Collective Dynamics

Signalized and Unsignalized Road Traffic Intersection Models: A Comprehensive Benchmark Analysis

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Received: 5 January 2023 / Last revision received: 25 November 2023 / Accepted: 26 November 2023
DOI: 10.17815/CD.2023.144

Abstract Road traffic flow models allow the development and testing of intelligent transportation solutions. Macroscopic intersection models are especially relevant for the simulation of large traffic networks. In this article, we study four first-order signalized and unsignalized intersection models. The two unsignalized approaches are the first-in-first-out (FIFO) model (roundabout-type intersection) and an optimal non-FIFO model (highway-type intersection). The optimal control operates upstream for the first signalized intersection model. It occurs downstream for the second signalized model. All four models satisfy the expected physical constraints of vehicle conservation, traffic demand, and assignment. The models are minimal and allow a comprehensible analysis of the results. We determine mathematical relationships between the intersection models and empirically analyze the performances using Monte Carlo simulations. The numerical simulations assume random demand, supply, and assignment. Besides average performances, the approach accounts for the flow ranges of variation. A benchmark analysis compares the intersection models. We observe that the optimal signalized intersection models overcome the performances of the FIFO model in congested states. They may even reach the performances of the idealistic non-FIFO model. Further applications for the four intersection models are discussed.

Keywords Road traffic intersection · first-order model · regulated intersection model · unregulated intersection model · Monte Carlo simulation

1 Introduction

Road traffic flows are complex systems of interaction agents. Modelling traffic flow is a challenging task, especially for intersections. Three families of continuous time road traffic models has been identified in the literature [1, 2]:

1. Microscopic models describing the trajectory of each vehicle individually using first or second order ordinary differential equations for the vehicle position, speed or acceleration.

2. Kinetic (mesoscopic) models by hyperbolic partial differential equation of first order for the distribution of the traffic performances. They are defined in analogy with statistical physics.

3. Macroscopic models based on hyperbolic or parabolic conservation laws for averaged traffic performance. Macroscopic models are mainly borrowed from fluid mechanics.

The major advantage of macroscopic models is their tractable mathematical structure, their low number of parameters, and their fast computational properties [3]. Several authors propose network traffic models including dynamics at intersections [4–7]. Macroscopic road traffic intersection models are flow functions. They depend on the traffic upstream (demand), the traffic downstream (supply), and the flow composition by direction [8]. The models are constrained by conservation laws. One of the main macroscopic road traffic models for unsignalized intersections are the First-In-First-Out (FIFO) model and non-FIFO model (optimal model), among [4, 9, 10]. FIFO models assume constant the flow composition over the intersection [4]. This makes the flows on the incoming and outgoing roads strongly inter-dependent. It corresponds to the situation where all intersecting vehicles share a common section without overtaking possibilities, like, e.g., a roundabout.

Non-FIFO models suppose independent flows by direction making the performance more efficient than the FIFO model in case of congestion [9, 10]. The situation for non-FIFO models corresponds to highway traffic intersection for which there exist specific roads by destination. Analysing the difference of FIFO and non-FIFO approaches is the topic of many research works [9–11]. Signalized intersection models include traffic lights and priority rules [5, 12–14]. They mainly rely on optimisation problems for the traffic light phase. Many recent solutions are based on deep reinforcement learning techniques [15–21] or again ARIMA processes [22].

Intersection models are combinations of merging, when several roads converge to a single one, and diverging models, when a single road is distributed over several outgoing roads [23]. The intersections are typically point-wise in macroscopic models, i.e., the intersection is reduced to a point without any physical dimensions. Intersection models are generally embedded in dynamic network loading (DNL) models to simulate the traffic on networks. The cell transmission model [4] and link transmission model [24] are for instance well-established discrete approaches based on FIFO intersection models.

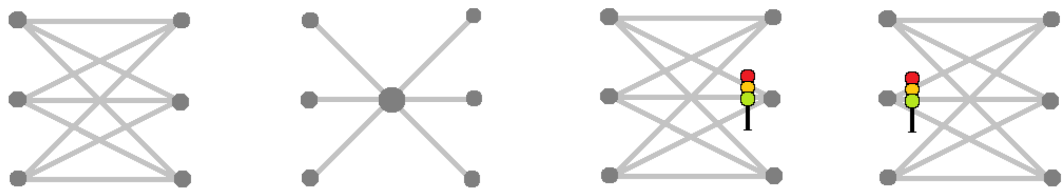

Figure 1 Illustrative scheme for the four types of intersection: Non-FIFO (optimal), FIFO, and the two regulated models with control on the incoming roads and on the outgoing roads, respectively.

In this contribution, we analyse and compare mathematically and by simulation four minimal signalized and unsignalized macroscopic intersection models of the first order (see Fig. 1). The objective is to provide a comprehensible benchmark analyse of different intersection models. The two unregulated models are the FIFO model with common crossing section and the optimal model for which exist distinct roads by direction. The control (i.e., traffic light) operates upstream on the incoming roads for the first regulated intersection model, while the control takes place downstream on the directions for the second regulated model. We demonstrate mathematical relationships between the intersection models and analyze the performances using Monte Carlo simulations.

The contribution is organised as follows. We review general physical constraints operating in intersection models in Sec. 2. We define the four signalized and unsignalized intersection models and provide some of their main properties in Sec. 3. We present simulation results in Sec. 4. Some conclusion and discussion are given in Sec. 5.

