30 October 1997

PHYSICS LETTERS B

ELSEVIER

Physics Letters B 412 (1997) 309–319

Effective action of the $N=2$ Maxwell multiplet in harmonic superspace [1]

I.L. Buchbinder [a], E.I. Buchbinder [b], E.A. Ivanov [b], S.M. Kuzenko [c,2], B. A. Ovrut [d]

[a] *Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia*
[b] *Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia*
[c] *Institut für Theoretische Physik, Universität Hannover, Appelstr. 2, 30167 Hannover, Germany*
[d] *School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA*
and Department of Physics, University of Pennsylvania, Philadelphia, PA 19104-6396, USA

Received 2 April 1997
Editor: M. Dine

Abstract

We present, in the $N=2$, $D=4$ harmonic superspace formalism, a general method for constructing the off-shell effective action of an $N=2$ abelian gauge superfield coupled to matter hypermultiplets. Using manifestly $N=2$ supersymmetric harmonic supergraph techniques, we calculate the low-energy corrections to the renormalized one-loop effective action in terms of $N=2$ (anti)chiral superfield strengths. For a harmonic gauge prepotential with vanishing vacuum expectation value, corresponding to massless hypermultiplets, the only non-trivial radiative corrections to appear are non-holomorphic. For a prepotential with non-zero vacuum value, which breaks the $U(1)$-factor in the $N=2$ supersymmetry automorphism group and corresponds to massive hypermultiplets, only non-trivial holomorphic [Bcorrections arise at leading order. These holomorphic contribution are consistent with Seiberg's quantum correction to the effective action, while the first non-holomorphic contribution in the massless case is the $N=2$ supersymmetrization of the Heisenberg-Euler effective Lagrangian. © 1997 Elsevier Science B.V.

$N=2$ supersymmetric field theories possess remarkable properties both at the classical and quantum levels. Applications of $N=2$ supersymmetry range from superstring theory to topological field theory, supergauge models and special geometry (see [1] for a modern review). Although the theory of $N=2$ supersymmetry has a long history, it still has properties yet to be explored.

During the last few years, quantum aspects of $N=2$ supersymmetric theories have excited considerable interest. This interest was inspired by the seminal papers of Seiberg and Witten [2] where the non-perturbative contribution to the low-energy effective action of the $N=2$, $SU(2)$ super Yang-Mills model were calculated

[1] To the memory of Professor V.I. Ogievetsky.
[2] Alexander von Humboldt Research Fellow. On leave from Department of Quantum Field Theory, Tomsk State University, Tomsk 634050, Russia.

PII S0370-2693(97)01025-3

exactly. The content of Refs. [2] is essentially based on the structure of the low-energy effective action proposed in Ref. [3] (see also [4]).

A key element of the whole approach of [2] is the statement that the leading contribution to the low-energy effective action of $N=2$ super Yang-Mills theory is represented by a single holomorphic function of the $N=2$ chiral superfield strength W. A detailed investigation of this statement, and the calculation of non-leading contributions to the low-energy effective action, have been undertaken in recent papers [5–9] [3].

As is well known, an adequate description of quantum N-extended supersymmetric field theories can be achieved in terms of unconstrained superfields given on an appropriate N-extended superspace. However, the analysis of Refs. [5–9], as well as the main statement of Ref. [3], were based on the formulation of $N=2$ supersymmetric theories in terms of $N=1$ superfields. Such formulations lack manifest $N=2$ supersymmetry which, in general, gets closed partly on-shell. Since these formulations do not keep $N=2$ supersymmetry manifest at all stages of the computation, they can lead to a number of obstacles. In this respect, the problem of calculating the effective action of $N=2$ theories in terms of unconstrained $N=2$ superfields appears to be of importance.

$N=2$ supersymmetric theories can be formulated in standard $N=2$ superspace in terms of constrained superfields. For a special $N=2$ matter multiplet (the so called relaxed hypermultiplet [11]) and the gauge multiplet the corresponding constraints were solved in [11,12]. However, these formulations look extremely complicated when the interaction is switched on and, in our opinion, are very difficult to use for the computation of the effective action.

A constructive and elegant approach to the description of theories with extended supersymmetry is based on the concept of harmonic superspace [13–16]. It allows one to investigate different extended supersymmetric models naturally and simply. As to $N=2$ models, their formulation using the harmonic superspace approach looks quite transparent.

In this letter we begin an investigation of the quantum aspects of $N=2$, $D=4$ supersymmetric field theories using the harmonic superspace approach. We study the low-energy structure of the Wilsonian effective action of an abelian gauge superfield coupled to matter superfields.

Because of $N=2$ supersymmetry and gauge invariance, which the harmonic superspace approach allows us to keep manifest, the effective action of the Maxwell multiplet is a non-local functional of the (anti)chiral superfield strengths W and $\overline{W}$ only. In the low-energy limit, when only the leading contribution in the space-time derivatives survives, we are left with a local effective superpotential depending only on W and $\overline{W}$.

