Universidad del Atlántico MATUA

Facultad de Ciencias Básicas
©Programa de Matemáticas
Vol 10 | No 1, (2023), p 11-19

Research Article

The spectrum of the power graph of a cyclic $p-$group and some characteristics of an orthogonal graph in an indefinite metric space

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Abstract. In this paper, among other results we find the spectrum of the power graph of a finite cyclic $p-$group, we show that the spectrum of the combinatorial Laplacian of the power graph of a finite group $P(G)$ has exactly $n-1$ positive eigenvalues being n the order of the group G, for this the basic concepts of group theory are included, certain theorems that support this study, the concept of graph, the essential results of graph theory, algebraic theory of graph and finally the concept of power graph of a finite group, which was presented for the first time in [1]. Finally, a characterization of the orthogonal graph of an indefinite metric space is made, which was introduced by the researchers in this article.
Keywords: Finite groups, graphs, power graph, adjacency matrix, Laplacian matrix, spectrum of a graph, isomorphic graphs, algebraic connectivity, space with indefinite metric, orthogonal graph

Resumen. En este trabajo, entre otros resultados encontramos el espectro del grafo potencia de un $p-$grupo cíclico finito, mostramos que el espectro del Laplaciano combinatorio del grafo potencia de un grupo finito $P(G)$ tiene exactamente $n-1$ valores propios positivos siendo n el orden del grupo G, para ello se incluyen los conceptos básicos de la teoría de grupos, ciertos teoremas que sustentan este estudio, el concepto de grafo, los resultados esenciales de la teoría de grafos, teoría algebraica de grafos y finalmente el concepto de grafo potencia de un grupo finito, que fue presentado por primera vez en [1]. Finalmente, se realiza una caracterización del grafo ortogonal de un espacio de métrica indefinida, la cual fue introducida por los investigadores en este artículo. Grupos finitos, grafos, grafo de potencias, matriz de adyacencia, matriz laplaciana, espectro de un grafo, grafos isomorfos, conectividad algebraica, espacio con métrica indefinida, grafo ortogonal

1. Introduction

By a graph, we understand a triple (V, E, ϕ), such that V, E are finite sets, $V \cap E = \emptyset$, $V \neq \emptyset$, y $\phi : E \to \binom{V}{2} \cup V$ the elements of $V = V(G)$ y $E = E(G)$ are called vertices and edges of G, respectively. Furthermore, we assume that H is a subgraph of G if each vertex of H is a

Date: Received: 05 10 2023; Accepted: 23 12 2023.

vertex of G and each edge of H is an edge of G, a complete graph is a graph where each pair of different vertices are connected by an edge. A complete graph of n vertices has $n(n-1)/2$ edges and is denoted K_n. A graph G is said to be connected if for each pair of vertices of G there is at least one path that joins them. A connected graph that does not have a cycle is called a tree. A set of two or more trees is called a forest. A spanning tree is a subgraph of G that is a tree whose vertices are all the vertices of graph G.

Currently, the study of the graphs that can be generated through groups has increased exponentially, as are the cases of the power graph of a finite group defined in [1], the center graph of a group [2], the $n-$th commutator graph of a finite group [3] and Prime Coprime Graph of a Finite Group [4]. Chakrabarty defines the power graph $P(G)$ as the graph whose set of vertices are the elements of the finite group G, namely $V(P(G)) = G$, and two vertices are adjacent if and only if one is the power of the other. In addition, the algebraic theory of graphs is a branch of mathematics that is responsible for studying the properties of graphs through linear algebra, because to each graph H we can associate different matrices, such as the adjacency matrix of size $n \times n$ denoted $A(H)$ and defined by:

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ is adjacent to } j \\ 0, & \text{if } i \text{ is not adjacent to } j \end{cases}$$

being i, j vertices of the graph H, the combinatorial Laplacian matrix of size $n \times n$ denoted $L(H)$ and defined by:

$$l_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -1, & \text{if } i \text{ is adjacent to } j \\ 0, & \text{in another case.} \end{cases}$$

where d_i is the degree of the i-th vertex, and the normalized laplacian matrix $\mathcal{L}(H)$ defined by:

$$\mathcal{L}_{ij} = \begin{cases} 1, & \text{if } i = j \\ \frac{-1}{\sqrt{d_i d_j}}, & \text{if } i \text{ is adjacent to } j \\ 0, & \text{in another case.} \end{cases}$$

Based on this we determine the eigenvalues of the matrices $A(P(G)), L(P(G))$, and $\mathcal{L}(P(G))$, being G a finite cyclic $p-$group, we also prove that the spectrum of the combinatorial laplacian matrix of the power graph $P(G)$ has at least one positive eigenvalue and finally we propose the definition of the orthogonal graph of an indefinite metric space, showing some characteristics of this graph.