2 Physical constraints

According to [5], the conditions that first-order macroscopic intersection models in dynamic network loading must conform to:

1. General applicability to any number of incoming and outgoing roads.
2. Non-negativity of the flow.
3. Conservation of vehicles.
4. Compliance with demand and supply constraints.
5. Flow maximisation from the user perspective.
6. Conservation of turning fractions.
7. Compliance with the invariance principle [8].

In addition to the physical constraints, we introduce the main variables for a point-wise intersection, which we use in the following to describe the system (see Fig. 2):

- α and β : the demand upstream and the supply downstream.
- ω and σ : the flows on the incoming roads and the outgoing roads.
- p : The assignment proportion of the demand by direction (route choice).
- q : Some priority rules (control).
- J : The total flow.

It holds for a $n\times m$ intersection: $\alpha,\omega\in\mathbb{R}_+^n$, $\beta,\sigma\in\mathbb{R}_+^m$, and $p\in M_+^{n\times m}$ with

$$\sum_{j=1}^{m} p_{ij} = 1, \quad \text{for all } i\in\{1,\ldots,n\}. \tag{1}$$

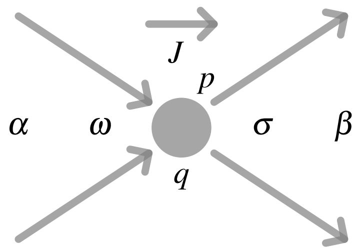

Figure 2 Notations for a point-wise intersection. α and β are the demand upstream and the supply downstream; ω and σ are the flows on the incoming roads and on the outgoing roads; p is the assignment of the demand by direction (route choice); q are some priority rules; J is the total flow.

An intersection model is a function specifying the incoming and outgoing flows according to the demand, supply, assignment by direction, and priority rules:

$$(\omega,\sigma) = F : (\alpha,\beta,p,q) \mapsto F(\alpha,\beta,p,q).$$

The fundamental physical constraints operating in point-wise intersection models are (see, e.g., [9, 10]):

(a) Flow (or vehicle) conservation (Kirchhoff's law):

$$J = \sum_{i=1}^{n}\omega_i = \sum_{j=1}^{m}\sigma_j. \tag{2}$$

(b) Flows positive and bounded by demand and supply:

$$\forall i\in\{1,\ldots,n\}, \quad 0\le\omega_i\le\alpha_i, \tag{3}$$

and

$$\forall j\in\{1,\ldots,m\}, \quad 0\le\sigma_j\le\beta_j. \tag{4}$$

(c) Assignment by direction:

i. Conserved flow assignment proportion

$$\forall j \in \{1,\dots,m\}, \quad \sigma_j = \sum_{i=1}^{n} p_{ij}\omega_i. \tag{5}$$

ii. Assignment constrained by the demand only

$$\forall j \in \{1,\dots,m\}, \quad \sigma_j \leq \sum_{i=1}^{n} p_{ij}\alpha_i. \tag{6}$$

The constraint **(c) i.** holds for FIFO models, while the more general constraint **(c) ii.** holds for non-FIFO models.

3 Unsignalized and signalized intersection models

3.1 Unsignalized intersection models

Many types of traffic intersections have been referred to as *unsignalized* or again *unregulated*. These include intersections with two-way stop control, intersections with all-way stop signage, or roundabouts [25]. In this section, we define two unregulated models, namely, the FIFO (First-in-First-out) model and a non-FIFO (optimal) model.

3.1.1 FIFO model

The vehicles are not able to overtake in FIFO models: the first vehicle going in the intersection is the first out. Consequently, the flow composition is conserved [4]. At diverging junctions, congestion on one outgoing road may impede the traffic on the other roads. Indeed, the flows are proportional and interdependent.

The FIFO model reads

$$\begin{cases} \omega_i &= \kappa\alpha_i, & i \in \{1,\dots,n\}, \\ \sigma_j &= \kappa\displaystyle\sum_{i=1}^{n} p_{ij}\alpha_i, & j \in \{1,\dots,m\}, \\ J_{\mathrm{F}} &= \kappa\displaystyle\sum_{i=1}^{n}\alpha_i, & \end{cases} \tag{7}$$

with

$$\kappa = \min\left\{1, \min_{j\in\{1,\dots,m\}} \frac{\beta_j}{\sum_{i=1}^{n} p_{ij}\alpha_i}\right\}$$

the proportion coefficient of the FIFO model.

3.1.2 Non-FIFO model

We assume for the non-FIFO model (or optimal model) that there exist distinct roads by direction [9]. Such an idealistic situation induces independence of the outgoing flow. It may correspond to large intersections on highways.

The non-FIFO model is:

$$\begin{cases} \omega_i &= \min\{\alpha_i, \gamma\} \qquad\qquad i \in \{1,\dots,n\}, \\ \sigma_j &= \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\}, \qquad j \in \{1,\dots,m\}, \\ J_{\text{NF}} &= \sum_{j=1}^{m} \sigma_j. \end{cases} \tag{8}$$

Here γ is the solution of

$$\sum_{i=1}^{n} \min\{\alpha_i, \gamma\} = J_{\text{NF}}. \tag{9}$$

Note that

$$\gamma \geq \max_{i\in\{1,\dots,n\}} \alpha_i \quad \text{if} \quad J_{\text{NF}} = \sum_{i=1}^{n} \alpha_i \ \ \text{(free state)},$$

while γ, by continuity and monotony, has a unique solution in

$$\gamma \in \big[J_{\text{NF}}/n, \max_{i\in\{1,\dots,n\}} \alpha_i\big[\quad \text{if} \quad J_{\text{NF}} < \sum_{i=1}^{n} \alpha_i \ \ \text{(congested state)}.$$

We have for instance

$$\gamma = \max\big\{J_{\text{NF}}/2,\ J_{\text{NF}} - \alpha_1,\ J_{\text{NF}} - \alpha_2\big\}$$

for a 2×1 converging congested intersection for which $J_{\text{NF}} < \alpha_1 + \alpha_2$ [10]. In fact, assuming $\alpha_1 < \alpha_2$ without loss of generality, then $\gamma = J_{\text{NF}} - \alpha_1$ and the congestion only affects the second incoming road if $\alpha_1 < J_{\text{NF}}/2$, while $\gamma = J_{\text{NF}}/2$ and the congestion affects both incoming roads if $\alpha_1 > J_{\text{NF}}/2$.