The Fayet-Sohnius massless hypermultiplet is described in harmonic superspace by an unconstrained analytic superfield $q^+(\zeta_A,u^+,u^-)$ [13], where $\zeta_A^M \equiv (x_A^m, \theta^{+\alpha}, \bar\theta^+_{\dot\alpha})$ are the coordinates of an analytic subspace of the whole $N=2$, $D=4$ harmonic superspace, $\theta^+_\alpha = \theta^i_\alpha u^+_i$, $\bar\theta^+_{\dot\alpha} = \bar\theta^i_{\dot\alpha} u^+_i$, $\|u^\pm_i\| \in SU(2)$, $i=1,2$. The most characteristic feature of the superfield q^+ is an infinite number of auxiliary fields coming from the harmonic expansions in u^+_i, u^-_i. This is the only possible way to describe the off-shell massless hypermultiplet within the framework of $N=2$ supersymmetry without central charges. The q^+ multiplet is universal, all known $N=2$ matter off-shell multiplets with finite numbers of auxiliary fields (e.g., the relaxed hypermultiplet [11]) are related to it via appropriate duality transformations [17].

The classical action for the hypermultiplet interacting with an abelian gauge superfield $V^{++}(\zeta_A,u^+,u^-)$ is given by

$$S\left[\breve{q}^{\,+},q^+,V^{++}\right] = \int d\zeta_A^{(-4)} du\, \breve{q}^{\,+} V^{++} q^+ . \tag{1}$$

[3] As was noted in [10], such non-leading contributions are described in terms of a real function of W and its conjugate $\overline{W}$.

Here $d\zeta_A^{(-4)} = d^4x_A d^2\theta^+ d^2\bar{\theta}^+$,

$$\nabla^{++} = D^{++} + iV^{++} , \tag{2}$$

and operation $\smile$ called 'smile' denotes the analyticity-preserving conjugation [13] ($\breve{q}^{+} \equiv \bar{q}^{*+}$). The explicit form of the operator D^{++} in the analytic basis, as well as all relevant notation, can be found in Ref. [13].

The $S[\breve{q}^{+},q^+,V^{++}]$ enters as part of the action of $N=2$ supersymmetric electrodynamics

$$S_{\rm SED} = \tfrac{1}{2}\int d^4x d^4\theta\, W^2 + \int d\zeta_A^{(-4)} du\, \breve{q}^{+} (D^{++} + iV^{++}) q^+ . \tag{3}$$

The chiral gauge invariant strength W and its conjugate $\overline{W}$ are expressed via V^{++} by the relation [13,14,16]

$$W = -\int du (\bar{D}^-)^2 V^{++}(x,\theta,u), \quad \overline{W} = -\int du (D^-)^2 V^{++}(x,\theta,u) \tag{4}$$

with $D^{\pm}_{\alpha} = D^i_{\alpha} u^{\pm}_i$, $\bar{D}^{\pm}_{\dot\alpha} = \bar{D}^i_{\dot\alpha} u^{\pm}_i$ the spinor covariant derivatives. For later use, we singled out in V^{++} a background part V_0^{++} and write $V^{++} = V_0^{++} + V_1^{++}$. V_0^{++} possesses a constant strength W_0 and can be chosen to be of the form

$$-(\theta^+)^2 \overline{W}_0 - (\bar{\theta}^+)^2 W_0, \quad W_0 = \text{const}. \tag{5}$$

For $V_0^{++} = 0$, the hypermultiplets are massless. What happens when $V_0^{++} \neq 0$? Whatever the origin of a non-vanishing V_0^{++} (and W_0) may be, such a V_0^{++} breaks the $U(1)$-factor in the $N=2$ superalgebra automorphism group $U(2)$ and gives q^+ a mass $m = |W_0|$ via generating a central charge proportional to the generator of gauge $U(1)$ symmetry [4].

Thus, this theory possesses two different phases associated with two physically different choices; $V_0^{++} = 0$ and $V_0^{++} \neq 0$. Because of the Bianchi identity $D^{\alpha i} D^j_{\alpha} W = \bar{D}^i_{\dot\alpha} \bar{D}^{\dot\alpha j} \overline{W}$, we have

$$\int d^4x d^4\theta (W_1 + W_0)^2 = \int d^4x d^4\theta\, W_1^2 . \tag{6}$$

Thus, $N=2$ Maxwell theory can be treated either as a theory of *massless* superfields q^+, $\breve{q}^{+}$ coupled to gauge superfield V_1^{++} (the first phase), or as a theory of *massive* q^+, $\breve{q}^{+}$ coupled to V_1^{++} (the second phase). We will consider both phases.

Note that an abelian theory with $W_0 = const$ naturally arises as an effective theory describing the spontaneous symmetry breaking phase in $N=2$ super Yang-Mills theory. In this case the classical potential vanishes at non-zero vacuum values of the scalar components of the gauge multiplet and only a $U(1)$-factor of the gauge group survives. In the superfield language, such a situation just corresponds to $W_0 = $ const. (see Ref. [25] for a generic discussion of spontaneous symmetry breakdown in $N=2$ super Yang-Mills theory).

The effective action $\Gamma[V^{++}]$ of the theory (1) is defined by the path integral

$$e^{i\Gamma[V^{++}]} = \int \mathscr{D}\breve{q}^{+} \mathscr{D}q^+ e^{iS[\breve{q}^{+},q^+,V^{++}]} \tag{7}$$

and can be formally written as

$$\Gamma[V^{++}] = i\,\mathrm{Tr}\ln \nabla^{++} . \tag{8}$$

We will calculate $\Gamma[V^{++}]$ starting with this relation, using a suitable definition of the right-hand side of (8).

[4] The fact that the hypermultiplet becomes massive follows from the dynamical equation $(D^{++} + iV_0^{++})q^+ = 0$ which implies $(\Box + m^2)q^+ = 0$, where $m = |W_0|$.