2. The power graph of a finite group

We start this section by recalling the concept of power graph and some relevant results of this theory, which serve as a reference for the results of this research work.

Theorem 2.1. *[13] Let G be a cyclic group of order n, then for each positive divisor d of n there is an unique subgroup of order d.*

Theorem 2.2 (Cauchy's theorem). *[12] Let G be a finite group of order n such that n is divisible by a prime p. Then G has an element of order p and therefore a subgroup of order p.*

Theorem 2.3. *[5] The group U_n is cyclical if and only if $n = 1,\ 2,\ 4,\ p^k$ or $2p^k$, where p is an odd prime, with $k \in \mathbb{N}$.*

Theorem 2.4. *[5] If $a \geq 2$ and $a^m + 1$ is prime, then a is even and m is a power of 2.*

Theorem 2.5. *[6] If the eigenvalues of two graphs do not match, then the graphs are not isomorphic.*

Theorem 2.6. *[6] Let G be a graph, and $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of the combinatorial laplacian of G. Then the number of spanning trees of G is given by*

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$

Theorem 2.7. *[7] Let G be a graph, $L(G)$ the combinatorial laplacian of the graph G and $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L(G)$. The graph G is connected if and only if $\lambda_2 > 0$.*

The result of the theorem above suggests an appropriate name for the eigenvalue λ_2 of $L(G)$. It is called the algebraic connectivity of G. From now on, it will be denoted by $a(G)$. More information about algebraic connectivity can be found in [7], [9] and [8]. In [9] we find previous and new results on the algebraic connectivity of graphs, giving a classification to the bounds of algebraic connectivity in function of other invariants of the graph, as well as the applications of the eigenvector related to algebraic connectivity, known as the Fiedler vector.

Proposition 2.1. *[11] If K_n is the complete graph of n vertices, then $Spec(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$, for all $n \in \mathbb{N}$.*

Proposition 2.2. *[11] The eigenvalues of the combinatorial Laplacian matrix associated with the complete graph K_n are given by $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = n$, for all $n \in \mathbb{N}$.*

Proposition 2.3. *[11] If G_1, G_2 are finite groups, such that $G_1 \cong G_2$, then $P(G_1)$ and $P(G_2)$ are isomorphic graphs.*

Proposition 2.4. *[1] Let S be a semigroup. Then $P(S)$ is complete if and only if the cyclic subsemigroups of S are ordered linearly with respect to the usual containing relation (that is, for any two cyclic subsemigroups S_1, S_2 de S, $S_1 \subseteq S_2$ o $S_2 \subseteq S_1$).*

Theorem 2.8. *[1] Let G be a finite group, the graph $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and a positive integer m.*

Theorem 2.9. *[1] The graph $P(U_n)$ is complete if and only if n takes any of the following values $n = 1, 2, 4, p, 2p$, where p is a Fermat prime. That is, $p = 2^{2^m} + 1$, for some integer $m \geq 0$.*

3. Space with indefinite metric

Definition 3.1. *Let $\mathcal{V}$ be a vector space over the field of complex numbers $\mathbb{C}$, an inner product in $\mathcal{V}$ is a function $\mathcal{Q} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C}$ such that:*

1. $\mathcal{Q}(x + y, z) = \mathcal{Q}(x, z) + \mathcal{Q}(y, z)$
2. $\mathcal{Q}(\alpha x, y) = \alpha \ \mathcal{Q}(x, y)$
3. $\mathcal{Q}(x, y) = \overline{\mathcal{Q}(y, x)}$ *for all* $x,\ y,\ z \in \mathcal{V}$ *and* $\alpha \in \mathbb{C}$.

If the inner product satisfies (1) and (2) it is called hermitian sesquilinear form if it also satisfies (3) it is said to be hermitian symmetric.

