Letter

Braid-protected topological band structures with unpaired exceptional points

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We demonstrate the existence of topologically stable unpaired exceptional points (EPs), and construct simple non-Hermitian (NH) tight-binding models exemplifying such remarkable nodal phases. While fermion doubling, i.e., the necessity of compensating the topological charge of a stable nodal point by an antidote, rules out a direct counterpart of our findings in the realm of Hermitian semimetals, here we derive how nonommuting braids of complex energy levels may stabilize unpaired EPs. Drawing on this insight, we reveal the occurrence of a single, unpaired EP, manifested as a non-Abelian monopole in the Brillouin zone of a minimal three-band model. This third-order degeneracy represents a sweet spot within a larger topological phase that cannot be fully gapped by any local perturbation. Instead, it may only split into simpler (second-order) degeneracies that can only gap out by pairwise annihilation after having moved around inequivalent large circles of the Brillouin zone. Our results imply the incompleteness of a topological classification based on winding numbers, due to non-Abelian representations of the braid group intertwining three or more complex energy levels, and provide insights into the topological robustness of non-Hermitian systems and their non-Abelian phase transitions.

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Following the experimental discovery of Dirac and Weyl semimetals in solid state materials [1–5], Bloch bands with topologically stable nodal points have become a major focus of research far beyond the field of condensed matter physics [1–12]. In crystalline systems, Bloch's theorem requires the periodicity of the band structure in reciprocal space. As a consequence, nodal points carrying a topological charge must be compensated for to allow the eigenstates to seamlessly fit together at the zone boundaries of the first Brillouin zone (BZ). A prominent example along these lines is provided by stable Weyl nodes [6,13,14], which are required to occur in pairs with opposite chirality. Under the name fermion doubling, such constraints have been discussed for decades [15,16].

In dissipative systems described by effective non-Hermitian (NH) operators, exceptional points (EPs) [17–22] represent the generic counterpart of diagonalizable degeneracies. Since stable EPs may carry topological charge in the sense of a relative winding in the complex energy plane, analogous constraints to the aforementioned fermion doubling have been reported based on $\mathbb{Z}$ discriminants [23]. However, as an additional twist, EPs are known to be of intrinsic non-Abelian nature, as characterized by the braid group [24–29]. This entails unique NH topological semimetal phases that elude elementary winding number descriptions [30–34]. The fusion of nodal points, a phase transition, is topologically path dependent [35]. Here, challenging the intuitive notion of fermion doubling, we reveal the possibility of topologically stable gapless phases in periodic systems that can host *unpaired* EPs, i.e., nodal points that are not gapped out when fusing. We showcase this in a tight-binding model hosting only two EPs in its BZ whose topological charges do not cancel. Instead of annihilating, these unpaired EP2s fuse to a single nontrivial threefold degeneracy (EP3). We find that these NH topological band structures are enabled by noncommuting braids of the complex energy levels along noncontractible loops in the BZ, which remove the need for the EPs' topological charges to pair up and cancel (see Fig. 1 for an illustration). The real part of the spectral gap closes along so-called Fermi arcs, which conventionally form open lines in the BZ that terminate in degeneracies. In the phases outlined here they can instead form closed and even noncontractible loops in the BZ. The only way to adiabatically gap out the nodal points is to move them along nontrivial loops in the BZ before annihilation (see Fig. 3). Transitions between these NH phases are thus topological, and correspond to braiding of nodal points in the BZ.

Admissible EPs by the braid group. In the vicinity of EPs, the eigenvalues are generally not single valued. This is manifest in the braiding of the complex energies when encircling an EP, and endows these degeneracies with a topological charge [25,26]. An illustrative example is an EP2 with dispersion $E = \pm\sqrt{k_x + ik_y}$, where two eigenvectors and eigenvalues coalesce. The two energy levels are swapped counterclockwise

