the authors and as the corresponding author, sincerely yours.

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[^0]:    Mathematics Subject Classification (2000): 41A05, 65D05, 42C05.

