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Szegő transformation and zeros of analytic perturbations of Chebyshev weights

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Abstract

In this contribution we present a method to find the closest zeros to ± 1 for orthogonal polynomials with respect to analytic perturbations of the Chebyshev weight functions in $[-1, 1]$. The error order we obtain is $\mathcal{O}(n^{-6})$.

Keywords: Analytic weights; Chebyshev measures; Zeros of para-orthogonal polynomials; Zeros of orthogonal polynomials.
2000 MSC: 33C45, 42C05.

1. Introduction

The study of zeros of orthogonal polynomials with respect to analytic weights on the unit circle has focused the attention of researchers after the classical paper by Nevai and Totik [6]. Special mention deserves the contributions of E. B. Saff et al. and B. Simon et al., among others.

In this contribution we deal with analytic properties of those orthogonal polynomials which lead to the zero distribution of para-orthogonal polynomials associated with such a kind of measures. Taking into account the Szegő transformation, we study a similar situation for measures supported on a bounded interval of the real line and, as a consequence, results concerning polynomials orthogonal with respect to analytic modifications of the Chebyshev measure of the first kind are deduced.

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Our main result provides a method to obtain an approximation of the nearest zeros to ± 1 of the orthogonal polynomials related to the analytic modifications of the Chebyshev measure of the first kind, that is, $d\mu_1(x) = \frac{w(x)}{\sqrt{1-x^2}}dx$. The approximation obtained with a low computational cost has order $\mathcal{O}(n^{-6})$. This method is extended to the first k-th zeros closest to ± 1. Furthermore, we discuss the case of analytic modifications of the other three Chebyshev measures. Finally, an illustrative list of examples is given.

2. Background

2.1. Transformation of modifications of the Chebyshev weights

Let $\omega(x)$ be a positive function on $[-1,1]$ such that the following modifications of the Chebyshev weights

$$d\mu_1(x) = \frac{\omega(x)}{\sqrt{1-x^2}}dx, d\mu_2(x) = \omega(x)\sqrt{1-x^2}dx,$$
$$d\mu_3(x) = \omega(x)\sqrt{\frac{1+x}{1-x}}dx \text{ and } d\mu_4(x) = \omega(x)\sqrt{\frac{1-x}{1+x}}dx$$

are finite positive Borel measures on $[-1,1]$ with finite moments of all order. We denote the sequences of monic orthogonal polynomials with respect to these measures by $\{P_{n,\mu_k}(x)\}_{n\geq 0}$ for $1 \leq k \leq 4$.

Let $d\nu(\theta) = \frac{1}{2}\omega(\cos\theta)d\theta$, with $z = e^{\imath\theta}$, be the corresponding measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ obtained through the Joukowsky transformation of the measure $d\mu_1$. If we denote by $\{\phi_n(z)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to $d\nu$ and by $\{\phi_n^*(z)\}_{n\geq 0}$ the sequence of reversed polynomials defined by $\phi_n^*(z) = z^n\overline{\phi_n}(\frac{1}{z})$, then in the next Proposition the relations between the polynomial sequences $\{\phi_n(z)\}_{n\geq 0}$ and $\{P_{n,\mu_k}(x)\}_{n\geq 0}, 1 \leq k \leq 4$, are given. Notice that in this situation $\phi_n(z)$ has real coefficients.

Proposition 1. *Let $\omega(x)$ be a positive function on $[-1,1]$ such that $d\mu_1, d\mu_2, d\mu_3,$ and $d\mu_4$, defined as aqbove, are finite positive Borel measures on $[-1,1]$ with finite moments of all order. Let $d\nu(\theta) = \frac{1}{2}\omega(\cos\theta)d\theta$, with $z = e^{\imath\theta}$ be the transformed measure of $d\mu_1$ through the Joukowsky transformation. If $x = \frac{z+z^{-1}}{2}$, then the following relations hold*

(i)

$$P_{n,\mu_1}(x) = \frac{1}{2^n(1+\phi_{2n}(0))}(z^{-n}\phi_{2n}(z) + z^n\phi_{2n}(\frac{1}{z})) = \frac{1}{2^n(1+\phi_{2n}(0))}\frac{\phi_{2n}(z)+\phi_{2n}^*(z)}{z^n}.$$

(ii)

$$P_{n,\mu_2}(x) = \frac{1}{2^n(1-\phi_{2n+2}(0))}\frac{z^{-(n+1)}\phi_{2n+2}(z) - z^{n+1}\phi_{2n+2}(\frac{1}{z})}{z-z^{-1}} = \frac{1}{2^n(1-\phi_{2n+2}(0))}\frac{\phi_{2n+2}(z)-\phi^*_{2n+2}(z)}{z^n(z^2-1)}.$$

(iii)

$$P_{n,\mu_3}(x) = \frac{1}{2^n(1+\phi_{2n+1}(0))}\frac{z^{-(n+1)+\frac{1}{2}}\phi_{2n+1}(z) + z^{n+1-\frac{1}{2}}\phi_{2n+1}(\frac{1}{z})}{z^{\frac{1}{2}}+z^{-\frac{1}{2}}} = \frac{1}{2^n(1+\phi_{2n+1}(0))}\frac{\phi_{2n+1}(z)+\phi^*_{2n+1}(z)}{z^n(z+1)}.$$

(iv)

$$P_{n,\mu_4}(x) = \frac{1}{2^n(1-\phi_{2n+1}(0))}\frac{z^{-(n+1)+\frac{1}{2}}\phi_{2n+1}(z) - z^{n+1-\frac{1}{2}}\phi_{2n+1}(\frac{1}{z})}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}} = \frac{1}{2^n(1-\phi_{2n+1}(0))}\frac{\phi_{2n+1}(z)-\phi^*_{2n+1}(z)}{z^n(z-1)}.$$

Proof. The four relations are well-known. The first and second ones can be seen in [9] and [2]. The two last relations can be seen in [1] and [2]. □

Since the previous formulas are related to para-orthogonal polynomials, we recall their definition as well as some properties. If ν is an arbitrary measure on $\mathbb{T}$ with monic orthogonal polynomial sequence $\{\phi_n(z)\}_{n\geq 0}$ and τ is a unimodular complex number, the para-orthogonal polynomials $W_n(z,\tau)$ are defined by $W_n(z,\tau) = \phi_n(z) + \tau\phi_n^*(z), n \geq 0$. It is very well-known that para-orthogonal polynomials have complex zeros which are simple and located on the unit circle (see [4]).