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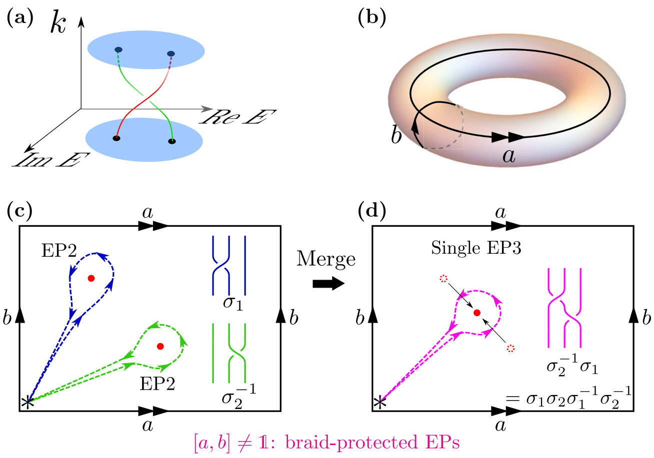

FIG. 1. Illustration of key results. (a) The braiding of complex energy levels as a function of lattice momentum is at the heart of our analysis. (b) In the two-dimensional Brillouin zone, NH phases are characterized by two braid elements (a, b) along the noncontractible loops. For gapped systems, a and b commute (*group* commutator $aba^{-1}b^{-1} = [a, b] = \mathbb{1}$). (c) Minimal model of a band structure with a nontrivial such commutator, hosting two unpaired EP2s. (d) While the unpaired EP2s may merge into a single EP3, the nontrivial braid topology implies that the system remains gapless. EPs are denoted by red dots and base points of loops by $*$ in all panels.

along a counterclockwise path around the EP. General N-fold degeneracies carry braid invariants with N strands. The corresponding braid group B_N is generated by elementary braids σ_j $(1 \leqslant j \leqslant N-1)$ that swap E_j and E_{j+1} counterclockwise in the complex plane. They satisfy the relations [36]

$$\sigma_{j+1}\sigma_j\sigma_{j+1} = \sigma_j\sigma_{j+1}\sigma_j, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| > 1). \tag{1}$$

In contrast to standard Hermitian degeneracies such as Weyl points [37], EPs are associated with non-Abelian groups, similar to topological defects in biaxial nematics [35,38] or multigap phases [39–43]. This comes from the non-Abelian fundamental group π_1 of the space $\mathcal{W}_N$ formed by gapped non-Hermitian Hamiltonians [25,26,44]. The combination or split of EPs can be described through the standard group action [35]. As it is non-Abelian, one needs to be careful in fixing a base point $*$ when doing the group product. We call an EP trivial when a small loop around it corresponds to a trivial braid of eigenvalues, and nontrivial otherwise. This terminology is motivated by the fact that only trivial EPs can be gapped out by small perturbations.

We denote the homotopy class of each loop γ_j around the EP j with a common base point $*$ as $[\gamma_j] \in B_N$. According to the topology of a torus, the total EPs in the BZ satisfy a non-Abelian sum rule [45,46] [illustrated in Fig. 2(a)],

$$[\gamma_1] \cdot [\gamma_2] \cdots [\gamma_n] = aba^{-1}b^{-1} \equiv [a, b], \tag{2}$$

where $[a, b]$ is the group commutator of two elements a, b in B_N. They represent how the energy is braided along the meridian and the longitude of the torus (see Fig. 1). As a result of this sum rule, models with a nontrivial commutator $[a, b]$ must contain degeneracies; they cannot be gapped.

As the group B_N is non-Abelian and not free, the sum rule can sometimes be hard to apply in practice. We can extract an easier necessary condition for EPs by taking the Abelianization $\pi_1/[\pi_1, \pi_1]$, which is the first homology group $\mathbf{H}_1(\mathcal{W}_N)$ [47]. Since all generators σ_j are conjugate to each other, the Abelianization of the braid group is $\mathbb{Z}$ [36]. The Abelianized element $[\gamma]_A$ can be intuitively understood as the number of band permutations along a braid (the total number of σ_j's in a braid). An nth root EP [48–50] has $[\gamma]_A = n-1$. The right-hand side of Eq. (2) is the zero element 0 in the homology group. The Abelianized sum rule is

$$\sum_j [\gamma_j]_A = 0, \tag{3}$$

where the sum represents the usual group product operation for Abelian groups. Equation (3) asserts that we cannot have a single EP with nth root dispersion in a BZ. Its Abelianized charge must be compensated, e.g., by a conjugate EP that has $[\gamma]_A = -(n-1)$.