Many authors instead of using the notation $\mathcal{Q}(x, y)$ write $\langle x, y \rangle$ or (x, y) or $[x, y]$ in our theory we we will reserve the notation $\langle x, y \rangle$ for inner products in Hilbert spaces. Note that $\mathcal{Q}(x, x)$ is a real number for this reason it can be a positive, negative or zero number. If a $\mathcal{Q}-$ metric defined in a vector space $\mathcal{V}$ takes positive and negative values, we will call the pair $(\mathcal{V}, \mathcal{Q})$ *Space with indefinite metric* or *Space with indefinite product*, for convenience we will write $\mathcal{Q}(x, y) = [x, y]$. A space with undefined inner product will be denoted by $(\mathcal{V}, [\cdot, \cdot])$ instead of $(\mathcal{V}, \mathcal{Q})$.

Proposition 3.1. *[10] If $\mathcal{V}$ is a space with indefinite inner product, then the polarization property is satisfied, that is, for all $x,\ y \in \mathcal{V}$ we have*

$$[x, y] = \frac{1}{4}[x + y, x + y] - \frac{1}{4}[x - y, x - y] + \frac{i}{4}[x + iy, x + iy] - \frac{i}{4}[x - iy, x - iy].$$

Since $[x, x] \in \mathbb{R}$ for all $x \in \mathcal{V}$, then the trichotomy property of real numbers motivates the following definition:

Definition 3.2. *Given a vector x in the space $(\mathcal{V}, [\cdot\,,\cdot])$ with inner product it is said that:*

(1) *x is positive, if $[x, x] > 0$;*
(2) *x is negative, if $[x, x] < 0$;*
(3) *x is neutral, if $[x, x] = 0$.*

Notice that x could be neutral even when $x \neq 0$.

Example 3.1. *Let's consider $\mathcal{V} = \mathbb{R}^2$ with the inner product $[\cdot, \cdot] : \mathbb{R}^2 \times \mathbb{R}^2$ given by:*

$$[(a, b), (c, d)] := ac - bd.$$

Clearly $(1, 1) \neq (0, 0)$, but $[(1, 1), (1, 1)] = 0$.

Thanks to the definition and the previous example we can identify the following sets:

$$\begin{aligned}
\mathcal{B}^{+} &= \{x \in \mathcal{V} : [x, x] \geq 0\};\\
\mathcal{B}^{++} &= \{x \in \mathcal{V} : [x, x] > 0 \text{ or } x = 0\};\\
\mathcal{B}^{++} &= \{x \in \mathcal{V} : [x, x] > 0 \text{ or } x = 0\};\\
\mathcal{B}^{--} &= \{x \in \mathcal{V} : [x, x] < 0 \text{ or } x = 0\};\\
\mathcal{B}^{0} &= \{x \in \mathcal{V} : [x, x] = 0\};\\
\mathcal{B}^{00} &= \{x \in \mathcal{V} : [x, x] = 0 \text{ and } x \neq 0\}.
\end{aligned}$$

It can be observed that $\mathcal{B}^0 \neq \emptyset$, since $0 \in \mathcal{B}^0$.

Definition 3.3. *An inner product space $(\mathcal{V}, [\cdot, \cdot])$ it is said to be:*

(1) *Space with indefinite inner product when it has both positive and negative elements, that is, there are $x, y \in \mathcal{V}$ such that $[x, x] > 0$ and $[y, y] < 0$.*
(2) *Space with semi-defined inner product when it is not indefinite.*
(3) *Space with positive semi-defined inner product when $[x, x] \geq 0$, for all $x \in \mathcal{V}$.*
(4) *Space with negative semi-defined inner product when $[x, x] \leq 0$, for all $x \in \mathcal{V}$.*
(5) *Space with defined inner product when $[x, x] = 0$ implies $x = 0$, for all $x \in \mathcal{V}$.*
(6) *Space with neutral inner product when $[x, x] = 0$, for all $x \in \mathcal{V}$.*

Definition 3.4. *Let $\mathcal{V}$ be a space with inner product and let $x, y \in \mathcal{V}$. We say that x, y are orthogonal vectors when $[x, y] = 0$ and it is denoted by $x[\perp]y$.*