An illustrative comparison for FIFO and non-FIFO models is presented Fig. 3 with a 1×2 diverging intersection.

3.1.3 Properties of the FIFO and non-FIFO intersection models

The demonstrations of the following propositions are straightforward.

Proposition 3.1. *The FIFO model Eq. 7 satisfies the constraints* (2), (3), (4), *and* (5).

Proof. • Constraint (2). We have by definition

$$\sum_{i=1}^{n} \omega_i = \kappa \sum_{i=1}^{n} \alpha_i =: J_{\text{F}},$$

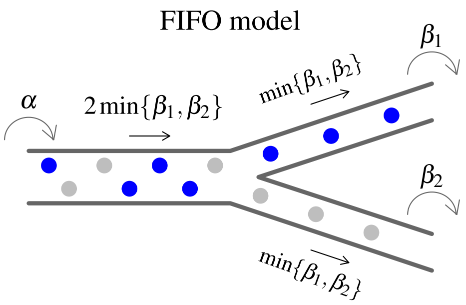

Figure 3 Illustrative comparison for FIFO and non-FIFO models on a 1×2 diverging intersection with $p=(0.5,0.5)$ and $\alpha \geq 2\min\{\beta_1,\beta_2\}$ (FIFO model, left panel) or $\alpha \geq \beta_1+\beta_2$ (non-FIFO model, right panel).

while

$$\sum_{j=1}^{m} \sigma_j = \sum_{j=1}^{m} \kappa \sum_{i=1}^{n} p_{ij}\alpha_i = \kappa \sum_{i=1}^{n} \alpha_i \underbrace{\sum_{j=1}^{m} p_{ij}}_{=1} = \kappa \sum_{i=1}^{n} \alpha_i =: J_{\mathrm{F}}.$$

Therefore

$$J_{\mathrm{F}} = \sum_{i=1}^{n} \omega_i = \sum_{j=1}^{m} \sigma_j$$

i.e., (2) holds.

- Constraints (3) and (4). On one hand, we have directly for any incoming road $i \in \{1,\ldots,n\}$

$$0 \leq \omega_i = \min\left\{\alpha_i, \min_k \frac{\alpha_i \beta_k}{\sum_{i=1}^{n} p_{ik}\alpha_i}\right\} \leq \alpha_i,$$

i.e., (3) holds. On the other hand, we have for any outgoing road $j \in \{1,\ldots,m\}$

$$0 \leq \sigma_j = \min\left\{\alpha_i, \min_k \frac{\beta_k}{\sum_{i=1}^{n} p_{ik}\alpha_i}\right\} \sum_{i=1}^{n} p_{ij}\alpha_i \leq \frac{\beta_j}{\sum_{i=1}^{n} p_{ij}\alpha_i} \sum_{i=1}^{n} p_{ij}\alpha_i = \beta_j,$$

i.e., (4) holds.

- Constraint (5). The assignment composition conservation (5) comes trivially from the definition of the FIFO model Eq. 7:

$$\sum_{i=1}^{n} p_{ij}\omega_i = \sum_{i=1}^{n} p_{ij}\kappa\alpha_i = \kappa \sum_{i=1}^{n} p_{ij}\alpha_i =: \sigma_j.$$

$\square$

Proposition 3.2. *The non-FIFO intersection model Eq. 8 satisfies the constraints (2), (3), (4), and (6).*

Proof.

- Constraint (2). We have by definition, see (8),

$$\sum_{j=1}^{m} \sigma_j =: J_{\mathrm{NF}},$$

while from (9) we obtain directly

$$\sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \min\{\alpha_i, \gamma\} =: J_{\mathrm{NF}}.$$

Therefore

$$J_{\mathrm{NF}} = \sum_{i=1}^{n} \omega_i = \sum_{j=1}^{m} \sigma_j$$

i.e., (2) holds.

- Constraints (3) and (4). We have directly for any incoming road $i \in \{1,\dots,n\}$

$$0 \le \omega_i = \min\{\alpha_i, \gamma\} \le \alpha_i,$$

i.e., (3) holds. The same holds for any outgoing road $j \in \{1,\dots,m\}$

$$0 \le \sigma_j = \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\} \le \beta_j,$$

i.e., (4) holds.