Another, basically equivalent version of the harmonic superspace description of the massless hypermultiplet makes use of an unconstrained analytic superfield $\omega(\zeta_A,u^+,u^-)$ [13]. It should be taken complex when coupled to the Maxwell gauge superfield. We now show that the effective action for the ω version of the hypermultiplet can be computed directly from the q^+ effective action $\Gamma[V^{++}]$. The classical action for ω interacting with the abelian V^{++} is given by

$$S\left[\breve{\omega},\omega,V^{++}\right]=\int d\zeta_A^{(-4)}du\,\nabla^{++}\breve{\omega}\,\nabla^{++}\omega \tag{9}$$

where

$$\nabla^{++}\omega=(D^{++}+iV^{++})\omega, \tag{10}$$

$$\nabla^{++}\breve{\omega}=(D^{++}-iV^{++})\breve{\omega}. \tag{11}$$

The effective action $\Gamma_\omega[V^{++}]$ of the theory (9) is defined by

$$e^{i\Gamma_\omega[V^{++}]}=\int\mathscr{D}\breve{\omega}\mathscr{D}\omega e^{iS[\breve{\omega},\omega,V^{++}]} \tag{12}$$

and can formally be written as

$$\Gamma_\omega[V^{++}]=i\,\mathrm{Tr}\ln(\nabla^{++})^2. \tag{13}$$

Eqs. (8) and (13) lead to the formal relation

$$\Gamma_\omega[V^{++}]=2\Gamma[V^{++}] \tag{14}$$

which, of course, needs justification.

In order to make the above considerations more precise, we consider a theory of two hypermultiplets q_i ($i=1,2$) with the action

$$\tilde{S}\left[\breve{q}^{+i},q_i^+,V^{++}\right]=\int d\zeta_A^{(-4)}du\,\breve{q}^{+i}\nabla^{++}q_i^+ \tag{15}$$

and introduce the corresponding effective action $\tilde{\Gamma}[V^{++}]$ defined by

$$e^{i\tilde{\Gamma}[V^{++}]}=\int\mathscr{D}\breve{q}^{+i}\mathscr{D}q_i^+e^{i\tilde{S}[\breve{q}^{+i},q_i^+,V^{++}]}=e^{2i\Gamma[V^{++}]}. \tag{16}$$

Let us also consider the following change of variables

$$\breve{q}^{+i}=u^{+i}\breve{\omega}+u^{-i}\breve{f}^{++},\quad q_i^+=u_i^+\omega+u_i^-f^{++} \tag{17}$$

with some analytic superfields f^{++}, $\breve{f}^{++}$. Transformation (17) has been introduced in Ref. [15] in order to prove the classical equivalence of the models (15) and (10) at $V^{++}=0$ [5]. The right-hand sides in (17) do not contain any dependence on V^{++} and, hence, the corresponding Jacobian is a constant. Now, putting (17) in path integral (16), and eliminating the auxiliary superfields f^{++} and $\breve{f}^{++}$, one readily finds

$$\tilde{\Gamma}[V^{++}]=\Gamma_\omega[V^{++}]. \tag{18}$$

Comparing this with (16) leads to (14). Thus to find the effective action of the theory (9), it is sufficient to calculate the effective action $\Gamma[V^{++}]$ for the theory (1)

[5] To avoid confusion, we point out that the single q^+ can also be traded for a single *real* ω hypermultiplet via Eq. (17) with $\breve{q}^{+i}=\epsilon^{ik}q_k^+$. In such a ω representation, however, the coupling to V^{++} contains explicit harmonics, which is inconvenient for practical calculations.

For the correct definition of the effective action $\Gamma[V^{++}]$, we consider the G 2) of operator ∇^{++}

$$\nabla_1^{++} G^{(1,1)}(1,2) = \delta_A^{(3,1)}(1,2) \tag{19}$$

where $1,2 \equiv (\zeta_{1,2A}, u_{1,2})$ and $\delta_A^{(3,1)}(1,2)$ is the appropriate analytic subspace δ-function [14]. Let us introduce an analytic superkernel $Q^{(3,1)}(1,2)$ which contains all information about the interaction and is defined by the rule

$$G_0^{(1,1)}(1,2) = \int d\zeta_{3A}^{(-4)} du_3 G^{(1,1)}(1,3) Q^{(3,1)}(3,2) \tag{20}$$

with $G_0^{(1,1)}$ the Greens function of the free hypermultiplet [14]

$$G_0^{(1,1)}(1,2) \equiv \langle \breve{q}^{+}(1) q^{+}(2) \rangle = -\frac{1}{\Box_1} (D_1^+)^4 (D_2^+)^4 \delta^4(x_1 - x_2) \delta^8(\theta_1 - \theta_2) \frac{1}{(u_1^+ u_2^+)^3}. \tag{21}$$

Then we have

$$Q^{(3,1)}(1,2) = \delta_A^{(3,1)}(1,2) + \mathrm{i} V^{++}(1) G_0^{(1,1)}(1,2). \tag{22}$$

With the use of $Q^{(3,1)}(1,2)$, effective action $\Gamma[V^{++}]$ can be defined in the form [6]

$$\Gamma[V^{++}] = \mathrm{i}\,\mathrm{Tr} \ln Q^{(3,1)}. \tag{23}$$

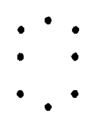
Here the operation Tr is understood in the sense

$$\mathrm{Tr}\,\mathscr{F}^{q,4-q} = \int d\zeta_A^{(-4)} du\, \mathscr{F}^{(q,4-q)}(1,2) \tag{24}$$

for any analytic superkernel $\mathscr{F}^{(q,4-q)}(1,2)$. Eqs. (22)–(24) show that the effective action (23) is well defined within perturbation theory.

