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Incomplete Information VCG Contracts
for Common Agency

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We study contract design for welfare maximization in the well-known "common agency" model of Bernheim and Whinston [1986]. This model combines the challenges of coordinating multiple principals with the fundamental challenge of contract design: that principals have *incomplete information* of the agent's choice of action. Our goal is to design contracts that satisfy truthfulness of the principals, welfare maximization by the agent, and two fundamental properties of individual rationality (IR) for the principals and limited liability (LL) for the agent.

Our results reveal an inherent impossibility. While for every common agency setting there exists a truthful and welfare maximizing contract, which we refer to as "incomplete information VCG contracts (IIVCG)", there is no such contract that also satisfies IR and LL for all settings. As our main results, we show that the class of settings for which there exists a contract that satisfies truthfulness, welfare maximization, LL, and IR is identifiable by a polynomial-time algorithm. Furthermore, for these settings, we design a polynomial-time computable contract: given valuation reports from the principals it returns, if possible for the setting, a payment scheme for the agent that constitutes a contract with all desired properties. We also give a sufficient graph-theoretic condition on the population of principals that ensures the existence of such a contract, and two truthful and welfare-maximizing contracts, where one satisfies LL and the other one satisfies IR.

Key words: VCG; Principal; Common Agent; Contract; Equilibrium; Limited Liability; Individual Rationality; Polynomial Complexity.

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1. Introduction

Markets are an ancient tool for allocating goods; but they are also a tool for the allocation of effort. The theory that models the allocation of effort is the celebrated principal-agent model, a.k.a. contract design, originating in the works of Hart, Hölmstrom and others. At the heart of the principal-agent model is a relationship between one entity (agent) who acts on behalf of another (principal); for example, corporate management and shareholders, a freelance worker and employers, the public regulator and citizens. In all of these examples, the agent (manager/freelancer/regulator) acts on behalf of *multiple* principals (shareholders/employers/citizens). The extension of the basic single-agent, single-principal setting to multiple principals mirrors the extension of the basic single-seller, single-buyer *auction* setting to multiple buyers. This extension was introduced in 1986 by Bernheim and Whinston (1986a), and is known as *common agency*.

The main challenge in the principal-agent model (with one or more principals) is the *agency problem*: Actions that generate good outcomes for the principal(s) are costly for the agent, who rationally acts in his own self-interest. To align the interests of the principals with those of the agent, the principals compensate the agent depending on the outcome of his action. This compensation scheme, designed to incentivize the agent to choose a desirable action, is called a *contract*. The agency problem has two necessary ingredients (if one of these is missing, the problem is resolved): (i) *Incomplete information* of the agent's actions; (ii) Principals are better *risk bearers* than the agent. Incomplete information means that the principals cannot directly observe the agent's choice of action, only its resulting (stochastic) outcome. The contractual payment scheme thus depends on the outcome, which serves as a noisy indicator of the action. This crucial feature of the model is known as *hidden action* or *moral hazard*. The second crucial feature is the asymmetry between principals and agents in their attitude towards risk. The principal typically has "deeper pockets" and can bear the risk that the agent's actions will lead to losses. The way this is incorporated into our model follows classic works like Innes (1990), Gollier et al. (1997), Carroll (2015): we focus on contracts guaranteeing *limited liability (LL)* for the agent, i.e., payments in the contractual

relationship flow only from the principals to the agent. We mention that other works like Bernheim and Whinston (1986a) capture the asymmetry in risk attitudes among principal and agent by explicitly modeling the agent's risk averseness. Without LL or risk averseness, the trivial but unrealistic solution of "buying the project" would apply (see, e.g., Carroll 2015).

Having multiple principals raises intriguing new challenges:

> "The principal-agent problem here is intensified: not only is there still asymmetric information between the principals and agent that can bring moral hazard, but there is also *asymmetric information between the principals* themselves [... S]ince *principals' interests often diverge*, they face incentives to advance their individual interests instead of the joint interests by all principals[...] As a result, introducing governance to align the interests of the principals with those of the agent is much more difficult" (Wikipedia contributors 2021).

This misalignment among the principals combined with their private information (their types) requires novel contractual mechanisms.