- Constraint (5). The last constraint (5) is straightforward

$$\sigma_j = \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\} \le \sum_{i=1}^{n} p_{ij}\alpha_i.$$

□

Proposition 3.3. *The FIFO intersection model Eq. 7 ensures conservation of the flow composition.*

Proof. Let us define the assignment proportion by direction p_j using the demands α_i

$$p_j = \frac{\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{i=1}^{n} \alpha_i}.$$

It follows directly

$$\frac{\sum_{i=1}^{n} p_{ij}\omega_i}{\sum_{i=1}^{n}\omega_i} = \frac{\sum_{i=1}^{n} p_{ij}\kappa\alpha_i}{\sum_{i=1}^{n}\kappa\alpha_i} = \frac{\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{i=1}^{n}\alpha_i} = p_j,$$

while

$$\frac{\sigma_j}{\sum_{k=1}^{m}\sigma_k} = \frac{\kappa\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{k=1}^{m}\kappa\sum_{i=1}^{n} p_{ik}\alpha_i} = \frac{\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{i=1}^{n}\alpha_i\sum_{k=1}^{m} p_{ik}} = \frac{\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{i=1}^{n}\alpha_i} = p_j.$$

Therefore, the flow composition in terms of assignment proportion by direction is conserved over the demands α_i, the incoming flows ω_i and the outgoing flows σ_j. □

Corollary 3.4. *The flow composition is not conserved through the intersection with the non-FIFO model Eq. 8.*

Proof. As counter-example, we can see that the flow composition of the demand is not equal to the flow composition on the incoming roads with the non-FIFO model Eq. 8 in a congested state for which $J_{\mathrm{NF}} < \sum_{i=1}^{n}\alpha_i$

$$\frac{\sum_{i=1}^{n} p_{ij}\alpha_i}{\sum_{i=1}^{n}\alpha_i} \neq \frac{\sum_{i=1}^{n} p_{ij}\omega_i}{\sum_{i=1}^{n}\omega i} = \frac{\sum_{i=1}^{n} p_{ij}\min\{\alpha_i,\gamma\}}{\sum_{i=1}^{n}\min\{\alpha_i,\gamma\}}$$

since it exists necessary i_0 such that $\omega_{i_0} = \gamma < \alpha_{i_0}$ if $J_{\mathrm{NF}} = \sum_{i=1}^{n}\omega_i < \sum_{i=1}^{n}\alpha_i$. □

Proposition 3.5. *The non-FIFO model systematically outperforms the FIFO model.*

Proof. We have by definition of the FIFO and non-FIFO intersection models Eqs. 7 and 8

$$\begin{aligned}
J_{\mathrm{NF}} &= \sum_{k=1}^{m}\min\{p_k\sum_{i=1}^{n}\alpha_i,\beta_k\} \\
&= \sum_{k=1}^{m} p_k\min\{\sum_{i=1}^{n}\alpha_i,\beta_k/p_k\} \\
&\geq \sum_{k=1}^{m} p_k\min_j\{\sum_{i=1}^{n}\alpha_i,\beta_j/p_j\} \\
&= \min_j\{\sum_{i=1}^{n}\alpha_i,\beta_j/p_j\} \;=\; J_{\mathrm{F}}.
\end{aligned}$$

□

Proposition 3.6. *The FIFO and non-FIFO models have identical free traffic states*

$$J_{\mathrm{F}}=\sum_{i=1}^{n}\alpha_i \quad\Leftrightarrow\quad J_{\mathrm{NF}}=\sum_{i=1}^{n}\alpha_i.$$

Proof. We have first

$$J_{\mathrm{F}}=\sum_{i=1}^{n}\alpha_i \quad\Rightarrow\quad \kappa=1 \quad\Rightarrow\quad \forall j\in\{1,\dots,m\},\ \beta_j\ge\sum_{i=1}^{n}p_{ij}\alpha_i \quad\Rightarrow$$
$$J_{\mathrm{NF}}=\sum_{j=1}^{m}\min\Big\{\sum_{i=1}^{n}p_{ij}\alpha_i,\beta_j\Big\}=\sum_{j=1}^{m}\sum_{i=1}^{n}p_{ij}\alpha_i=\sum_{i=1}^{n}\alpha_i.$$

The reciprocal is straightforward

$$J_{\mathrm{NF}}=\sum_{i=1}^{n}\alpha_i \quad\Rightarrow\quad \forall j\in\{1,\dots,m\},\ \beta_j\ge\sum_{i=1}^{n}p_{ij}\alpha_i \quad\Rightarrow\quad \kappa=1 \quad\Rightarrow\quad J_{\mathrm{F}}=\sum_{i=1}^{n}\alpha_i.$$

□

3.2 Signalized intersection models

We consider intersections regulated by priority rules (i.e., traffic lights). We design by lack of simplicity the priorities using coefficients [26, 27]

$$0\le q_i^{\mathrm{in}},q_j^{\mathrm{out}}\le 1 \quad \forall i\in\{1,\dots,n\},j\in\{1,\dots,m\}, \qquad \sum_{i=1}^{n}q_i^{\mathrm{in}}=\sum_{j=1}^{m}q_j^{\mathrm{out}}=1.$$

Such a continuum approximation allows obtaining understandable results. It takes however not account for the phase of the traffic lights [28]. Note that assuming strict priorities such that $\sum_{i=1}^{n}q_i^{\mathrm{in}}=\sum_{j=1}^{m}q_j^{\mathrm{out}}=1$ allows avoiding conflict for crossing flows. Two optimal control intersection models are investigated. The control operates on the outgoing roads for the first model while the regulation is done on the incoming roads for the second model (see Fig. 4).