We can write the effective action $\Gamma[V^{++}]$ as a perturbation series in powers of the interaction as

$$\Gamma[V^{++}] = \sum_{n=1}^{\infty} \Gamma_n[V^{++}] = \mathrm{i}^2$$ $$-\frac{1}{2}\mathrm{i}^3$$ $$+$$

$$+\frac{1}{3}\mathrm{i}^4$$  $$+ \ldots + \frac{(-1)^{n+1}}{n}\mathrm{i}^{n+1}$$  $$+ \ldots \tag{25}$$

where the n-th term $\Gamma_n[V^{++}]$ is depicted by a supergraph with n external V^{++} -legs.

Eq. (23) leads to the following structure for $\Gamma_n[V^{++}]$

$$\Gamma_n[V^{++}] = \mathrm{i} \frac{(-1)^{n+1}}{n} \mathrm{Tr} \left(\mathrm{i} V^{++} G_0^{(1,1)} \right)^n. \tag{26}$$

Taking into account the antisymmetry of $G_0^{(1,1)}$ [14], one observes that all the coefficients Γ_n with odd n are vanishing. Therefore, only the supergraphs with even numbers of legs contribute to the effective action. $\Gamma[V^{++}]$ can be shown to be gauge invariant. Hence, each coefficient Γ_n (26) can, in fact, only depend on the strengths W, $\overline{W}$ in the low-energy limit.

[6] From a formal point of view, this definition means that $\Gamma[V^{++}] = -\mathrm{i}\,\mathrm{Tr} \ln (G^{(1,1)} / G_0^{(1,1)})$ where we have used the fact that the effective action is always defined up to a constant.

As was previously pointed out, the theory under consideration possesses two different phases corresponding to the cases $V_0^{++}=0$ and $V_0^{++}\neq 0$. First let us discuss the $V_0^{++}=0$ case.

We begin with a direct calculation of the term $\Gamma_2[V^{++}]$ which, in the central basis, reads

$$\Gamma_2[V^{++}] = -\frac{\mathrm{i}^3}{2}\int \mathrm{d}^4x_1\mathrm{d}^4\theta_1^+\mathrm{d}u_1\mathrm{d}^4x_2\mathrm{d}^4\theta_2^+\mathrm{d}u_2\frac{1}{\Box_1}(D_1^+)^4(D_2^+)^4[\delta^4(x_1-x_2)\delta^8(\theta_1-\theta_2)]$$
$$\times\frac{1}{\Box_2}(D_2^+)^4(D_1^+)^4[\delta^4(x_2-x_1)\delta^8(\theta_2-\theta_1)]\frac{V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)}{(u_1^+u_2^+)^3(u_2^+u_1^+)^3} \tag{27}$$

where the explicit form of $G_0^{(1,1)}$ (20) has been used. [7] Let us restore the full Grassmann measure $\mathrm{d}^8\theta_1\mathrm{d}^8\theta_2$ [15], make use of the relation between V^{++} and V^{--} [16]

$$V^{--}(x,\theta,u) = \int \mathrm{d}u_1\frac{V^{++}(x,\theta,u_1)}{(u^+u_1^+)^2},$$

and perform the Fourier transform. As a result one obtains

$$\Gamma_2[V^{++}] = -\frac{\mathrm{i}}{2}\frac{1}{(2\pi)^8}\int \mathrm{d}^4p\mathrm{d}^8\theta\mathrm{d}u\, V^{++}(p,\theta,u)V^{--}(-p,\theta,u)\Pi(p) \tag{28}$$

where

$$\Pi(p) = \int\frac{\mathrm{d}^4q}{q^2(q-p)^2}. \tag{29}$$

Regularizing $\Gamma[V^{++}]$ by the dimensional regularization prescription

$$\Pi(p) \quad\to\quad \Pi_{\mathrm{reg}}(p) = \mu^{2\varepsilon}\int\frac{\mathrm{d}^Dq}{q^2(q^2-p^2)} \tag{30}$$

with $D=4-2\varepsilon$ and μ the normalization parameter, and subtracting the ultraviolet divergence

$$\Gamma_{div}[V^{++}] = \frac{1}{32\pi^2\varepsilon}\int \mathrm{d}^4x\mathrm{d}^4\theta\, W^2 \tag{31}$$

one ends up with the two-leg correction to the renormalized effective action $\Gamma_R[V^{++}]$

$$\Gamma_{2\,R}[V^{++}] = -\frac{1}{32\pi^2}\int \mathrm{d}^4x\mathrm{d}^4\theta\, W\ln\left(-\frac{\Box}{\mu^2}\right)W. \tag{32}$$

An analogous quantum correction has been found in $N=1$ super Yang-Mills theory in [19]. Eq. (32) can be treated as the leading term in the effective action for a weak but rapidly varying gauge superfield. However, for this correction is problematical in the low-energy limit where $p^2\to 0$. To overcome this, we introduce an infrared cutoff Λ^2 using the rule

$$\Pi_{\mathrm{reg}}(0) = \mu^{2\varepsilon}\int_{\Lambda^2}\frac{\mathrm{d}^Dq}{q^4} = \mathrm{i}\pi^2\left(\frac{1}{\varepsilon}+\ln\frac{\mu^2}{\Lambda^2}\right). \tag{33}$$

[7] We use the following notation: $(D^\pm)^2=\frac{1}{4}D^{\pm\alpha}D^\pm_\alpha$, $(\bar D^\pm)^2=\frac{1}{4}\bar D^\pm_{\dot\alpha}\bar D^{\pm\dot\alpha}$ and $(D^+)^4=(D^+)^2(\bar D^+)^2$.