In particular, under the hypothesis of Proposition 1, if $\tau = 1$, then $\phi_n(z)+\phi_n^*(z)$ has simple and unimodular complex zeros as well as, when n is odd, -1 is also a zero. In any case the zeros different from -1 are pairwise conjugated. If $\tau = -1$, then the para-orthogonal polynomial $\phi_n(z) - \phi_n^*(z)$ has also simple and unimodular complex zeros as well as, when n is even, ± 1 are also zeros and, when n is odd, 1 is a zero. In any case the zeros different from ± 1 are pairwise conjugated.

Since the polynomials in the numerators of the right-hand side of (i), (ii), (iii), and (iv) are para-orthogonal polynomials, then the above properties hold. In the four cases each pair of conjugated zeros determine a zero (the real part of both complex numbers) of the corresponding polynomials P_{n,μ_i}. Hence if we approximate the zeros of these para-orthogonal polynomials, then we are able to approximate the zeros of the polynomials P_{n,μ_i}. This property will be used to obtain the zeros of P_{n,μ_i} which are closer to 1 and -1.

2.2. Analytic weights on the unit circle and zeros of para-orthogonal polynomials

Analytic weights on the circle were introduced by Nevai and Totik in [6] and they have been subsequently studied by several authors, (see [5], [7]). In the sequel we present some properties that we believe are generally known.

We recall that an analytic measure ν on the circle is associated with an analytic function on an open annulus, with radii $1/r$ and r, $r < 1$, which is positive on $\mathbb{T}$. In this situation the Szegő function $\Pi(z, \nu)$, given by

$$\Pi(\nu, z) = \exp\{-\frac{1}{4\pi}\int_0^{2\pi} \log \nu'(\theta) \frac{e^{\imath\theta} + z}{e^{\imath\theta} - z} d\theta\}$$

is analytic up to $|z| < \frac{1}{r}$ and the asymptotic results given in the next proposition hold. These results are an extension of those related to the well known Szegő theory. In what follows we consider the normalized Szegő function $\frac{\Pi(\nu,z)}{\Pi(\nu,0)}$ and, for simplicity, we continue denoting by $\Pi(\nu, z)$ this last function. Furthermore, to simplify the notation we write $\Pi(z)$ instead of $\Pi(\nu, z)$.

Proposition 2. *Let ν be an analytic weight on $\mathbb{T}$, $\{\phi_n(z)\}$ the monic orthogonal polynomial sequence related to ν, $\{W_n(z, \tau)\}_{n\geq 0}$ the sequence of para-orthogonal polynomials and $\Pi(z)$ the normalized Szegő function. Then the following relations hold*

(i) $\lim_{n\to\infty} \phi_n^*(z) = \Pi(z)$ *uniformly on $\mathbb{T}$. Moreover $\phi_n^*(z) = \Pi(z) + \mathcal{O}(r^n)$ for $z \in \mathbb{T}$.*

(ii) $\lim_{n\to\infty} \frac{\phi_n(z)}{z^n} = \overline{\Pi}(\frac{1}{z})$ *uniformly on $\mathbb{T}$. Moreover $\phi_n(z) = z^n\overline{\Pi}(\frac{1}{z}) + \mathcal{O}(r^n)$ for $z \in \mathbb{T}$.*

(iii) $\lim_{n\to\infty} (\phi_n^*)'(z) = \Pi'(z)$ *uniformly on $\mathbb{T}$. Moreover $(\phi_n^*)'(z) = \Pi'(z) + \mathcal{O}(n^2r^n)$ for $z \in \mathbb{T}$.*

(iv) $\lim_{n\to\infty} \frac{\phi_n'(z)}{nz^{n-1}} = \overline{\Pi}(\frac{1}{z})$ *uniformly on $\mathbb{T}$. Moreover $\phi_n'(z) = nz^{n-1}\overline{\Pi}(\frac{1}{z}) + \mathcal{O}(1)$ for $z \in \mathbb{T}$.*

(v) There exist positive real numbers D and E such that for every $z \in \mathbb{T}$ and for all n it holds $|W_n(z, \tau)| \leq D$, $nE \leq |W_n'(z, \tau)| \leq nD$. Moreover $W_n'(z, \tau) = nz^{n-1}\overline{\Pi}(\frac{1}{z}) + \mathcal{O}(1)$.

Notice that the preceding properties are uniformly satisfied on $\mathbb{T}$.

Proof. We give an sketch of the proofs in order to make the paper self-contained.

(i) The first part is well-known. For the second statement we use the Szegő's recurrence relation $\phi_{n+1}^*(z) - \phi_n^*(z) = \overline{\phi_{n+1}(0)}z\phi_n(z)$, (see [9]). From the properties of the orthogonality measure we know that $\phi_n^*(z)$ converges uniformly to $\Pi(z)$ for $|z| < \frac{1}{r}$ and, therefore, $\phi_n^*(z)$ and

$\phi_n(z)$ are bounded for $|z|=1$. Moreover, it holds that $|\phi_n(0)| = \mathcal{O}(r^n)$, (see [6]). Then for $z \in \mathbb{T}$ we have $|\phi_{n+1}^*(z) - \phi_n^*(z)| \leq Cr^{n+1}$ and, therefore, $|\phi_{n+m}^*(z) - \phi_n^*(z)| \leq C\frac{(1-r^m)}{(1-r)}r^{n+1}$. Hence, if we take limits when $m \to \infty$ we obtain $|\Pi(z) - \phi_n^*(z)| \leq C\frac{r}{1-r}r^n$ for $z \in \mathbb{T}$.