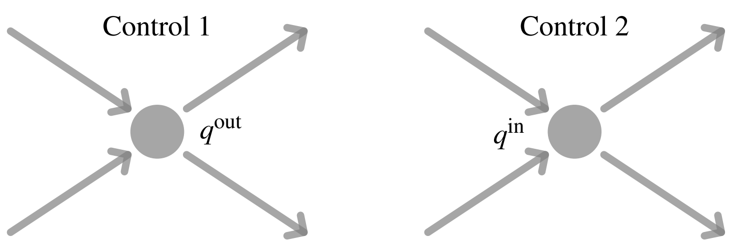

Figure 4 Illustrative scheme for the first signalized model Eq. 10 for which the control operates on the outgoing roads (left panel) and for the second signalized model Eq. 11 for which the regulation operates on the incoming roads (right panel).

3.2.1 Priorities by direction

In the first signalized model, the control operates on the outgoing roads, i.e., on the directions (see Fig. 4, left panel). The first signalized intersection model reads

$$\begin{cases} \omega_i &= \min\{\alpha_i, \gamma\}, & i \in \{1,\dots,n\}, \\ \sigma_j &= \min\{\sum_{i=1}^{n} p_{ij}\alpha_i, q_j^{\mathrm{out}}C, \beta_j\}, & j \in \{1,\dots,m\}, \\ J_{\mathrm{C1}} &= \sum_{j=1}^{m} \sigma_j. \end{cases} \tag{10}$$

Here $C > 0$ is the road capacity, which, for simplicity, we assume to be the same for each incoming and outgoing road, while the number γ, as in the non-FIFO model Eq. 8, is such that

$$\sum_{i=1}^{n} \min\{\alpha_i, \gamma\} = J_{\mathrm{C1}}.$$

3.2.2 Priorities on the incoming roads

The control operates on the incoming roads for the second signalized model (see Fig. 4, right panel). This second signalized intersection model is given by

$$\begin{cases} \omega_i &= \min\{q_i^{\mathrm{in}}C, \alpha_i, \gamma\} & i \in \{1,\dots,n\}, \\ \sigma_j &= \min\{\sum_{i=1}^{n} p_{ij} \min\{q_i^{\mathrm{in}}C, \alpha_i\}, \beta_j\} & j \in \{1,\dots,m\}, \\ J_{\mathrm{C2}} &= \sum_{j=1}^{m} \sigma_j \end{cases} \tag{11}$$

Here γ is such that

$$\sum_{i=1}^{n} \min\{q_i^{\mathrm{in}}C, \alpha_i, \gamma\} = J_{\mathrm{C2}},$$

i.e.,

$$\gamma \geq \max_{i\in\{1,\dots,n\}} \min\{q_i^{\mathrm{in}}C, \alpha_i\} \quad \text{if} \quad J_{\mathrm{C2}} = \sum_{i=1}^{n} \min\{q_i^{\mathrm{in}}C, \alpha_i\}, \ \text{(free state)},$$

while γ, by continuity and monotony, has a unique solution in

$$\gamma \in [J_{\mathrm{C2}}/n, \max_{i\in\{1,\dots,n\}} \min\{q_i^{\mathrm{in}}C, \alpha_i\}[, \quad \text{if} \quad J_{\mathrm{C2}} < \sum_{i=1}^{n} \min\{q_i^{\mathrm{in}}C, \alpha_i\} \ \text{(congested state)}.$$

Remark 1. *Both models are non-FIFO by construction, i.e. flows are not proportional and flow composition is not conserved.*

Remark 2. *Assuming $C \geq \max\{\sum_{i=1}^{n} \alpha_i, \beta_1, \dots, \beta_n\}$, the two special cases $q_i^{in} = 1$ for all $i \in \{1,\dots,n\}$ in the first signalized model Eq. 10 and $q_j^{out} = 1$ for all $j \in \{1,\dots,m\}$ in the second signalized model Eq. 11 correspond to the non-FIFO model Eq. 8.*

3.2.3 Properties of the signalized intersection models

The proofs of the following propositions are straightforward.

Proposition 3.7. *If we assume that the priority rules are strict, i.e.,*

$$\sum_{i=1}^{n} q_i^{in} = \sum_{j=1}^{m} q_j^{out} = 1,$$

then

$$\max\{J_{\mathrm{C1}}, J_{\mathrm{C2}}\} \leq C,$$

i.e. the regulated intersection models only allow the streaming of a single road (exclusive green phase of the traffic light).

Proof. We have

$$J_{\mathrm{C1}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, q_j^{\mathrm{out}}C, \beta_j\Big\} \leq \sum_{j=1}^{m} q_j^{\mathrm{out}}C \leq C$$

and

$$J_{\mathrm{C2}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\min\{q_i^{\mathrm{in}}C, \alpha_i\}, \beta_j\Big\} \leq \sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}q_i^{\mathrm{in}}C = \sum_{i=1}^{n} q_i^{\mathrm{in}}C\sum_{j=1}^{m} p_{ij} = C$$

So we have

$$\max\{J_{\mathrm{C1}}, J_{\mathrm{C2}}\} \leq C.$$

□

Proposition 3.8. *The non-FIFO model Eq. 8 overcomes the performances of the regulated models Eq. 10 and Eq. 11:*

$$\max\{J_{\mathrm{C1}}, J_{\mathrm{C2}}\} \leq J_{\mathrm{NF}}.$$

Proof. We have directly

$$J_{\mathrm{C1}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, q_i^{\mathrm{in}}C, \beta_j\Big\} \leq J_{\mathrm{NF}} \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\},$$

and

$$J_{\mathrm{C2}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\min\{q_i^{\mathrm{in}}C, \alpha_i\}, \beta_j\Big\} \leq J_{\mathrm{NF}} \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\}$$

Therefore

$$\max\{J_{\mathrm{C1}}, J_{\mathrm{C2}}\} \leq J_{\mathrm{NF}}$$

holds □

Proposition 3.9. *If one or both of the two regulated models Eq. 10 and 11 describe a free traffic states with no congestion (I.e., the total flow is the sum of the demands), then the FIFO and non-FIFO Eq. 7 and 8 also describe a free traffic*

$$J_{\mathrm{C1}} = J_{\mathrm{C2}} = \sum_{i=1}^{n} \alpha_i \quad \Rightarrow \quad J_{\mathrm{F}} = J_{\mathrm{NF}} = \sum_{i=1}^{n} \alpha_i.$$

The reciprocal is false.