However, if the net number of elementary braids around a set of EPs is zero, they satisfy the Abelianized sum rule and can occur in a BZ. Depending on whether a and b commute, their topological charges can nevertheless be nontrivial and need not cancel out. This is a significant difference from Abelian degeneracies, where periodic boundary conditions necessitate a trivial bulk. In the following parts, we explicitly show the existence of nontrivial unpaired EPs. In the Supplemental Material we further present tight-binding models that feature topologically trivial single EPs of arbitrary order in their BZ [44]. They can be created locally and under perturbation either split into nontrivial paired EPs or get gapped out.

Unpaired EPs. EPs without partners compensating their braid charge can only exist for models with three or more bands, since the group B_2 for two-band systems is Abelian, rendering Eqs. (2) and (3) identical. In the case of three bands, all braids are generated by σ_1, σ_2 and the relation (1). Those that can be written as a commutator contain equal numbers of crossings σ_i and inverse crossings σ_j^{-1}. A simple nontrivial such braid is $B = \sigma_2^{-1}\sigma_1$. By relation (1), it can be written as a commutator,

$$\begin{aligned}[\sigma_1, \sigma_2] &= \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_2^{-1}(\sigma_2\sigma_1\sigma_2)\sigma_1^{-1}\sigma_2^{-1} \\ &= \sigma_2^{-1}(\sigma_1\sigma_2\sigma_1)\sigma_1^{-1}\sigma_2^{-1} = \sigma_2^{-1}\sigma_1.\end{aligned} \tag{4}$$

A simple model Hamiltonian carrying these braids is

$$H_B = \begin{pmatrix} 1 - e^{ik_x} & -i(1 + e^{ik_x}) & 0 \\ -i(1 + e^{ik_x}) & e^{ik_x} - e^{ik_y} & -i(1 + e^{ik_y}) \\ 0 & -i(1 + e^{ik_y}) & -1 + e^{ik_y} \end{pmatrix}, \tag{5}$$

for k_x, k_y periodic, parametrized on $[-\pi, \pi]$. It represents the momentum-space Bloch Hamiltonian of a real-space model with nearest-neighbor hoppings on a two-dimensional square lattice, where each unit cell contains three orbitals. The spectrum along $(k_x, -\pi)$ corresponds to $a = \sigma_1$, and to $b = \sigma_2$ along (π, k_y). As shown in Eq. (4), these boundary braids do not commute. The total BZ boundary braid, along the composition of the two great circles [Fig. 1(b)], is the nontrivial braid $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$. We show the spectrum on the BZ boundary in Fig. 2(b). In the Supplemental Material [44], we extend the constructive method of Ref. [51] to show that there is always a way to construct a given braid within a BZ-continuous model.