Then, the low-energy correction reads

$$\Gamma_{2R}[V^{++}] = -\frac{1}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}\int d^4x d^4\theta W^2 . \tag{34}$$

Eq. (31) constitutes the only divergence in the theory under consideration. All contributions $\Gamma_n[V^{++}]$ for $n > 2$ are automatically ultraviolet-finite. Clearly, Eq. (34) corresponds to a holomorphic contribution to the effective action.

The next stage is the calculation of the four-leg contribution $\Gamma_4[V^{++}]$ in the low-energy limit. We start with general relation (26) for $n = 4$ and restore the full Grassmann measure $d^8\theta$. As the result, we get

$$\begin{aligned}\Gamma_4[V^{++}] = &-\frac{i}{4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^8\theta_1 d^8\theta_2 du_1 du_2 du_3 du_4 \frac{1}{\Box_1}(D_1^+)^4(D_2^+)^4\\ &\times\left[\delta^4(x_1-x_2)\delta^8(\theta_1-\theta_2)\right]\left(\frac{1}{\Box_2}\delta^4(x_3-x_4)\right)\\ &\times\frac{1}{\Box_3}(D_3^+)^4(D_4^+)^4\left[\delta^4(x_3-x_4)\delta^8(\theta_1-\theta_2)\right]\left(\frac{1}{\Box_4}\delta^4(x_4-x_1)\right)\\ &\times\frac{V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)V^{++}(x_3,\theta_3,u_3)V^{++}(x_4,\theta_4,u_4)}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3(u_3^+u_4^+)^3(u_4^+u_1^+)^3} .\end{aligned} \tag{35}$$

Here we have used the explicit form of $G_0^{(1,1)}$ (20) and integrated over two Grassmann coordinates.

After performing the Fourier transformation of δ-function, the previous expression can be rewritten in the form

$$\begin{aligned}\Gamma_4[V^{++}] = &-\frac{i}{4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^8\theta_1 d^8\theta_2 du_1 du_2 du_3 du_4 \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16}p_1^2p_2^2p_3^2p_4^2}\exp(ip_1(x_1-x_2))\\ &\times\exp(ip_2(x_2-x_3))\exp(ip_3(x_3-x_4))\exp(ip_4(x_4-x_1))\\ &\times\delta^8(\theta_1-\theta_2)V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)\left[D_2^+(-p_1)\right]^4\left[D_1^+(p_1)\right]^4\\ &\times\frac{\left[V^{++}(x_3,\theta_2,u_3)V^{++}(x_4,\theta_1,u_4)\left[D_3^+(p_3)\right]^4\left[D_4^+(-p_3)\right]^4\delta^8(\theta_1-\theta_2)\right]}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3(u_3^+u_4^+)^3(u_4^+u_1^+)^3} .\end{aligned} \tag{36}$$

We have omitted the terms obtained by the action of $[D_1^+(p_1)]^4$ on $V^{++}(x_1,\theta_1,u_1)$ and $[D_2^+(-p_1)]^4$ on $V^{++}(x_2,\theta_2,u_2)$ because they do not contribute in the local limit.

Our aim is to find the local low-energy contribution to $\Gamma_4[V^{++}]$. Due to the supergauge invariance, it should be composed only from the superfield strengths W and $\overline{W}$ at the same point (x,θ). This means that we are led to consider W, $\overline{W}$ as independent functional arguments of $\Gamma_4[V^{++}]$, neglecting all space-time derivatives of these superfields. Taking into account the relation between W and V^{++}, Eq. (4), there is only one possible way to convert all V^{++} into the superfield strengths. It is necessary to distribute eight spinor derivatives among the external lines so as to have an equal number of the derivatives D^+ and $\overline{D}^+$ acting on the Grassmann δ-function; otherwise the result will be zero. It is evident that we get both W and $\overline{W}$ in this manner and, hence, a non-holomorphic contribution.

Let us briefly discuss the possibility to obtain holomorphic contributions. Such a contribution is defined by an integral over the chiral subspace which can be obtained by the rule $\int d^4x d^8\theta \sim \int d^4x d^4\theta \overline{D}^4$. Then we could throw only four spinor derivatives on the external legs and distribute the remainder among the δ-functions. The total number of these derivatives formally suffices to obtain a non-zero result. Unfortunately, all derivatives acting on the external legs should have the same chirality in order to finally get the expression depending only on W. This means that the numbers of D^+'s and $\bar{D}^+$'s acting on the δ-function do not match each other and the final result must vanish. Thus, there is no holomorphic contribution to $\Gamma_4[V^{++}]$.