Proof. Both equalities

$$J_{\mathrm{C1}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, q_j^{\mathrm{out}} C, \beta_j\Big\} = \sum_{i=1}^{n} \alpha_i$$

or

$$J_{\mathrm{C2}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij} \min\{q_i^{\mathrm{in}} C, \alpha_i\}, \beta_j\Big\} = \sum_{i=1}^{n} \alpha_i$$

implies that

$$\sum_{i=1}^{n} p_{ij}\alpha_i \leq \beta_j \quad \text{for any} \quad j \in \{1, \dots, m\}.$$

So we obtain

$$J_{\mathrm{NF}} = \sum_{j=1}^{m} \min\Big\{\sum_{i=1}^{n} p_{ij}\alpha_i, \beta_j\Big\} = \sum_{i=1}^{n} \alpha_i$$

as well. Moreover, we observe in Prop. 3.6 that $J_{\mathrm{F}} = \sum_{i=1}^{n} \alpha_i$ and $J_{\mathrm{NF}} = \sum_{i=1}^{n} \alpha_i$ are equivalent. This completes the proof of the first part of the proposition.

It is enough to choose a particular signalling such as $\sum_{j=1}^{m} p_{ij}\alpha_i \geq q_j^{\mathrm{out}} C$ (model Eq. 10) or $\alpha_i \geq q_i^{\mathrm{in}} C$ (model Eq. 11) to see that the reciprocal is false. □

4 Simulation analysis

4.1 Setting up the simulations

We realise 1e6 Monte Carlo simulations for a 2×2 intersection assuming random demand, supply, and direction assignments. The four roads have identical capacity $C > 0$. The demands on the two incoming roads α_1 and α_2 and the two supplies β_1 and β_2 on the outgoing roads are uniformly distributed on $[0, C]$ and independent. The vehicles are randomly affected to the two destinations with probability $p \in]0, 1[$, i.e.,

$$p_{11} = p_{21} = p, \qquad p_{12} = p_{22} = 1 - p$$

The four models for a 2×2 intersection are:

- Unsignalized models

– FIFO model Eq. 7

$$J_{\mathrm{F}} = \min\big\{\alpha_1 + \alpha_2, \beta_1/p, \beta_2/(1-p)\big\}. \tag{12}$$

– Non-FIFO model Eq. 8

$$J_{\mathrm{NF}} = \min\big\{p(\alpha_1+\alpha_2), \beta_1\big\} + \min\big\{(1-p)(\alpha_1+\alpha_2), \beta_2\big\}. \tag{13}$$

- Optimal signalized models with $\tilde{q}_1, \tilde{q}_2 \in [0,1]^2$ the optimal priority control such that

$$\tilde{q}_k = \arg\max_{q\in[0,1]} J_{\mathrm{C}_k}(q), \qquad k = 1,2,$$

– First signalized model Eq. 10

$$J_{\mathrm{C}_1}(\tilde{q}_1) = \min\big\{p(\alpha_1+\alpha_2), \tilde{q}_1 C, \beta_1\big\} + \min\big\{(1-p)(\alpha_1+\alpha_2), (1-\tilde{q}_1)C, \beta_2\big\}. \tag{14}$$

– Second signalized model Eq. 11

$$\begin{aligned} J_{\mathrm{C}_2}(\tilde{q}_2) = \min\big\{&p\big[\min\{\tilde{q}_2 C, \alpha_1\} + \min\{(1-\tilde{q}_2)C, \alpha_2\}\big], \beta_1\big\} \\ + \min\big\{&(1-p)\big[\min\{\tilde{q}_2 C, \alpha_1\} + \min\{(1-\tilde{q}_2)C, \alpha_2\}\big], \beta_2\big\}. \end{aligned} \tag{15}$$

The existence of the optimal priority control hold thanks to the extreme value theorem, the flow functions J_{C_1} and J_{C_2} being real-value and continuous.

The simulations are carried out with the R software. The optimisation of the priority coefficient in the signalized models Eq. 14 and Eq. 15 is done numerically using the function `optim` with the "L-BFGS-B" method, that is a quasi-Newton approach allowing box constraints [29]. The R code allowing to compute the data is publicly available online at the following Jupyter/Colab project.

4.2 Simulation results

The stochastic approach allows accounting for the average performances and their range of variation. Indeed, reliable intersections should describe "regular" performances (i.e., with small variations) [30, 31]. The mean flow and standard deviation of the Monte Carlo simulations for the four intersection models are given in Tab. 1. We observe that the optimal unsignalized intersection model Eq. 13 overcome the performances of the FIFO model Eq. 12 on average by a factor of approximately 25%, when the regulated models C1 Eq. 14 and C2 Eq. 15 are, respectively, 4 and 7% smaller. The differences between FIFO and non-FIFO models are especially high for congested states (up to 35%), i.e., when the total flow is strictly smaller than the sum of the demands.