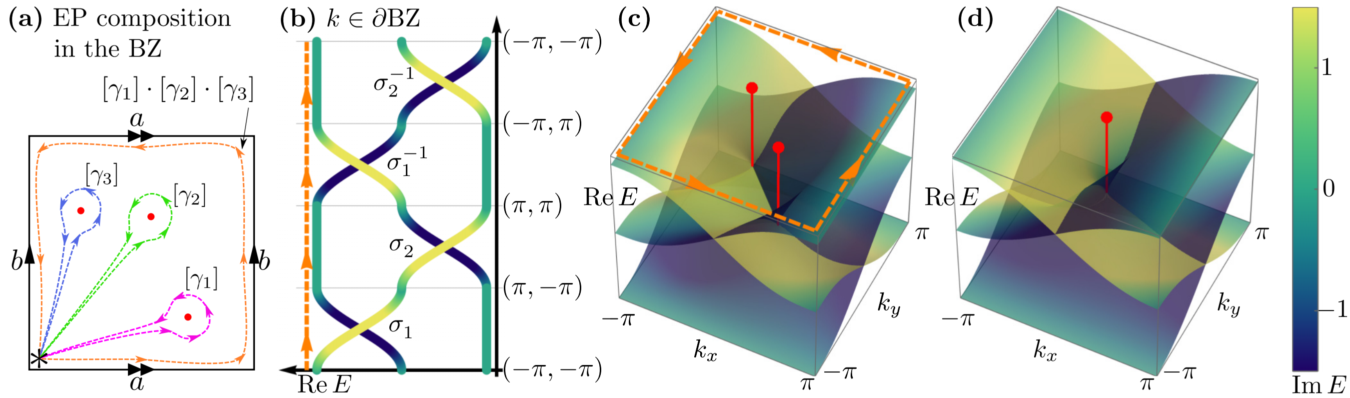

FIG. 2. Composition rule for braids. (a) Schematic: Braids around degeneracies must add up to the boundary braid. (b) Boundary braid: Spectrum of Eq. (5) shown along the BZ boundary, counterclockwise from $(-\pi, -\pi)$. Associating σ_i to a counterclockwise exchange of eigenvalues E_i, E_{i+1} in the complex plane allows us to read off the boundary braid as $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$. (c) Spectrum of Eq. (5) shown on the entire BZ. The orange path corresponds to the path traversed in (a) and (b) that gives rise to the boundary braid. The two degeneracies marked in red carry charges σ_2^{-1} and σ_1, which add up to the boundary charge [see Eq. (4)]. (d) Tuning this model to fuse the two EPs [see Eq. (6)] cannot annihilate them if the perturbation conserves the boundary charge, which is why we call these degeneracies unpaired. Instead, a single nontrivial EP3 remains as a non-Abelian monopole that compensates the entire braid charge of the boundary.

The entire spectrum of H_B is shown in Fig. 2(c). It hosts two EP2s (which occur generically in two dimensions) that carry braid charges σ_2^{-1} and σ_1, compensating the boundary braid according to the sum rule (2) or (4) explicitly. While they have opposite Abelianization, their braid charges are not mutual inverses: These EPs are *unpaired*.

To illustrate this further, we tune this Hamiltonian to

$$H_\delta^{\text{EP3}} = H_B + \delta\, 2\sqrt{2}\,\text{diag}(1, 0, -1), \quad 0 \leqslant \delta \leqslant 1, \tag{6}$$

which leaves the boundary braids unchanged but merges the EP2s as $\delta \to 1$. We show the corresponding spectrum in Fig. 2(d). As the EP2s are unpaired, they cannot annihilate, but fuse to a *single* EP3 at $k_x = k_y = 0$.

While fine tuned, this EP3 is significantly distinct from the simple third-root EPs commonly implemented in topologically trivial spaces [23,33,49]. First, it can exist alone under the BZ periodic boundary condition. Second, its dispersion consists of the roots of a third-order polynomial which forbids a simple expression. Third, it lies at the intersection of two Fermi arcs, along which one of the real gaps $\text{Re}(E_i - E_j)$ closes. These Fermi arcs extend through the BZ, roughly following $(k_x, 0)$, and $(0, k_y)$. In particular they form extended noncontractible loops, unlike the common