Definition 3.5. *Let $\mathcal{V}$ be a space with inner product and let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{V}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal sets when $a[\perp]b$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and it is denoted by $\mathcal{A}[\perp]\mathcal{B}$.*

Remark 3.1. *If $[x, y] = 0$, for all $y \in \mathcal{Y}$, we will write $[x, \mathcal{Y}] = 0$.*

Definition 3.6. *Let $(\mathcal{V}, [\cdot, \cdot])$ be a space with inner product and E a subset of $\mathcal{V}$. The orthogonal companion of E, denoted by $E^{[\perp]}$, is the set*

$$E^{[\perp]} = \{x \in \mathcal{V} : [x, E] = 0\}.$$

Proposition 3.2. *[11] Let $(\mathcal{V}, [\cdot, \cdot])$ be a space with indefinite inner product, and $E \subseteq \mathcal{V}$. Then $E^{[\perp]}$ is vector subspace of $\mathcal{V}$.*

4. The spectrum of the power graph of a cyclic $p-$group

Proposition 4.1. *Let G_1, G_2 be two finite groups. If, $G_1 \cong G_2$ then we have that $Spec(P(G_1)) = Spec(P(G_2))$.*

Proof. Let G_1, G_2 be two isomorphic finite groups, then by the Proposition 2.3 we have that $P(G_1)$ is isomorphic to $P(G_2)$. Now, from the Theorem 2.5 it follows that $\mathrm{Spec}(P(G_1)) = \mathrm{Spec}(P(G_2))$. ☐

Proposition 4.2. *If G is a finite cyclic $p-$group, then we have*

$$Spec(P(G)) = \begin{pmatrix} -1 & |G|-1 \\ |G|-1 & 1 \end{pmatrix}.$$

Proof. Let G be a finite cyclic $p-$group, then $|G| = p^k$, for some $k \in \mathbb{N}$ and some prime number p. Then by Theorem 2.8, $P(G)$ is a complete graph, such that $P(G) = K_{p^k}$. By Proposition 2.1 it follows that,

$$Spec(P(G)) = \begin{pmatrix} -1 & p^k-1 \\ p^k-1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & |G|-1 \\ |G|-1 & 1 \end{pmatrix}.$$

☐

Theorem 4.1. *If p is a Fermat prime number, then the spectrum of $P(U_p)$ and $P(U_{2p})$ are given by*

$$Spec(P(U_p)) = \begin{pmatrix} -1 & p-2 \\ p-2 & 1 \end{pmatrix}, \quad Spec(P(U_{2p})) = \begin{pmatrix} -1 & \phi(2p)-1 \\ \phi(2p)-1 & 1 \end{pmatrix}.$$

Proof. Let p be a Fermat prime number, then by the Theorem 2.9 we have that $P(U_p)$ and $P(U_{2p})$ are complete graphs. Also, $|U_p| = \phi(p) = p-1$, therefore $P(U_p) = K_{p-1}$. Besides, from Proposition 2.1 we get that

$$Spec(P(U_p)) = \begin{pmatrix} -1 & p-2 \\ p-2 & 1 \end{pmatrix}.$$

On the other hand, $|U_{2p}| = \phi(2p)$, hence $P(U_{2p}) = K_{\phi(2p)}$. Finally, by using Proposition 2.1 it follows that

$$Spec(P(U_{2p})) = \begin{pmatrix} -1 & \phi(2p)-1 \\ \phi(2p)-1 & 1 \end{pmatrix}.$$

☐

Corolary 4.1. *Let p be a Fermat prime number, then:*

(1) $Spec(P(U_{2p})) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, *if $p = 2$.*

(2) $Spec(P(U_p)) = Spec(P(U_{2p}))$, *if $p > 2$.*

Proof. If $p = 2$, then $\phi(2p) = \phi(4) = \phi(2^2) = 2$, so by Theorem 4.1 we have

$$Spec(P(U_{2p})) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

On the other hand, if $p > 2$ then

$$\phi(2p) = 2p\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{p}\right) = p - 1.$$