Generally speaking, the FIFO model describes poor performances in case of congestion, while the non-FIFO and the two signalized models perform similarly in case of free or congested traffic. The variability of the flow is significantly smaller with the two signalized models compared to the unsignalized models, with a factor up to 25%. This is

Table 1 Mean flow and standard deviation of the Monte Carlo simulations for the four intersection models: FIFO J_{F} Eq. 12, non-FIFO J_{NF} Eq. 13, and the two regulated models J_{C1} Eq. 14 and J_{C2} Eq. 15 in capacity unit C. The traffic state is free if $J_{\mathrm{NF}} = \sum_{i=1}^{n} \alpha_i$ and congested if $J_{\mathrm{NF}} < \sum_{i=1}^{n} \alpha_i$.

	Flow model	J_{F}	J_{NF}	J_{C1}	J_{C2}
	Mean value	0.519	0.648	0.624	0.603
	Standard deviation	0.338	0.313	0.271	0.262
Free state	Mean value	0.656	0.656	0.632	0.632
	Standard deviation	0.314	0.314	0.273	0.273
Congested state	Mean value	0.473	0.646	0.622	0.594
	Standard deviation	0.333	0.312	0.270	0.257

mainly due to the fact that the total flow in the two signalized models Eq. 14 and Eq. 15 are bounded by the capacity C of a single road. Scatter plots for 1e3 flow simulations of the 2×2 intersection are presented in Fig. 5. The histograms of the optimal priority coefficient $\tilde{q}_1$ and $\tilde{q}_2$ for the regulated models Eq. 14 and erefC2n are presented Fig. 6 and 7, respectively. Most of the optimum are reached at equilibrium $\tilde{q} = 0.5$ (respectively 65% of the cases for the model Eq. 14 and 55% for the model Eq. 15). The concentration of the results on $\tilde{q}_1 = 0.5$ or $\tilde{q}_2 = 0.5$ corresponds to congested cases for which $J_{\mathrm{C_1}} = C$ or $J_{\mathrm{C_2}} = C$. As the capacity of the intersection is that of a single road, these cases are over-represented. The distributions around these modal values describe nontrivial shapes for both signalized models (see the subplots in Figs. 6 and 7).

4.3 Empirical properties of the four intersection models observed by simulation

Several empirical properties of the four intersection models can be picked up from the simulation results.

- Regarding FIFO Eq. 12 and non-FIFO Eq. 13 models:

$$J_{\mathrm{F}} \leq J_{\mathrm{NF}} \leq C + J_{\mathrm{F}}/2.$$

- Regarding the two regulated models, Eq. 14 and Eq. 15,

$$J_{\mathrm{C_2}} \leq J_{\mathrm{C_1}} \leq 2J_{\mathrm{C_2}}.$$

- Regarding the unregulated and regulated models:

$$J_{\mathrm{C_1}} = \min\{J_{\mathrm{NF}}, C\}, \quad J_{\mathrm{C_2}} \leq J_{\mathrm{NF}} \leq 2J_{\mathrm{C_2}},$$

  and

$$\min\{J_{\mathrm{F}}, C\} \leq \min\{J_{\mathrm{C_1}}, J_{\mathrm{C_2}}\}.$$

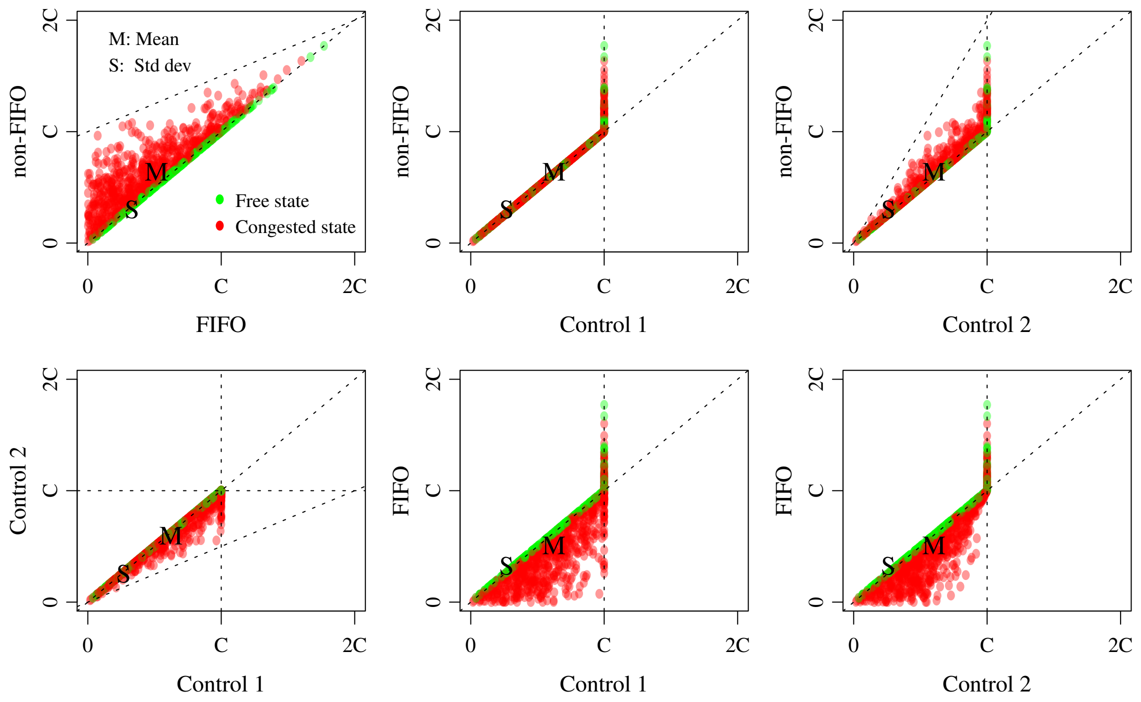

Figure 5 Scatter plots for 1e3 flow simulations for random demand, supply and assignment on a 2×2 intersection with FIFO Eq. 12, non-FIFO Eq. 13, and the two regulated models Control 1 and Control 2 Eqs. 14 and 15, respectively.