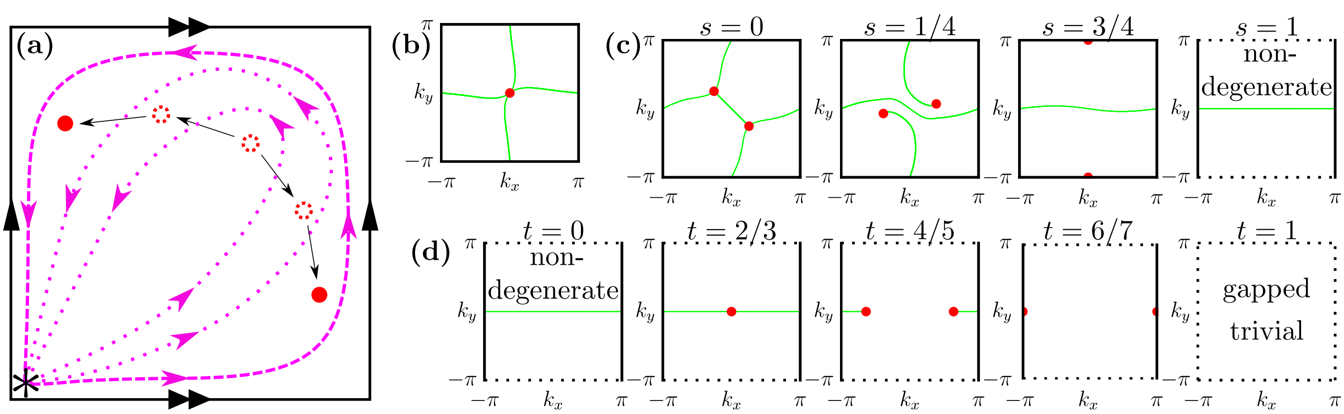

FIG. 3. Non-Hermitian phase transition. (a) To gap out the system, any contractible loop (fuchsia) in the BZ must carry a trivial braid. The only way to change nontrivial braids (dashed) into trivial ones (dotted) continuously is to move an EP across the loop. (b) Schematic representation of the spectrum of Eq. (6) at $\delta = 1$. The single nontrivial EP3 is marked in red, and the green lines illustrate the real-gap closings which form nontrivial loops around the BZ. (c) Schematic representation of the spectrum for the interpolation Eq. (7) between the unpaired model hosting unpaired EPs and a gapped phase. Initially the BZ boundaries (solid black lines) carry braid charges σ_1 (σ_2) in the k_x (k_y) direction. At $s = 3/4$ the two EPs merge at the boundary, breaking the k_x braid charge into a trivial braid; the corresponding boundary is then marked with dotted lines. For $s > 3/4$, all braids along k_y = const are trivial, while loops along k_x = const correspond to braids of type σ_2. The boundary braid in that regime is then $\sigma_2\sigma_2^{-1} = \mathbb{1}$. (d) Same as (c) for interpolation Eq. (8) between nontrivial and trivial gapped phases for increasing interpolation parameter t. A single twofold degeneracy is created at $t = 2/3$ and subsequently splits into two EP2s. These merge at the boundary for $t = 6/7$. For $t > 6/7$ the system is topologically trivial and gapped; all loops in this system correspond to the trivial braid. Note that the last panel in (c) and the first panel in (d) describe the same system.

line-segment Fermi arcs which are known to appear in topologically trivial models.

Topological robustness of the gapless phase. The topological robustness of the EP phase arises from the nontrivial braids along contractible loops on the BZ torus, especially the loop successively traveling as $aba^{-1}b^{-1}$. We show first that all contractible loops in a gapped area must carry trivial braids: We take a deformation retraction γ_s ($0 \leqslant s \leqslant 1$) between some loop γ_0 in the BZ and a constant loop γ_1 (a point). If the Hamiltonian is gapped throughout this deformation retraction, then there is a well-defined braid of eigenvalues $[\gamma_s]$ corresponding to each γ_s, defining a continuous deformation of braids. Such continuous deformation without intersecting strands does not affect the braid group element, which means $[\gamma_0] = [\gamma_1]$. As the constant loop carries the trivial braid this concludes the proof. This implies conversely that if any contractible loop in the BZ carries a nontrivial braid, the system must be gapless.

This argument implies robustness of the gaplessness under perturbation. In the BZ of H_B, all contractible loops enclosing either or both of the EPs carry nontrivial braids. The only way to break the braid along a loop is to move EPs across it [Fig. 3(a)]. When perturbing H_B, the braids along loops far away from the EP are unaffected. They are of the same type as labeled by the great circles $aba^{-1}b^{-1}$. So the system remains gapless and contains multiple EP2s or a single EP3, whose braids combine to $\sigma_2^{-1}\sigma_1$.