(ii) The first part is well-known. The second part follows in a straightforward way from (i).

(iii) The first statement follows from (i). To prove the second part we denote by $\varepsilon_n(z)$ the analytic function on the disk $|z| < \frac{1}{r}$,

$$\varepsilon_n(z) = \phi_n^*(z) - \Pi(z). \tag{1}$$

If $\varepsilon_n(z) = \sum_{k=0}^{\infty} A_{k,n} z^k$ then the coefficients are bounded in the following form. By applying Cauchy's theorem we get

$$A_{k,n} = \left| \frac{1}{2\pi\imath} \int_{\mathbb{T}} \frac{\phi_n^*(z) - \Pi(z)}{z^{k+1}} dz \right| \leq C_1 r^n,$$

for every k, that is, $A_{k,n} = \mathcal{O}(r^n)$.

Moreover, for $k \geq n+1$ $A_{k,n} \leq C_2 r^k$, that is, $A_{k,n} = \mathcal{O}(r^k)$. Therefore, since $\varepsilon_n'(z) = (\phi_n^*)'(z) - \Pi'(z) = \sum_{k=1}^{\infty} k A_{k,n} z^{k-1}$, it is clear that $(\phi_n^*)'(z) = \Pi'(z) + \mathcal{O}(n^2 r^n)$.

(iv) From (1) we get

$$\phi_n(z) = z^n \overline{\Pi}(\frac{1}{z}) - z^n \overline{\varepsilon_n}(\frac{1}{z}). \tag{2}$$

Then, if we take derivatives we obtain

$$\phi_n'(z) = nz^{n-1}\left(\overline{\Pi}(\frac{1}{z}) - \overline{\varepsilon_n}(\frac{1}{z})\right) + z^n\left[\left(\overline{\Pi}(\frac{1}{z})\right)' - \left(\overline{\varepsilon_n}(\frac{1}{z})\right)'\right],$$

from which the statement follows in a straightforward way.

(v) From (i) and (ii) we get that there exists a positive constant D such that $|W_n(z,\tau)| \leq D$ for every n and for all z. From the Bernstein inequality it follows that $|W_n'(z,\tau)| \leq nD$ for every n and for all $z \in \mathbb{T}$. The bounded from below and the last part are direct consequences of (iii) and (iv).

$\square$

Next we present a result about the location of zeros of para-orthogonal polynomials.

Proposition 3. *Let $W_n(z,\tau)$ be a nth para-orthogonal polynomial with respect to the measure ν analytic on $\mathbb{T}$ and with Szegő function $\Pi(z)$ analytic up to $|z| < \frac{1}{r}$. with $r<1$. Let α_1 and α_2 be two consecutive zeros of $W_n(z,\tau)$. If we assume that $\alpha_1 = e^{\imath\theta_1}$ and $\alpha_2 = e^{\imath\theta_2}$, then*

$$|\theta_1 - \theta_2| - \frac{2\pi}{n} = \mathcal{O}(\frac{1}{n^2}). \tag{3}$$

Furthermore, each zero α_i of $W_n(z,\tau)$ has associated a zero β_i of $z^n\overline{\Pi}(\frac{1}{z})+\tau\Pi(z)=0$ in such a way that $|\beta_i-\alpha_i|=\mathcal{O}(r^n)$.

Proof. Maybe the simplest way to prove the result is to illustrate it as a problem of mobile bodies around a circular velodrome $\mathbb{T}$, with θ playing the role of the time.
To solve the equation $W_n(z,\tau)=0$, that is, $\phi_n(z)+\tau\phi_n^*(z)=0$, we write it in the equivalent form

$$z^n=-\frac{\tau\phi_n^*(z)}{\frac{\phi_n(z)}{z^n}}. \tag{4}$$

If $z=e^{\imath\theta}$, the left hand side of (4) can be considered as a body moving with constant velocity $v_1=n$ on the unit circle, which turns in the counterclockwise sense as θ increases. The right hand side of (4) can be considered as a body moving with velocity $v_2=d(-\frac{\tau z^n\phi_n^*(z)}{\phi_n(z)})/d\theta$, which is bounded by a constant A. If n is large enough, then $|v_2|<A<n$.

Thus, if $z_1=e^{\imath\theta_1}$ is a zero of $W_n(z,\tau)$, that is, if both bodies are at the same position $e^{\imath n\theta_1}$ on the unit circle, then they will be again at the same position for the first time at $\theta=\theta_2$, with $\theta_2\in[\theta_1+\frac{2\pi}{n}-\frac{2\pi}{n(n+A)},\theta_1+\frac{2\pi}{n}+\frac{2\pi}{n(n-A)}]$. Hence (3) is proved.

To prove the second statement we use that equation (4) for n large enough is very similar to

$$z^n=-\frac{\tau\Pi(z)}{\overline{\Pi}(\frac{1}{z})}. \tag{5}$$

Now, the same arguments can be used as above. Thus, if $\alpha_1=e^{\imath\theta_1}$ is a zero of (4) and we replace in (5), we get

$$\alpha_1^n+\tau\frac{\Pi(\alpha_1)}{\overline{\Pi}(\frac{1}{\alpha_1})}=\alpha_1^n+\frac{\tau\phi_n^*(\alpha_1)+\mathcal{O}(r^n)}{\frac{\phi_n(\alpha_1)}{\alpha_1^n}+\mathcal{O}(r^n)}=\alpha_1^n+\frac{\tau\phi_n^*(\alpha_1)}{\frac{\phi_n(\alpha_1)}{\alpha_1^n}}+\mathcal{O}(r^n)=\mathcal{O}(r^n).$$

Hence, the difference between the position of the mobile bodies in (5) at the time θ_1 is $\mathcal{O}(r^n)$.