Therefore, by using Theorem 4.1 we obtain that

$$Spec(P(U_{2p})) = \begin{pmatrix} -1 & p-2 \\ p-2 & 1 \end{pmatrix}.$$

Hence, $Spec(P(U_p)) = Spec(P(U_{2p}))$. ☐

Lemma 4.1. *If G is a finite cyclic $p-$group, then the spectrum of $L(P(G))$ is given by*

$$Spec(L(P(G))) = \begin{pmatrix} 0 & |G| \\ 1 & |G|-1 \end{pmatrix}.$$

Proof. Let G be a finite cyclic $p-$group, then $|G| = p^k$, for some prime p and $k \in \mathbb{N}$. Then $P(G)$ is complete, this is $P(G) = k_{p^k}$. From Proposition 2.2 we have that the eigenvalues of the combinatorial Laplacian matrix of $P(G)$ are $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{p^k-1} = p^k = |G|$. So,

$$Spec(L(P(G))) = \begin{pmatrix} 0 & |G| \\ 1 & |G|-1 \end{pmatrix}.$$

$\square$

Proposition 4.3. *If G is a finite cyclic $p-$group, then the number of expansive trees of $P(G)$ are $|G|^{|G|-2}$.*

Proof. Let G be a finite cyclic $p-$group, then $|G| = p^k$, for some $k \in \mathbb{N}$ and some prime number p. Then, $P(G) = K_{|G|} = K_{p^k}$ and employing Proposition 2.2 the eigenvalues of the Laplacian combinatorial matrix of $P(G)$ are $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{p^k-1} = p^k$. Therefore, by Theorem 2.6, the number of expansive trees of $P(G)$ is given by:

$$\frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{p^k} = \frac{p^k\cdot p^k\cdots p^k}{p^k} = \frac{(p^k)^{p^k-1}}{p^k} = (p^k)^{\left(p^k-2\right)} = |G|^{|G|-2}.$$ $\square$

Theorem 4.2. *Let G be a finite group such that $|G| = n > 1$, then $L(P(G))$ has exactly $n-1$ positive eigenvalues.*

Proof. Let $G = \{v_1, v_2, \cdots, v_n\} = V(P(G))$. So, $v_i = e$, for some $i \in \{1, \cdots, n\}$, where e is the neutral element of G. Then for every v_k we have that $v_k^{|G|} = v_k^n = e$. Hence, v_k is adjacent to e. So, for each pair of elements of G there is a path that joins them, that is, $P(G)$ is a connected graph. Thus, from Theorem 2.7 we obtain $\lambda_2(L(P(G))) > 0$. Given that

$$\lambda_2(L(P(G))) \leq \lambda_3(L(P(G))) \leq \cdots \leq \lambda_n(L(P(G))),$$

then

$$\lambda_3(L(P(G))) > 0, \lambda_4(L(P(G))) > 0, \cdots, \lambda_n(L(P(G))) > 0.$$

Therefore, it is proved that the combinatorial Laplacian of the graph $P(G)$ has exactly $n-1$ positive eigenvalues. $\square$

Theorem 4.3. *Let G be a finite cyclic $p-$group, then we have*

$$Spec(\mathcal{L}(P(G))) = \begin{pmatrix} \frac{|G|-2}{|G|-1} & 2 \\ |G|-| & 1 \end{pmatrix}.$$

Proof. Let G be a finite cyclic $p-$group, then $|G| = p^k$, for some prime p and $k \in \mathbb{N}$, then $P(G)$ is complete, that is $P(G) = K_{p^k}$. The normalized Laplacian matrix associated with the graph $P(G)$ is given by:

$$\begin{aligned} \mathcal{L} &= D^{\frac{-1}{2}}LD^{\frac{-1}{2}} \\ &= \begin{pmatrix} 1 & \frac{1}{p^k-1} & \frac{1}{p^k-1} & \cdots & \frac{1}{p^k-1} \\ \frac{1}{p^k-1} & 1 & \frac{1}{p^k-1} & \cdots & \frac{1}{p^k-1} \\ \frac{1}{p^k-1} & \frac{1}{p^k-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{p^k-1} \\ \frac{1}{p^k-1} & \frac{1}{p^k-1} & \cdots & \frac{1}{p^k-1} & 1 \end{pmatrix}. \end{aligned}$$