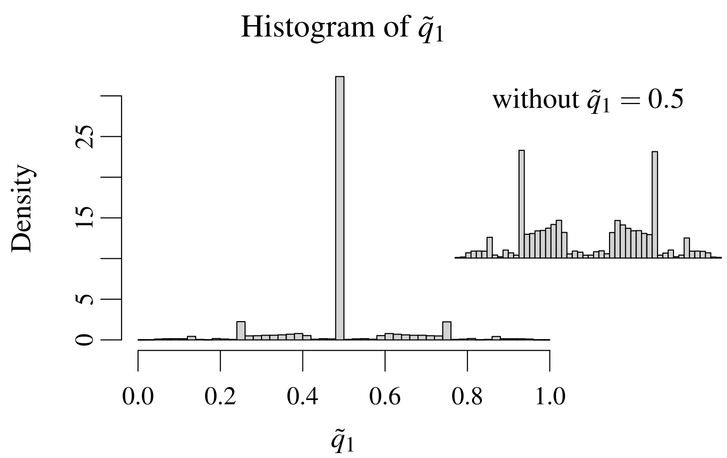

Figure 6 Histogram of the optimal priority coefficient $\tilde{q}_1$ for the first regulated model Eq. 14. The sub-plot top-right presents the distribution without the modal value $\tilde{q}_1 = 0.5$.

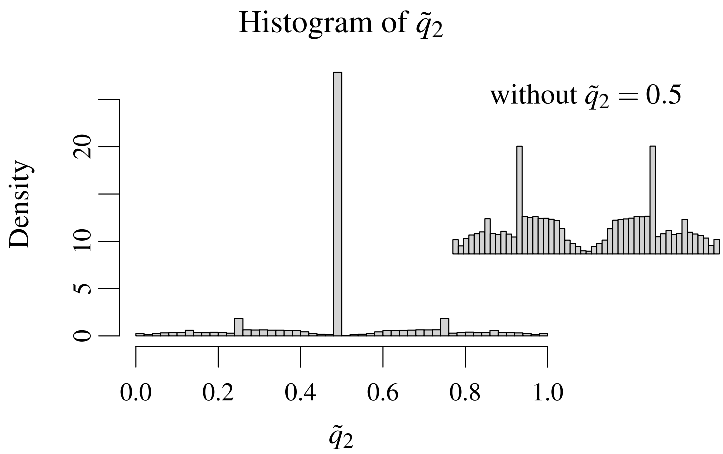

Figure 7 Histogram of the optimal priority coefficient $\tilde{q}_2$ for the second regulated model Eq. 15. The sub-plot top-right presents the distribution without the modal value $\tilde{q}_2 = 0.5$.

5 Conclusion and discussion

The four minimal pointwise intersection models developed in this article cover different types of dynamics, namely FIFO, non-FIFO (optimal) and optimal signalized intersection models. They can be coupled with macroscopic flow models for modelling traffic networks. The intersection models provide boundary conditions for the flow models of the network links, which can be continuous macroscopic models through systems of partial differential equations, like the classical first-order Lighthill-Whitham-Richards models [32, 33], or discrete lattice models such as the cell transmission model [4] and its stochastic variant as in [34].

The main characteristics of the four intersection models are summarised in Tab. 2. The FIFO intersection model is not efficient in case of congestion. The performances of the two optimal signalized models are close to the flow of the idealistic non-FIFO model. They ensure strict priority by bounding the total flow to the capacity of a single road. This makes the optimal signalized approach more relevant regulation strategy than the FIFO regulation. More complex intersections including more than two incoming or outgoing roads could reveal different behaviors and remain to be investigated.

The optimal traffic light systems overcome the performance of a roundabout-type intersection in case of congestion. The main drawbacks of signalized systems arise in free state, where no regulations are required. These results suggest the development of a roundabout with dynamic traffic lights operating exclusively in case of congestion. Such regulation systems remain to be tested and validated using microscopic traffic models.

Table 2 Qualitative evaluation of the four intersection models: FIFO (F) Eq. 7, non-FIFO (NF) Eq. 8, regulation on the outgoing roads (C1) Eq. 10, and regulation on the incoming roads (C2) Eq. 11.

Model	F	NF	C1	C2
Free state	++	++	+	+
Congested state	−	++	++	+
Variability	−	−	+	+

Acknowledgement

The authors acknowledge financial support through Bergisch.Smart.Mobility, a research consortium funded by the ministry for economy, innovation, digitalization and energy (MWIDE) of the state North Rhine Westphalia under the grant-no. DMR-3-2.

Author contributions

Ibrahima Ba performed writing – original draft, formal analysis, review and editing. Antoine Tordeux performed conceptualization, methodology, software, supervision and writing, review and editing.

Data availability

The R code allowing to compute the data is publicly available online at the following Jupyter/Colab project: https://colab.research.google.com/drive/1OfCsdEU4OMrTZR61VP7jq97jmWIHoL3W?usp=sharing

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