Since the body z^n in (5) moves with velocity of order n and the other body has bounded velocity, then they find one to each other at a time belonging to $[\theta_1-\mathcal{O}(r^n),\theta_1+\mathcal{O}(r^n)]$. □

3. The largest zero of $P_{n,\mu_1}(x)$.

Let $\omega(x)$ be an analytic and positive function on $[-1,1]$ and let us consider the measures $d\mu_1(x)=\dfrac{\omega(x)}{\sqrt{1-x^2}}$ and $d\nu(\theta)=\frac{1}{2}\omega(\cos\theta)d\theta$, with $z=e^{\imath\theta}$. Let $\{P_{n,\mu_1}(x)\}_{n\geq0}$ be the monic orthogonal polynomial sequence associated with μ_1 and let $\{W_n(z,1)\}_{n\geq0}$ be the sequence of para-orthogonal polynomials with respect to ν. As we have seen before, the Szegő function $\Pi(z)$ has analytic extension outside the unit disk up to $\frac{1}{r}$, $r<1$, and its Taylor expansion has real coefficients.

Based on the ideas given above, the zero of $P_{n,\mu_1}(x)$ which is closest to 1 is associated with the pairwise zeros of $W_{2n}(z,1)$ which are closest to 1. Moreover, by using Proposition 3, these zeros are very close to the pairwise zeros of $z^{2n}\overline{\Pi}(\frac{1}{z})+\Pi(z)=0$ closer to 1. Indeed, taking into account this previous result, the distance on the arc between these last zeros of $W_{2n}(z,1)$ is at most $\frac{2\pi}{2n}+\mathcal{O}(\frac{1}{(2n)^2})$. Consequently, the argument of the zero in the upper plane is $\frac{\pi}{2n}+\mathcal{O}(\frac{1}{(2n)^2})$.

Now our aim is to obtain a better approximation of this zero on $\mathbb{T}$ by solving the equation $z^{2n}\overline{\Pi}(\frac{1}{z})+\Pi(z)=0$, for which we solve the equivalent equation $-z^{2n}=\frac{\Pi(z)}{\Pi(\frac{1}{z})}$. If we take $z=e^{\imath\theta}$, then it can be rewritten as

$$-e^{\imath 2n\theta}=\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}, \tag{6}$$

and by taking logarithms in (6) we get

$$\pi+2n\theta=\log\left(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}\right)/\imath+2k\pi.$$

Since $\log\left(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}\right)=\imath\arg(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})})$, then if we denote by $A(\theta)=\arg\left(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}\right)$, then the previous equation can be rewritten as

$$\pi+2n\theta=A(\theta)+2k\pi. \tag{7}$$

Here π represents the argument of the initial position at the time $\theta=0$ of the body which moves with velocity $2n$, $2n\theta$ is the traveled arc by this body, $A(\theta)$ means the position of the second body and $2k\pi$ means the number of times that the first body meets the second one. Thus, for $k=1$ we have the zero nearest to $\theta=0$ and for $k=2$ we get the second zero nearest to $\theta=0$.

To solve (7), first we point out some properties satisfied by $A(\theta)$:

(i) $A(\theta)$ takes real values for every θ.

(ii) $A(\theta)$ is an odd function in θ, that is, $A(-\theta)=-A(\theta)$.

(iii) $A(\theta)$ has derivatives of every order (in the region where we are dealing with, that is, θ near 0) and all of them are bounded.

(iv) The even derivatives vanish at 0, that is, $A^{(2k)}(0)=0$ for every k.

To approximate the zeros of (7) for $k=1$, that is, $A(\theta)-2n\theta+\pi=0$, we substitute $A(\theta)$ by its Taylor polynomial of fourth order around 0 and thus we propose to solve the following equation

$$\frac{A'''(0)}{6}\theta^3 + (A'(0) - 2n)\theta + \pi = 0. \tag{8}$$

This cubic polynomial takes positive values for $\theta = 0$ and, for n large enough, it is easy to see that it takes negative values for $\theta = \frac{\pi}{n}$. Hence, for n large enough, each cubic polynomial has a positive real zero near $\frac{\pi}{n}$. Moreover, it is easy to see that the two other zeros have modulus of order $\sqrt{n}$.

To identify the zero which is nearest to 0, we apply Cardano formulas as follows

(8) may have only one or three real solutions depending on the sign of $A'''(0)$ and if we denote $A'''(0) = a$ and $A'(0) = b$, then it can be rewritten

$$\theta^3 + 6\frac{(b-2n)}{a}\theta + \frac{6\pi}{a} = 0. \tag{9}$$

Indeed, if $a < 0$, then there exists only one real zero, given by

$$\theta_1 = \sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(b-2n)^3}{9a^3} + \frac{\pi^2}{a^2}} - \frac{\pi}{a}} - \sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(b-2n)^3}{9a^3} + \frac{\pi^2}{a^2}} + \frac{\pi}{a}}. \tag{10}$$

Otherwise, if $a > 0$, then (9) has three real zeros. Two of them are of order $\sqrt{n}$ and the third one, which is the nearest to zero, can be obtained in an easy way by using the trigonometric expressions of the three zeros. This leads to

$$\theta_1 = \frac{e^{-\frac{2\imath\pi}{3}}}{\sqrt[3]{2}}\sqrt[3]{-\frac{6\pi}{a} + \imath 2\sqrt{\left|\frac{8\ (b-2n)^3}{\tilde{a}^3} + \frac{9\pi^2}{a^2}\right|}} + \frac{e^{\frac{2\imath\pi}{3}}}{\sqrt[3]{2}}\overline{\sqrt[3]{-\frac{6\pi}{a} + \imath 2\sqrt{\left|\frac{8\ (b-2n)^3}{a^3} + \frac{9\pi^2}{a^2}\right|}}}. \tag{11}$$

Notice that the zero of (7) for $k = 1$ and the zero θ_1 of (8) nearest to 0 are very similar.