It is instructive to consider how the degeneracies evolve when we transition from the gapless phase to a gapped phase. We interpolate between Eq. (5) and a model $H_B(\pi, k_y)$ without k_x dependence,

$$H_s(k_x, k_y) = (1 - s)H_B(k_x, k_y) + sH_B(\pi, k_y). \tag{7}$$

Physically this corresponds to decoupling the unit cells in the x direction. The results are summarized in Fig. 3(c). As s increases, the two EPs move away from each other through the BZ towards $(0, \pm\pi)$. Their combined trajectory traces over the meridian along a large nontrivial loop involving the boundary. Finally, for $s = 3/4$ they fuse at $(0, \pi)$. Further increasing s annihilates the two EPs and we obtain a gapped phase.

We may also understand this process and the robustness of the phase via the Fermi arcs. The initial model hosts two topologically nontrivial Fermi arcs, one along each great circle. These arcs can only end in degeneracies, and they change continuously under continuous deformation of H_B. By removing dependence on k_x, we remove the Fermi arc that follows k_y, correspondingly, two degeneracies must follow the meridian around the torus.

The $s = 1$ phase, while nondegenerate, is still a topologically nontrivial phase. One of the three bands is effectively decoupled from the others, but the other two are braided in k_y direction. They share a corresponding extended Fermi arc of $\mathrm{Re}(E_2 - E_3)$ closing along $(k_x, 0)$. The spectrum along the BZ boundary corresponds to $(a, b) = (\mathbb{1}, \sigma_2)$. These two braid elements commute and the combined braid along the boundary is trivial, consistent with previous classification [25,26]. We also present how this topologically nontrivial gapped phase can be switched into a fully topologically trivial phase $H_0 = [2 + \cos(k_x)]\mathrm{diag}(1, 0, -1)$. We choose the following interpolation,

$$H_t = (1 - t)H_B(\pi, k_y) + tH_0. \tag{8}$$

The evolution of the spectrum is shown in Fig. 3(d). At $t = 2/3$, a new degeneracy is created at $\mathbf{k} = (0, 0)$. It splits into a pair of EP2s for increasing t, which then separate further in the k_x direction. Their trajectory traces over a complete longitude, until they merge and annihilate into the trivial gapped phase at $t = 6/7$.

The above series of figures showcases how the topology of EP-annihilation trajectories corresponds to different phase transitions. In the H_s interpolation (7), the EP2s travel over the meridian to cross all contractible loops enclosing the EP3 and break braid a. In the Supplemental Material [44], we prove that if the two EP2s trace out a simple loop, it must be in the noncontractible homotopy class, based on the universal covering map $\mathbb{R}^2 \to T^2$ and the Jordan-Schoenflies theorem [52,53]. In the H_t interpolation (8), the newly created EPs must cross all meridians in order to render the b braid trivial.

Conclusion. We have constructed non-Hermitian phases with unpaired exceptional degeneracies and explained their topological robustness, illustrated in a short-range tight-binding lattice model.

The non-Abelian nature of three or more complex energy bands allows for overall nontrivial topological charges and even non-Abelian monopoles under periodic boundary conditions in reciprocal space. In nontrivial phases these charges are unpaired and cannot be annihilated locally by perturbations. Instead they must travel nontrivially around the BZ torus in order to be gapped out. This endows the gapless phases presented here with topological robustness, as the movement of degeneracies generally is continuous when deforming a model.

Conversely, topologically inequivalent ways of moving EPs around the BZ lead to distinct gapped and gapless phases. The movement of EPs thus encodes the topological information of the system. This differs from the situation on a plane [54], where at least three EPs are required to exhibit non-Abelian features.

Under some tuning, our model exhibits an isolated nontrivial higher-order degeneracy with dispersion beyond the nth root behaviors constructed previously in tight-binding models. This EP3 is accompanied by extensive Fermi arcs that form noncontractible loops, instead of open line segments. These topological Fermi arcs do not have counterparts in Hermitian Weyl nodes, and are purely a feature of non-Hermitian systems.

Note added. Recently, we became aware of two experimental works [55,56] that realized non-Hermitian multiband systems with nontrivial energy braids in acoustic metamaterials, which corroborates the direct experimental relevance of our work.

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