The characteristic polynomial associated with $\mathcal{L}$ is given by:

$$\begin{aligned}
|\mathcal{L}-\lambda I| &= |-(\lambda I-\mathcal{L})| \\
&= (-1)^{p^k}|\lambda I-\mathcal{L}| \\
&= (-1)^{p^k}\begin{vmatrix} \lambda-1 & \frac{-1}{p^k-1} & \frac{-1}{p^k-1} & \cdots & \frac{-1}{p^k-1} \\ \frac{-1}{p^k-1} & \lambda-1 & \frac{-1}{p^k-1} & \cdots & \frac{-1}{p^k-1} \\ \frac{-1}{p^k-1} & \frac{-1}{p^k-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{-1}{p^k-1} \\ \frac{-1}{p^k-1} & \frac{-1}{p^k-1} & \cdots & \frac{-1}{p^k-1} & \lambda-1 \end{vmatrix} \\
&= (-1)^{p^k}\left(\frac{1}{p^k-1}\right)^{p^k}(p^k-1)(\lambda-2)\left[(\lambda-1)(p^k-1)+1\right]^{p^k-1}.
\end{aligned}$$

Hence, if $|\mathcal{L}-\lambda I|=0$ then:

$$(-1)^{p^k}\left(\frac{1}{p^k-1}\right)^{p^k}(p^k-1)(\lambda-2)\left[(\lambda-1)(p^k-1)+1\right]^{p^k-1}=0.$$

From the equality above, we get that $\lambda=2$ or $\lambda=\dfrac{p^k-2}{p^k-1}$, that is, $\lambda=2$ or $\lambda=\dfrac{|G|-2}{|G|-1}$. Thus, we can conclude that the normalized Laplacian spectrum $Spec(\mathcal{L}(P(G)))$ is given by:

$$\text{Spec}(\mathcal{L}(P(G)))=\begin{pmatrix} \frac{|G|-2}{|G|-1} & 2 \\ |G|-| & 1 \end{pmatrix}.$$

$\square$

5. Some characteristics of an orthogonal graph in an indefinite metric space

Definition 5.1. *Let $(\mathcal{V},[\cdot,\cdot])$ be an indefinite metric space and $\mathcal{K}\subseteq\mathcal{V}$, with $\mathcal{K}\neq\emptyset$. We define the orthogonal graph in $\mathcal{K}$, denoted by $\Omega_{[\perp]}(\mathcal{K})$ as the graph whose set of vertices are elements of $\mathcal{K}$ and two different vertices $x,y\in\mathcal{K}$ are adjacent if and only if $[x,y]=0$. If we have that, $K=\mathcal{V}$, the graph $\Omega_{[\perp]}(\mathcal{V})$ is called an orthogonal graph in the indefinite metric space $(\mathcal{V},[\cdot,\cdot])$.*

Remark 5.1. *[11] Let $(\mathcal{V},[\cdot,\cdot])$ be a space with semi-defined inner product. For any $x\in B^0$ and $y\in\mathcal{V}$, we have that x is adjacent to y in the graph $\Omega_{[\perp]}(\mathcal{V})$.*

Theorem 5.1. *Let $(\mathcal{V},[\cdot,\cdot])$ be an indefinite metric space, the graph $\Omega_{[\perp]}(\mathcal{V})$ has a complete induced subgraph.*

Proof. Let $x,y\in B^0$, then we have $[x,x]=0$ and $[y,y]=0$. From the polarization identity we have:

$$[x,y]=\frac{1}{4}[x+y,x+y]-\frac{1}{4}[x-y,x-y]+\frac{i}{4}[x+iy,x+iy]-\frac{i}{4}[x-iy,x-iy]=0.$$

Therefore, x is adjacent to y, thus the graph $\Omega_{[\perp]}(B^0)$ is complete. Also the graph $\Omega_{[\perp]}(B^0)$ is an induced subgraph of $\Omega_{[\perp]}(\mathcal{V})$. $\square$

Remark 5.2. *Let $(\mathcal{V},[\cdot,\cdot])$ be an indefinite metric space, two elements $x,y\in\mathcal{V}$ are adjacent in $\Omega_{[\perp]}(\mathcal{V})$ if and only if their respective additive inverses are adjacent.*