If we consider the Taylor expansion of fourth order of $A(\theta)$ around 0, then we obtain

$$A(\theta) - 2n\theta = \frac{A'''(0)}{6}\theta^3 + (A'(0) - 2n)\theta + \pi + \frac{A^{(v)}(\xi)}{5!}\theta^5 \tag{12}$$

for some ξ. Since $A(\theta_1) - 2n\theta_1 = \frac{A^{(v)}(\xi)}{5!}\theta_1^5$, then $A(\theta_1) - 2n\theta_1$ has order n^{-5}. By using again the argument of the velodrome, the true zero of (7) must be $\theta_1 + \mathcal{O}(n^{-6})$.

If we want to obtain the zero of $P_n(x, \mu_1)$ which is nearest to -1 it suffices to consider $w_1(x) = w(-x)$ which leads to the Szegő function $\Pi_1(z) = \Pi(-z)$, and determine, in the same way as above, the zero θ_1 which is nearest to 1 with an approximation of order $\mathcal{O}(n^{-6})$. In this situation $\cos(\pi \pm \theta_1)$ is the approximation that we are looking for.

In summary, we have proved the following theorem

Theorem 1. *Let $w(x)$ be an analytic and positive function on $[-1,1]$ and let us consider the measure $d\mu_1(x) = \frac{w(x)}{\sqrt{1-x^2}}dx$ with monic orthogonal polynomial sequence $\{P_{n,\mu_1}(x)\}_{n\geq 0}$. Let $d\nu(\theta) = \frac{1}{2}w(\cos\theta)d\theta$ for $z = e^{\imath\theta}$ and $\Pi(z)$ be the normalized Szegő function of ν.*

1. *If we denote $a = \frac{d^3}{d\theta^3}\left(\log\left(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}\right)\right)\Big|_{\theta=0}$ and $b = \frac{d}{d\theta}\left(\log\left(\frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})}\right)\right)\Big|_{\theta=0}$ then*
   (a) *If $a>0$ and $\cos(\theta_1)$ is the zero of $P_n(x,\mu_1)$ nearest to 1, then θ_1 can be approximated by*
$$\frac{e^{-\frac{2\imath\pi}{3}}}{\sqrt[3]{2}}\sqrt[3]{-\frac{6\pi}{a}+i2\sqrt{\left|\frac{8\ (b-2n)^3}{a^3}+\frac{9\pi^2}{a^2}\right|}}\sqrt[3]{2}+\frac{e^{\frac{2i\pi}{3}}}{\sqrt[3]{2}}\ \overline{\sqrt[3]{-\frac{6\pi}{a}+i2\sqrt{\left|\frac{8\ (b-2n)^3}{a^3}+\frac{9\pi^2}{a^2}\right|}}} \quad (13)$$
   with an error of order $\mathcal{O}(n^{-6})$.
   (b) *If $a<0$ and $\cos(\theta_1)$ is the zero of $P_n(x,\mu_1)$ nearest to 1, then θ_1 can be approximated by*
$$\sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(b-2n)^3}{9a^3}+\frac{\pi^2}{a^2}}-\frac{\pi}{a}}-\sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(b-2n)^3}{9a^3}+\frac{\pi^2}{a^2}}+\frac{\pi}{a}}. \quad (14)$$
   with an error of order $\mathcal{O}(n^{-6})$.
2. *If we denote $\tilde{a} = \frac{d^3}{d\theta^3}\left(\log\left(\frac{\Pi(-e^{\imath\theta})}{\Pi(-e^{-\imath\theta})}\right)\right)\Big|_{\theta=0}$ and $\tilde{b} = \frac{d}{d\theta}\left(\log\left(\frac{\Pi(-e^{\imath\theta})}{\Pi(-e^{-\imath\theta})}\right)\right)\Big|_{\theta=0}$, then*
   (a) *If $\tilde{a}>0$ and $\cos(\theta_2)$ is the zero of $P_n(x,\mu_1)$ nearest to -1, then θ_2 can be approximated by*
$$\frac{e^{-\frac{2i\pi}{3}}}{\sqrt[3]{2}}\sqrt[3]{-\frac{6\pi}{\tilde{a}}+i2\sqrt{\left|\frac{8\ (\tilde{b}-2n)^3}{\tilde{a}^3}+\frac{9\pi^2}{\tilde{a}^2}\right|}}+\frac{e^{\frac{2i\pi}{3}}}{\sqrt[3]{2}}\ \overline{\sqrt[3]{-\frac{6\pi}{\tilde{a}}+i2\sqrt{\left|\frac{8\ (\tilde{b}-2n)^3}{\tilde{a}^3}+\frac{9\pi^2}{\tilde{a}^2}\right|}}} \quad (15)$$
   with an error of order $\mathcal{O}(n^{-6})$.
   (b) *If $\tilde{a}<0$ and $\cos(\theta_2)$ is the zero of $P_n(x,\mu_1)$ nearest to -1, then θ_2 can be approximated by*
$$\sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(\tilde{b}-2n)^3}{9\tilde{a}^3}+\frac{\pi^2}{\tilde{a}^2}}-\frac{\pi}{\tilde{a}}}-\sqrt[3]{3}\sqrt[3]{\sqrt{\frac{8(\tilde{b}-2n)^3}{9\tilde{a}^3}+\frac{\pi^2}{\tilde{a}^2}}+\frac{\pi}{\tilde{a}}}. \quad (16)$$
   with an error of order $\mathcal{O}(n^{-6})$.