The only part of $\Gamma_4[V^{++}]$ which contains eight spinor derivatives on external lines can be singled out as follows:

$$\begin{aligned}\Gamma_4[V^{++}] \Rightarrow & -\frac{i}{4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^8\theta_1 d^8\theta_2 du_1 du_2 du_3 du_4 \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16} p_1^2 p_2^2 p_3^2 p_4^2} \\ & \times \exp(ip_1(x_1-x_2))\exp(ip_2(x_2-x_3))\exp(ip_3(x_3-x_4))\exp(ip_4(x_4-x_1)) \\ & \times \delta^8(\theta_1-\theta_2)[D_3^+(p_3)]^4[D_4^+(-p_3)]^4\delta^8(\theta_1-\theta_2) \\ & \times \frac{V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)[D_2^+(-p_1)]^4V^{++}(x_3,\theta_2,u_3)[D_1^+(p_1)]^4V^{++}(x_4,\theta_1,u_4)}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3(u_3^+u_4^+)^3(u_4^+u_1^+)^3}.\end{aligned} \tag{37}$$

After performing the D-algebra and integrating over θ_2 one gets in the low-energy limit

$$\begin{aligned}\Gamma_4[V^{++}] = & -\frac{i}{8(2\pi)^4}\int_{\Lambda^2}\frac{d^4p}{p^8}\int d^4x d^8\theta \int du_1 (D_1^-)^2 V^{++}(x,\theta,u_1)\int du_2 (\bar{D}_2^-)^2 V^{++}(x,\theta,u_2) \\ & \times \int du_3 (\bar{D}_3^-)^2 V^{++}(x,\theta,u_3)\int du_4 (D_4^-)^2 V^{++}(x,\theta,u_4),\end{aligned} \tag{38}$$

with Λ^2 the infrared cutoff. Now let us use the relations (4) which allow us to represent Eq. (38) in a manifestly gauge invariant form

$$\Gamma_4[V^{++}] = \frac{1}{(16\pi)^2\Lambda^4}\int d^4x d^8\theta W^2\bar{W}^2. \tag{39}$$

This result has a simple physical interpretation. Let us keep as non-vanishing only the electromagnetic field components $F_{\mu\nu}$ of W and $\bar{W}$. Then $\Gamma_4[V^{++}]$ turns into

$$\Gamma_4[V^{++}] = \frac{1}{(64\pi)^2}\frac{1}{\Lambda^4}\int d^4x\left\{\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right\}, \tag{40}$$

where $\tilde{F}^{\mu\nu}$ is the dual of $F_{\mu\nu}$. Eq. (40) is, in fact, the first time a non-linear quantum correction to the electromagnetic Lagrangian has been presented for the $N=2$ theories under consideration. This type of correction was originally discussed by Heisenberg and Euler (see, for instance, [20]). Therefore, $\Gamma_4[V^{++}]$ can be interpreted as the $N=2$ supersymmetric generalization of Heisenberg-Euler Lagrangian. By construction, $\Gamma_4[V^{++}]$ is given in a manifestly $N=2$ supersymmetric and gauge covariant form.

It is worth noticing (see [5,10]) that the functional $\int d^4x d^8\theta \bar{W}^2 W^2$ rewritten in terms of $N=1$ superfields contains a contribution with four spinor derivatives of chiral matter superfields. This kind of one-loop quantum correction to the effective action has been found in Ref. [21] and called the effective potential of auxiliary fields (see also [6]).

The above consideration can be generalized to give the $2n$-leg contribution $\Gamma_{2n}[V^{++}]$, for $n=3,4,\ldots$, in the low-energy approximation

$$\Gamma_{2n}[V^{++}] \sim \int d^4x d^8\theta\left(\frac{\bar{W}W}{2\Lambda^2}\right)^n \quad n>1. \tag{41}$$

Eqs. (34) and (41) specify the general form of low-energy effective action. We see that the effective action has both holomorphic and non-holomorphic parts. The holomorphic contribution is simple and stipulated by the ultraviolet divergence. The non-holomorphic contribution has a very special structure; i.e. it depends on W and $\overline{W}$ only via the combination $\overline{W}W$.

To fix the dependence on the arbitrary parameter μ we should, as usual, impose some renormalization conditions. The infrared cutoff Λ, unlike μ, is a physical parameter which, in accordance with the status of the Wilsonian effective action [24], defines the physical scale where we study the low energy phenomena.

We now turn to the calculation of the low-energy effective action for the case when $V_0^{++} \neq 0$. We start from the four-leg contribution (36). In order to obtain a holomorphic contribution one should throw two derivatives D^+ and two derivatives $\overline{D}^+$ on the external lines. The only term which gives a contribution in the local limit is

$$\begin{aligned}\Gamma_4[V^{++}] \Rightarrow & -\frac{i}{4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^8\theta_1 d^8\theta_2 du_1 du_2 du_3 du_4 \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16} p_1^2 p_2^2 p_3^2 p_4^2} \\ & \times \exp(ip_1(x_1-x_2))\exp(ip_2(x_2-x_3))\exp(ip_3(x_3-x_4))\exp(ip_4(x_4-x_1)) \\ & \times \delta^8(\theta_1-\theta_2)\left[\overline{D}_2^+(-p_1)\right]^2\left[D_1^+(p_1)\right]^2\left[D_3^+(p_3)\right]^4\left[D_4^+(-p_3)\right]^4\delta^8(\theta_1-\theta_2) \\ & \times \frac{V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)\left[D_2^+(-p_1)\right]^2 V^{++}(x_3,\theta_2,u_3)\left[\overline{D}_1^+(p_1)\right]^2 V^{++}(x_4,\theta_1,u_4)}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3(u_3^+u_4^+)^3(u_4^+u_1^+)^3}.\end{aligned} \tag{42}$$