Proof. Let $x,y\in\mathcal{V}$ be adjacent in $\Omega_{[\perp]}(\mathcal{V})$, then $[x,y]=0$. Using inner product properties, we have $[-x,-y]=0$. Therefore, $-x$ and $-y$ are adjacent in $\Omega_{[\perp]}(\mathcal{V})$.
Reciprocally, let $x,y\in\mathcal{V}$ such that $-x$ and $-y$ are adjacent in $\Omega_{[\perp]}(\mathcal{V})$, then $[-x,-y]=0$. Again, using inner product properties is easy to see $[x,y]=0$. Thus, we concluded that x and y are adjacent in $\Omega_{[\perp]}(\mathcal{V})$. $\square$

Proposition 5.1. *Let* $(\mathcal{V}, [\cdot,\cdot])$ *be an indefinite metric space. If* $\mathcal{K}$ *is a nonempty subset of* $\mathcal{V}$*, then the graph* $\Omega_{[\perp]}(\mathcal{K} \cup \{0\})$ *is connected.*

Proof. Let $x \in \mathcal{K}$. Then

$$[0, x] = [x + (-x), x] = [x, x] + [-x, x] = [x, x] - [x, x] = 0.$$

So, if $x \in \mathcal{K} \cup \{0\}$, then, x is adjacent to 0 in the graph $\Omega_{[\perp]}(\mathcal{K} \cup \{0\})$. Therefore, for every $a, b \in \mathcal{K} \cup \{0\}$ there is a path that join them, that is, the graph $\Omega_{[\perp]}(\mathcal{K} \cup \{0\})$ is connected. $\square$

Example 5.1. *Consider the indefinite metric space* $(\mathbb{R}^2, [\cdot,\cdot])$*, in which it is defined* $[(a, b), (c, d)] = ac - bd$*. If*

$$\mathcal{K} = \{(2, 8), (4, 1), (3, 3), (4, 4), (5, 5), (6, 6), (9, 9)\},$$

then the graph $\Omega_{[\perp]}(\mathcal{K})$ *(see Figure 1) is orthogonal.*

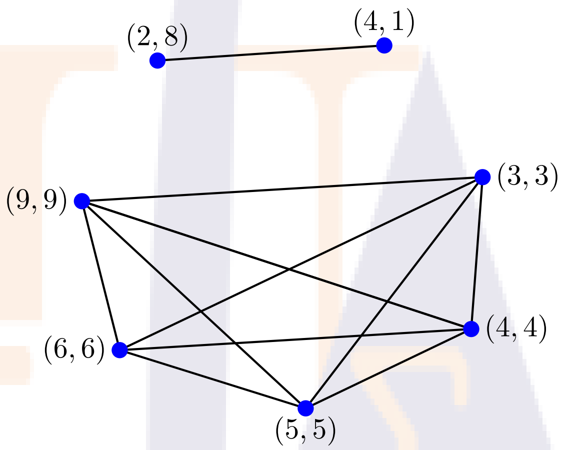

FIGURE 1. Orthogonal graph $\Omega_{[\perp]}(\mathcal{K})$.

Theorem 5.2. *Let* $(\mathcal{V}, [\cdot,\cdot])$ *be an indefinte metric space,* E_1, E_2 *subsets of* $\mathcal{V}$ *such that* $E_1 \subseteq E_2$*. Then the graph* $\Omega_{[\perp]}(E_2^{[\perp]})$ *is an induced subgraph of* $\Omega_{[\perp]}(E_1^{[\perp]})$*.*

Proof. Let $x \in E_2^{[\perp]}$, then $x[\perp]y$, for all $y \in E_2$. Given that $E_1 \subseteq E_2$ it follows that $x[\perp]y$, for all $y \in E_1$. This is $x \in E_1^{[\perp]}$. Thus, $E_2^{[\perp]} \subseteq E_1^{[\perp]}$. From Proposition 3.2 we have that $E_1^{[\perp]}$ and $E_2^{[\perp]}$ are subspaces of $\mathcal{V}$. Hence, $\left(E_1^{[\perp]}, [\cdot,\cdot]\right)$ and $\left(E_2^{[\perp]}, [\cdot,\cdot]\right)$ are indefinite metric spaces, as $E_2^{[\perp]} \subseteq E_1^{[\perp]}$ then $V\left(\Omega_{[\perp]}(E_2^{[\perp]})\right) \subseteq V\left(\Omega_{[\perp]}(E_1^{[\perp]})\right)$. Now let x, y be adjacent vertices in the graph $\Omega_{[\perp]}(E_2^{[\perp]})$. Then, by definition of the orthogonal graph, $x, y \in E_2^{[\perp]}$ and $[x, y] = 0$. So, $x, y \in E_1^{[\perp]}$. Thus, x, y also are adjacent in the graph $\Omega_{[\perp]}(E_1^{[\perp]})$. Whereby, $E\left(\Omega_{[\perp]}(E_2^{[\perp]})\right) \subseteq E\left(\Omega_{[\perp]}(E_1^{[\perp]})\right)$. In conclusion, the graph $\Omega_{[\perp]}(E_2^{[\perp]})$ is an induced subgraph of the graph $\Omega_{[\perp]}(E_1^{[\perp]})$. $\square$