Remark 1. *As we have said before, the equation that we have solved*

$$\frac{A'''(0)}{6}\theta^3 + (A'(0) - 2n)\theta + \pi = 0$$

comes from a more general one

$$\frac{A'''(0)}{6}\theta^3 + (A'(0) - 2n)\theta + (2k-1)\pi = 0,$$

where k denotes the order of the zero according to its distance to 1. For fixed values of k and n large enough, formulas (13), (14), (15) and (16) can be adapted to determine the zero number k nearest to 1 and -1 by changing $\frac{\pi}{a}$ by $\frac{(2k-1)\pi}{a}$ and $\frac{\pi}{\tilde{a}}$ by $\frac{(2k-1)\pi}{\tilde{a}}$, respectively.

Remark 2. *Proceeding in a similar way, we can obtain the corresponding theorems for the other three measures μ_2, μ_3, and μ_4. Now we need to use the relations given in (ii), (iii),and (iv) of Proposition 1.*

For the measure μ_2 the equation equivalent to (6) is

$$e^{\imath(2n+2)\theta} = \frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})},$$

and, therefore, $(2n+2)\theta = A(\theta) + 2k\pi$ holds. Thus one has to solve

$$\theta^3 + 6\frac{(b-(2n+2))}{a}\theta + 6\frac{2k\pi}{a} = 0.$$

For the measure μ_3 the equation equivalent to (6) is

$$-e^{\imath(2n+1)\theta} = \frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})},$$

and, therefore, $\pi + (2n+1)\theta = A(\theta) + 2k\pi$ holds. Thus one has to solve

$$\theta^3 + 6\frac{(b-(2n+1))}{a}\theta + 6\frac{(2k-1)\pi}{a} = 0.$$

For the measure μ_4 the equation equivalent to (6) is

$$e^{\imath(2n+1)\theta} = \frac{\Pi(e^{\imath\theta})}{\Pi(e^{-\imath\theta})},$$

and, therefore, $(2n+1)\theta = A(\theta) + 2k\pi$ holds. Thus one has to solve

$$\theta^3 + 6\frac{(b-(2n+1))}{a}\theta + 6\frac{2k\pi}{a} = 0.$$

Measure	Change $\frac{8(b-2n)^3}{a^3}$ in (13) and (14) by	Change $\frac{\pi}{a}$ in (13) and (14) by
μ_1	$\dfrac{8(b-2n)^3}{a^3}$	$\dfrac{(2k-1)\pi}{a}$
μ_2	$\dfrac{8(b-(2n+2))^3}{a^3}$	$\dfrac{2k\pi}{a}$
μ_3	$\dfrac{8(b-(2n+1))^3}{a^3}$	$\dfrac{(2k-1)\pi}{a}$
μ_4	$\dfrac{8(b-(2n+1))^3}{a^3}$	$\dfrac{2k\pi}{a}$

Table 1: Changes in (13) and (14) to obtain $\cos\theta_1$, the kth zero of de $P_{n,\mu_i}(x)$ nearest to 1

Thus the changes needed in (13) and (14) to obtain the kth zero of $P_{n,\mu_i}(x)$, nearest to 1 are given in Table 1.

Proceeding in the same way, formulas (15) and (16) can be adapted for the measures μ_i in order to obtain the kth zero of $P_{n,\mu_i}(x)$ nearest to -1. The changes corresponding to μ_1 and μ_2 are evident, while for μ_3 and μ_4 one must take into account that $\mu_3(-z)$ is $\mu_4(z)$ and viceversa. See Table 2.

4. Numerical examples

In this section we give several examples to illustrate how the method just presented can be implemented.

Example 1. *We consider a rational modification of the Chebyshev measure of the first kind by using the weight function* $w(x) = \dfrac{1}{p(x)}, p(x)$ *a positive polynomial in* $[-1, 1]$*. In this case, the transformed measure* $d\nu(\theta)$ *is a Bersntein-Szegő measure given by* $d\nu(\theta) = \frac{1}{2}w(\theta)d\theta = \frac{1}{2|Q(z)|^2}, z = e^{\imath\theta}$*, where* $Q(z)$ *is a polynomial well determined from* $p(x)$*, (see [9]). It is well known that the zeros of* $Q(z)$ *are located outside the unit disk. Indeed, they are obtained by applying the Joukowsky transformation to the zeros of* $p(x)$*. The monic orthogonal polynomials related to* ν *are given by* $\phi_n(z) = z^{n-degQ}\frac{Q^*(z)}{\overline{Q(0)}}$ *and the normalized Szegő function is* $\Pi(z) = \frac{Q(z)}{Q(0)}$*.*

In this first example we have chosen $p(x) = (x+4)(x^2+\frac{4}{3}x+\frac{2}{3})$*. Since its zeros are* -4*,* $\frac{\sqrt{2}}{\imath-\sqrt{2}}$*, and* $\frac{\sqrt{2}}{\imath-\sqrt{2}}$*, then the zeros of* $Q(z)$ *are* $-4-\sqrt{15}$*,* $-1-\sqrt{2}\imath$*, and* $-1+\sqrt{2}\imath$*. The normalized*