The expression we are interested in can be picked out from (42). Using the fact that $V^{++} = V_0^{++} + V_1^{++}$ with V_0^{++} given by Eq. (5), we can conclude that the local holomorphic contribution comes from the following piece of $\Gamma_4[V^{++}]$

$$\begin{aligned}\Gamma_4[V^{++}] = & -\frac{i}{4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^8\theta_1 d^8\theta_2 du_1 du_2 du_3 du_4 \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16} p_1^2 p_2^2 p_3^2 p_4^2} \\ & \times \exp(ip_1(x_1-x_2))\exp(ip_2(x_2-x_3))\exp(ip_3(x_3-x_4))\exp(ip_4(x_4-x_1)) \\ & \times \frac{\delta^8(\theta_1-\theta_2)\left[\overline{D}_2^+(-p_1)\right]^2\left[D_1^+(p_1)\right]^2\left[D_3^+(p_3)\right]^4\left[D_4^+(-p_3)\right]^4\delta^8(\theta_1-\theta_2)}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3(u_3^+u_4^+)^3(u_4^+u_1^+)^3} \\ & \times V^{++}(x_1,\theta_1,u_1)V^{++}(x_2,\theta_2,u_2)\Big[(u_2^+u_3^+)^2\overline{W}_0\left[\overline{D}_1^+(p_1)\right]^2 V_1^{++}(x_4,\theta_1,u_4) \\ & + (u_4^+u_1^+)^2 W_0\left[D_2^+(-p_1)\right]^2 V_1^{++}(x_3,\theta_2,u_3) + (u_2^+u_3^+)^2\overline{W}_0(u_4^+u_1^+)^2 W_0\Big].\end{aligned} \tag{43}$$

In the low-energy limit Eq. (43) gives rise to the gauge invariant contribution

$$\Gamma_4[V^{++}] = -\frac{i}{4}\int\frac{d^4p}{(2\pi)^4 p^6}\int d^4x d^4\theta W^3\overline{W}_0 + \frac{i}{8}\int\frac{d^4p}{(2\pi)^4 p^6}\int d^4x d^4\theta W^2\overline{W}_0 W_0 + \text{h.c.}, \tag{44}$$

where the identity

$$\int d^4x d^8\theta du V^{++}V^{--}K(W) = \int d^4x d^4\theta W^2 K(W) \tag{45}$$

for arbitrary an holomorphic function $K(W)$ has been used.

Analogously, in the $2n$-th order we have

$$\Gamma_{2n}[V^{++}] = -\frac{i}{2n}\int\frac{d^4p}{(2\pi)^4 p^{2n+2}}\int d^4x d^4\theta W^{n+1}\overline{W}_0^{n-1}$$

$$+\frac{\mathrm{i}}{4n}\int\frac{\mathrm{d}^4p}{(2\pi)^4p^{2n+2}}\int\mathrm{d}^4x\mathrm{d}^4\theta W^2\left(\overline{W}_0W_0\right)^{n-1}+\mathrm{h.c.} \tag{46}$$

To calculate the total one-loop effective action we should sum up all contributions (46). This leads to the expression

$$\Gamma[V^{++}]=\frac{1}{32\pi^2}\int\mathrm{d}^4x\mathrm{d}^4\theta\frac{W}{\overline{W}_0}\int\mathrm{d}p^2\ln\left(1+\frac{W\overline{W}_0}{p^2}\right)$$
$$-\frac{1}{64\pi^2}\int\mathrm{d}^4x\mathrm{d}^4\theta\frac{W^2}{W_0\overline{W}_0}\int\mathrm{d}p^2\ln\left(1+\frac{W_0\overline{W}_0}{p^2}\right)+\mathrm{h.c.} \tag{47}$$

After renormalization and doing the momentum integral, one gets

$$\Gamma_R[V^{++}]=\int\mathrm{d}^4x\mathrm{d}^4\theta\mathscr{F}(W)+\mathrm{h.c.}, \tag{48}$$

where

$$\mathscr{F}(W)=\frac{1}{64\pi^2}W^2\left(1-\ln\frac{W^2}{\mu^2}\right). \tag{49}$$

Here all the dependence on W_0, $\overline{W}_0$ has been absorbed in the normalization point μ.

Eqs. (48) and (49) are two of the main results of our paper. We see that the massive branch of the theory, unlike the massless one, allows one to obtain non-trivial holomorphic contribution to the low-energy effective action. This holomorphic contribution does not depend on the infrared cutoff and, hence, it is automatically infrared-finite. To fix the ultraviolet normalization point we should impose, as usual, some renormalization condition such as

$$\mathscr{F}(W)|_{W^2=M^2}=0. \tag{50}$$

It means that the quantum correction to the classical Lagrangian $\frac{1}{2}W^2$ is absent at the scale M. The above condition fixes the normalization point μ and allows us to rewrite Eq. (49) in the form

$$\mathscr{F}(W)=-\frac{1}{64\pi^2}W^2\ln\frac{W^2}{M^2}. \tag{51}$$

It is interesting to note that Eq. (51) coincides, up to sign and numerical coefficient, with the perturbative holomorphic quantum correction to the classical Lagrangian of $N=2$ super Yang-Mills theory which was found by Seiberg [3], based on non-manifestly $N=2$ supersymmetric considerations. The difference in sign and the coefficient is due to two reasons. Firstly, we compute the quantum correction coming from matter superfields, not gauge ones, which leads to the opposite sign of the β-function. Secondly, the present model describes different degrees of freedom as compared to the $N=2$ super Yang-Mills model.