The following result is an immediate consequence of the previous theorem.

Corolary 5.1. *Let* $(\mathcal{V}, [\cdot,\cdot])$ *be an indefinite metric space and* E *a subset of* $\mathcal{V}$*. Then the graph* $\Omega_{[\perp]}(E^{[\perp][\perp][\perp]})$ *is an induced subgraph of* $\Omega_{[\perp]}(E^{[\perp]})$*.*

6. CONCLUSIONS

In this work, we present the definition of the power graph of a finite group, developing the most relevant results of the subject. Based on the fact that each graph can be represented in a matrix form, we study the behavior of the eigenvalues of the power graph $P(G)$. Also, we associate a graph to a space with indefinite metric, which we call orthogonal graph of $(\mathcal{V}, [\cdot,\cdot])$.

From this definition we draw a new line of research, and we study some important characteristics of the graph $\Omega_{[\perp]}(\mathcal{V})$.

Acknowledgements

The authors are grateful to Centro de Modelación Matemática of Universidad del Atlántico, Colombia.

References

[1] Chakrabarty, I. and Ghosh, S. and Sen, M.K, *Undirected power graphs of semigroups*, Semigroup Forum, vol. 78, Springer, 2009, pp. 410–426. Cited on pp. 11, 12, and 13.

[2] P. Balakrishnan, M. Sattanathan, and R. Kala, *The center graph of a group*, South Asian J. Math 1 (2011), no. 1 Cited on p. 12.

[3] L. Xiu and X. Ma, *The commutator graph of a finite group*. South Asian J Math 5 (2015), no. 3, 141–146. Cited on p. 12.

[4] A. Adhikari and S. Banerjee, *Prime coprime graph of a finite group* Novi Sad J. Math, Novi Sad Journal of Mathematics, https://doi. org/10.30755/NSJOM 11151 (2021). Cited on p. 12.

[5] G.A. Jones and J.M. Jones, *Elementary number theory,*. Springer Science & Business Media, 2005. Cited on p. 12.

[6] Butler, S. K, *Eigenvalues and structures of graphs*. University of California, San Diego, 2008. Cited on pp. 12 and 13.

[7] Fiedler, M, *Algebraic connectivity of graphs*, Czechoslovak mathematical journal 23 (1973), no. 2, 298–305. Cited on p. 13.

[8] R. Grone and R. Merris, *Algebraic connectivity of trees*, Czechoslovak Mathematical Journal 37 (1987), no. 4, 660–670. Cited on p. 13.

[9] N.M.M. De Abreu, *Old and new results on algebraic connectivity of graphs*, Linear algebra and its applications 423 (2007), no. 1, 53–73. Cited on p. 13.

[10] J. Bognár, *Indefinite inner product spaces*, vol. 78, Springer Science Business Media, 2012. Cited on p. 13.

[11] E. Aldana, *El espectro del grafo potencia de un grupo finito y algunas características del grafo ortogonal de un espacio con métrica indefinida*, Master's thesis, 10 2022, p. 72. Cited on pp. 13, 14, and 17.

[12] J.J. Rotman, *An introduction to the theory of groups*,vol. 148, Springer Science & Business Media, 2012. Cited on p. 12.

[13] D. S. Malik, J.M Mordeson and M. K. Sen, *Fundamentals of abstract algebra,*, The McCGraw-HILL Companies, Inc. New York st. Louis, san Francisco, printed in Singapore (1997). Cited on p. 12.

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