Measure	Change $\frac{8(\tilde{b}-2n)^3}{\tilde{a}^3}$ in (15) and (16) by	Change $\frac{\pi}{\tilde{a}}$ in (15) and (16) by
μ_1	$\dfrac{8(\tilde{b}-2n)^3}{\tilde{a}^3}$	$\dfrac{(2k-1)\pi}{\tilde{a}}$
μ_2	$\dfrac{8(\tilde{b}-(2n+2))^3}{\tilde{a}^3}$	$\dfrac{2k\pi}{\tilde{a}}$
μ_3	$\dfrac{8(\tilde{b}-(2n+1))^3}{\tilde{a}^3}$	$\dfrac{2k\pi}{\tilde{a}}$
μ_4	$\dfrac{8(\tilde{b}-(2n+1))^3}{\tilde{a}^3}$	$\dfrac{(2k-1)\pi}{\tilde{a}}$

Table 2: Changes in (15) and (16) to obtain $\cos\theta_1$, the kth zero of $P_{n,\mu_i}(x)$ nearest to -1

Szegő function is $\Pi(z) = (1+\frac{z}{4+\sqrt{15}})(1+\frac{z}{1+\sqrt{2}\imath})(1+\frac{z}{1-\sqrt{2}\imath})$.

By applying formula (13), adapted as in the last remark, we have obtained the five zeros closest to 1 *of* $p_{500,\mu_1}(x)$*. Based on these approximations, we have proceeded to refine them by using the FindRoot command from Mathematica, obtaining the following table 3.*

zero number	Value of θ_1 applying (13)	Refined value	Error
1	0.003146497210659052	0.003146497210659047	-4.21110 E-18
2	0.009439491547931159	0.009439491547930136	-1.02331 E-15
3	0.015732485633065297	0.015732485633052137	-1.31601 E-14
4	0.022025479297969563	0.022025479297898782	-7.07806 E-14
5	0.028318472374552163	0.028318472374303474	-2.48688E-13

Table 3: Using Table 1 to adapt (13) for obtaining $\cos\theta_1$, the kth zero of $P_{500,\mu_1}(x)$ nearest to 1

To show numerically that the errors evolve with the suggested order we have repeated the experiment using $n = 5000$*, that is, multiplying* n *by 10 and we have obtained Table 4.*

Notice that the above errors are less than the former ones multiplied by 10^{-6}

Example 2. *For this second example we consider the measure* $d\mu_1(x) = \frac{1}{\sqrt{1-x^2}}w_{RS}^{(q)}(x)dx$*, that is, we consider the modification of the Chebyshev measure of the first kind by the weight function*

$$w_{RS}^{(q)}(x) = 4\pi(2\pi a)^{-\frac{1}{2}}\sum_{j=\infty}^{\infty} e^{\frac{-(\arccos x - 2\pi j)^2}{2a}}, \text{ with } a = \log\left(\frac{1}{q}\right), \text{ and } q \in (0,1).$$

zero number	Value of θ_1 applying (13)	Refined value	Error
1	0.000314208242149350	0.000314208242149350	-4.175773 E-24
2	0.000942624726439695	0.000942624726439695	-1.014713 E-21
3	0.001571041210704967	0.001571041210704967	-1.304929 E-20
4	0.002199457694928451	0.002199457694928451	–7.018227 E-20
5	0.002827874179093432	0.002827874179093432	-2.465755E-19

Table 4: Using Table 1 to adapt (13) for obtaining $\cos\theta_1$, the kth zero of $P_{5000,\mu_1}(x)$ nearest to 1

The Joukowsky transformation of the measure μ_1 gives the well known Rogers-Szegő measure, (see [7])

$$d\nu_q(\theta) = 2\pi(2\pi a)^{-\frac{1}{2}} \sum_{j=\infty}^{\infty} e^{\frac{-(\theta-2\pi j)^2}{2a}},$$

which can also be described by its moments $c_n = q^{\frac{n^2}{2}}$ and also by the Verblunsky coefficients $\alpha_n = (-1)^n q^{\frac{(n+1)}{2}}$.

The monic orthogonal polynomials are given by $\phi_n(z) = \sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{(n-j)}{2}} z^j$ and it is also well known that the normalized Szegő function is given by

$$\Pi(z) = \frac{1}{(-q^{\frac{1}{2}}z;q)_\infty} = \frac{1}{\prod_{j=0}^{\infty}(1+zq^{j+\frac{1}{2}})}.$$

Now we are in conditions to apply formulas (14) and (15) as well as their modifications. We have chosen $q = 0.5$ and $n = 1000$ and we have applied (14) and (15) adapted following the last remark to approximate the five zeros closer to 1 and -1 of $P_{1000,\mu_2}(x)$. Based on these approximations, we have proceed to refine the results by using the FindRoot command from Mathematica, obtaining the following table 5.

zero number	Value of θ_1 applying (14)	Refined value	Error
1	0.00313906378824043883390501	0.00313906378824043866844337	-1.654616E-19
2	0.00627812757320344212021217	0.00627812757320343682543901	-5.294773E-18
3	0.00941719135161157434554271	0.00941719135161153413835115	-4.020719E-17
4	0.01255625512018740006495599	0.01255625512018723063213537	-1.694328E-16
5	0.01569531887565348393616855	0.01569531887565296686805296	-5.170681E-16

Table 5: Using Table 1 to adapt (14) for obtaining $\cos\theta_1$, the kth zero of $P_{1000,\mu_2}(x)$ closest to 1.