Let us summarize the results. We have developed a general approach to the problem of computing the effective action of the $N=2$, $D=4$ abelian gauge superfield coupled to massless and massive off-shell hypermultiplets (with the mass arising as an effect of the non-zero vacuum expectation value of the gauge superfield). This approach is based on the formulation of $N=2$ supersymmetric theories in harmonic superspace and guarantees manifest $N=2$ supersymmetry at each step of the computation. We have demonstrated that the $N=2$ supergraph techniques of Refs. [14,15] are suitable for the investigation of a broad class of $N=2$ supersymmetric theories in the same way, and with the same degree of efficiency, as the well known $N=1$ supergraph techniques (see, for instance, Refs. [22,23]).

Theory (1) possesses two different phases corresponding to massless and massive hypermultiplets. The renormalized Wilsonian effective action of the Maxwell multiplet was considered for both phases of the theory. We calculated its explicit form, which depends only on the superfield strengths W and $\overline{W}$, in the low-energy limit where all derivatives on the superfield strengths can be neglected. In the massless case, we found that the effective action contains the trivial holomorphic contribution which is stipulated by the ultraviolet divergence and the non-trivial non-holomorphic contributions (39) and (41). These non-holomorphic contributions are automatically ultraviolet-finite and depend on an infrared cutoff Λ defining a physical scale in the theory under consideration. The simplest non-holomorphic contribution (39) leads to the $N=2$ supersymmetric extension of the well-known Heisenberg-Euler lagrangian. The massive branch occurs when the hypermultiplet is coupled to a background gauge superfield V_0^{++} with the constant strength $W_0 \neq 0$. V_0^{++} can be associated with the breakdown of the $U(1)$ factor in the automorphism group $U(1) \times SU(2)$ of $N=2$ supersymmetry. In the massive case, the structure of the effective action is changed drastically as compared to the massless case. Here the effective action contains non-trivial holomorphic contributions. Moreover, their structure is analogous to the low-energy perturbative effective action for $N=2$ super Yang-Mills theory obtained by Seiberg by integrating the R-anomaly [3].

Acknowledgements

I.L.B., E.A.I. and S.M.K. acknowledge a partial support from the RFBR-DFG project No 96-02-00180. Research of I.L.B. and S.M.K. was partly supported by RFBR under the project No 96-02-16017. Research of E.A.I. was partly supported by RFBR under the project No 96-02-17634, by INTAS under the project INTAS-94-2317 and by a grant of the Dutch NWO organization. I.L.B. is very grateful to the Department of Physics, University of Pennsylvania where part of this research was carried out, and for support from the Research Foundation of the University of Pennsylvania. B.A.O. acknowledges the DOE Contract No. DE-AC02-76-ER-03072 for partial support. We thank B.M. Zupnik for collaboration at the early stage of this work and valuable discussions.

References

[1] P. Fre, P. Soriani, The N = 2 Wonderland (World Scientific, Singapore, 1995).
[2] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19; B 430 (1994) 485.
[3] N. Seiberg, Phys. Lett. B 206 (1988) 75.
[4] S.J. Gates, Nucl. Phys. B 238 (1984) 349.
[5] B. de Wit, M.T. Grisaru, M. Roček, Phys. Lett. B 374 (1996) 297.
[6] A. Pickering, P. West, Phys. Lett. B 383 (1996) 54.
[7] M.T. Grisaru, M. Roček, R. von Unge, Phys. Lett. B 383 (1996) 415.
[8] T.E. Clark, S.T. Love, Phys. Lett. B 388 (1996) 577.
[9] U. Lindström, F. Gonzalez-Rey, M. Roček, R. von Unge, Phys. Lett. B 388 (1996) 581.
[10] M. Henningson, Nucl. Phys. B 458 (1996) 445.
[11] P.S. Howe, K.S. Stelle, P.K. Townsend, Nucl. Phys. B 214 (1983) 519.
[12] L. Mezincescu, JINR preprint P2-12572 (1979), (in Russian);
W. Siegel, S.J. Gates, Nucl. Phys. B 189 (1981) 295;
P.S. Howe, K.S. Stelle, P.K. Townsend, Nucl. Phys. B 236 (1984) 125.
[13] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 1 (1984) 469.
[14] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 2 (1985) 601.
[15] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 2 (1985) 617.
[16] B. Zupnik, Teor. Mat. Fiz. 69 (1986) 207 (in Russian); Phys. Lett. B 183 (1987) 175.
[17] A. Galperin, E. Ivanov, V. Ogievetsky, Nucl. Phys. B 282 (1987) 74.
[18] N. Ohta, Phys. Rev. D 32 (1985) 1467;
N. Ohta, H. Sugata, H. Yamaguchi, Ann. Phys. 172 (1986) 26.
[19] S.M. Kuzenko, J.V. Yarevskaya, Yad. Fiz. 56 (1993) 193 (English translation: Phys. At. Nucl. 56 (1993) 396).
[20] C. Itzykson, J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).
[21] I.L. Buchbinder, S.M. Kuzenko, J.V. Yarevskaya, Nucl. Phys. B 411 (1994) 665; Yad. Fiz. 56 (1993) 202 (English translation: Phys. At. Nucl. 56 (1993) 680).
[22] S.J. Gates, M.T. Grisaru, M. Roček, W. Siegel, Superspace (Benjamin-Cummings, Reading, MA, 1983).
[23] I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity (IOP Publ., Bristol and Philadelphia, 1995).
[24] K.J. Wilson, J. Kogut, Phys. Rep. 12 (1974) 75.
[25] L. Alvarez-Gaume, S.F. Hassan, Preprint CERN-TH/96-371, hep-th/9701069.