The second table contains a similar experiment, but now with the zeros of $P_{1000,\mu_2}(x)$ closest to -1 .

zero number	Value of θ_2 applying (15)	Refined value	Error
1	3.138455139396180461250	3.138455139396180469349	8.099632E-18
2	3.135317625184155535532	3.135317625184155794718	2.591858E-16
3	3.132180110935306311722	3.132180110935308279884	1.968161E-15
4	3.129042596631220638073	3.129042596631228931710	8.293636E-15
5	3.1259050822534863595977	3.125905082253511669060	2.530946E-14

Table 6: Using table 1 to adapt (15) for obtaining $\cos\theta_1$, the kth zero of $P_{1000,\mu_2}(x)$ closer to -1

Example 3. *Let us consider the following modification of the Chebyshev measure $d\mu_1(x) = \frac{1}{\sqrt{1-x^2}}e^{p(x)}dx$, where $p(x)$ is a polynomial with real coefficients. If we consider the expansion of $p(x)$ with respect to the Chebyshev polynomials of the first kind, $p(x) = \sum_{j=0}^{m} a_j T_j(x)$, then we denote by $P(z)$ the polynomial $P(z) = \sum_{j=0}^{m} a_j z^j$.*

If we apply the Joukowsky transformation to the measure μ_1, then we get

$$d\nu(\theta) = \frac{1}{2}e^{p(\cos\theta)}d\theta = \frac{1}{2}e^{\frac{P(z)}{2}}e^{\frac{P(\bar{z})}{2}}d\theta, \quad z = e^{\imath\theta},$$

and the Szegő function is

$$\Pi(z) = \frac{e^{\frac{-P(z)}{2}}}{e^{\frac{-P(0)}{2}}}$$

since

$$w(\theta) \propto \frac{1}{\Pi(z)\Pi(\frac{1}{z})}, \quad z = e^{\imath\theta}.$$

For this third example we have taken $p(x) = -x^2$, that is, $p(x) = -\frac{1}{2}T_2(x) - \frac{1}{2}$. Then $P(z) = -\frac{1}{2} - \frac{1}{2}z^2$ and $d\nu(\theta) = \frac{1}{2}e^{-(\frac{1}{4}+\frac{z^2}{4})}e^{-(\frac{1}{4}+\frac{1}{4z^2})}$. In particular, the Szegő function is $\Pi(z) = \frac{e^{\frac{1}{4}z^2+\frac{1}{4}}}{e^{\frac{1}{4}}}$. Thus we can apply (14) and (16) adapted following the last remark. First, we have obtained the arguments of the first five zeros of P_{500,μ_3} closest to 1 . Based on these approximations, we have proceeded to refine the results by using the FindRoot command from Mathematica, obtaining the following table 7. Notice that we apply the previous command to $z^{2n} = -\frac{\Pi(z)}{\Pi(\frac{1}{z})}$ and take into account Proposition 3.

The second table contains a similar experiment for the first five zeros of P_{500,μ_4} closest to -1.

zero number	Value of θ_1 applying (14)	Refined value	Error
1	0.001570796325502968	0.001570796325502969	6.375408 E-19
2	0.004712388945502629	0.004712388945502784	1.549221 E-16
3	0.007853981472483468	0.007853981472485460	1.992303 E-15
4	0.010995573844432959	0.010995573844443674	1.071502 E-14
5	0.014137165999338603	0.014137165999376249	3.764542 E-14

Table 7: Using table 1 to adapt (14) for obtaining $\cos(\theta_1)$, the kth zero of $P_{500,\mu_3}(x)$ closest to 1

zero number	Value of θ_2 applying (16)	Refined value	Error
1	3.140021857264290270	3.140021857264290269	- 6.375408 E-19
2	3.136880264644290609	3.136880264644290454	-1.549221 E-16
3	3.133738672117309769	3.133738672117307777	-1.992303 E-15
4	3.130597079745360279	3.130597079745349564	-1.071502 E-14
5	3.127455487590454634	3.127455487590416989	-3.764542 E-14

Table 8: Using table 2 to adapt (16) for obtaining $\cos\theta_1$, the kth zero of $P_{500,\mu_4}(x)$ closest to -1

Obviously the second part of the example could be solved taking into account that the kth zero of $P_{n,\mu_3}(x)$ closest to 1 is just the opposite of the kth zero of $P_{n,\mu_4}(x)$ closest to -1 . Thus the arguments are supplementary angles.

Note 1. *The FindRoot command of Mathematica does not necessarily converge to a zero, even using a good start. We can observe this by executing*

$FindRoot[z^{10000} - 1, z, 1.01]$.

So, by evaluating the corresponding equations in the refined values we have done an extra test. In all cases, the equation was satisfied with negligible errors.

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References

[1] E. Berriochoa, A. Cachafeiro, J. García-Amor, *Connection between orthogonal polynomials on the unit circle and bounded interval*, J. Comput. Appl. Math. **177** (1) (2005),

205-223.

[2] J. García-Amor, *Ortogonalidad Bernstein-Chebyshev en la recta real*, Doctoral Dissertation, Universidad de Vigo, 2003 (in Spanish).

[3] L. B. Golinskii, *Quadrature formula and zeros of para-orthogonal polynomials on the unit circle*, Acta Math. Hungar. **96** (3) (2002), 169–186.

[4] W. B Jones, O. Njastad, W. J. Thron, *Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989), 113–152.

[5] A. Martínez-Finkelshtein, K.T.-R. McLaughlin and E.B. Saff, *Szegő orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics*, Constr. Approx. **24** (3) (2006), 319-363.

[6] P. Nevai, V. Totik, *Orthogonal polynomials and their zeros*, Acta Sci. Math. Szeged **53** (1989), 99–104.

[7] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1, and Part 2*, Amer. Math. Soc. Colloq. Publi., Vol. **54**, Amer. Math. Soc., Providence, RI, 2005.

[8] B. Simon, *Fine structure of the zeros of orthogonal polynomials, I. A tale of two pictures*, Electr. Trans. on Numer. Analysis **25** (2006), 328–368.

[9] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. **23**, 4th ed., Amer. Math. Soc., Providence, RI, 1975.

[10] M. L. Wong, *First and second kind paraorthogonal polynomials and their zeros*, J. Approx. Theory **146** (2) (2007), 282-293.