Research challenge. In this paper we focus on the design of *welfare-maximizing* contracts. We seek for contracts that satisfy the following desiderata. (i) Incentives. This concerns the principals' and agents' willingness to participate, and principals' willingness to provide private information. Since principals' values are privately-known, we require that principals are incentivized to report their values truthfully in dominant strategies. We also require *individual rationality* (IR) so that principals' participation is a dominant strategy, and as explained above, we also require the LL property so the agent faces no risk. We mention that one could also consider relaxing the LL constraint by requiring not that *every* contractual payment is non-negative but rather that only the *aggregate* payment from the principals to the agent is non-negative for every outcome; this does not change our results (see Remark 2). (ii) Social welfare: The payment schemes should incentivize the agent to choose a welfare maximizing action. We leave revenue maximization for future work, noting that the concept of "revenue" in a multi-principal settings may require new definitions. (iii) Computational efficiency: Our contracts should be polynomial-time computable.

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Our results. As our first conceptual contribution, in Section 2 we formulate a game that diverges from classic literature by considering more complex contracts. Specifically, followed by Lavi and Shamash (2019), we instantiate an abstract concept known as *contractible contracts* (see, e.g., Peters and Szentes 2012). Compared to classic contracts, principal ℓ's payment for an outcome is no longer just her own (reported) value from this outcome, but is a function of the other principals' reported values as well. We first prove that even under this general definition of contracts, there exists no contract that satisfies our entire desiderata of truthfulness, welfare maximization, limited liability and individual rationality (see Theorem 1).

At the cornerstone of our proof for the impossibility result, and the positive results that follow, is a new class of contracts which we define, referred to as *Incomplete Information VCG contracts (IIVCG)*. We show that – similar to their namesake, this is the *unique* class of truthful and welfare maximizing contracts. We use this characterization of truthful and welfare maximizing contracts to prove Theorem 1. Specifically, we show that there exists no IIVCG contract that satisfies both LL and IR simultaneously.

We mention that while the definition of IIVCG contracts shares important features with the celebrated VCG mechanism, it also departs from it in a significant way: In the classic VCG model there is a social planner that takes the welfare maximizing choice (with respect to the participants' reported valuations). Here, the social planner is replaced with the strategic agent who acts in his own self-interest. Therefore, our contract needs to *incentivize the agent* to take the welfare maximizing choice, on top of the need to incentivize the principals to report truthfully.

The definition of IIVCG contracts is also useful since it exactly defines our "design space". In particular, it translates to linear constraints that allow to overcome the impossibility result using the following algorithmic construction, which is our main contribution: We give a polynomial-time algorithm (Algorithm 2) that determines for a given common agency setting whether or not there is an IIVCG contract for this setting that is both LL and IR. Such settings are referred to as *applicable* settings. If such a contract exists, we design another polynomial-time algorithm (Algorithm 1)

that implements it, i.e., it computes "on the fly" contractual payments for every principal. The algorithmic basis of this approach gives us flexibility that may not exist when trying to design a contract with a closed-form formula for payments.

In the appendix we also give two instantiations of the IIVCG family called *Auction-Inspired IIVCG* and *Weighted IIVCG*. The former satisfies IR and the latter satisfies LL. For Weighted IIVCG, we also characterize the class of settings for which they are also IR – namely, settings that satisfy a graph-theoretic condition on the graph whose nodes are the principals. This completes the picture, showing the full range within IIVCG: IR and not LL (Auction-Inspired IIVCG), LL and not IR (Weighted IIVCG), and both LL and IR whenever possible (i.e., for applicable settings). One avenue we do not explore in this paper but leave as an open question is contracts that guarantee LL and IR always, but only approximately achieve the optimal social welfare.

1.1. Additional Related Work

As mentioned above, our solution concept arises from recent work of Lavi and Shamash (2019). Their work studies a model with multiple agents and *complete* information about the agents' choice of action, and thus no agency problem. To demonstrate how different the complete and incomplete information models are note that with full information, the principal can simply tell the agent to work and pay him his cost, thus maximizing welfare and extracting it fully as her expected utility. Example 1 shows that this is no longer possible when there is incomplete information. Further evidence of the difference between the two models is the state of the literature in each. For complete information, the problem of welfare maximization with multiple principals and a single agent is largely settled by Bernheim and Whinston (1986b) in a paper preceding their "Common Agency" paper (Bernheim and Whinston 1986a). Lavi and Shamash (2019) address a generalization of this problem to multiple agents (the generalization is due to Prat and Rustichini (2003)). They instantiate an abstract concept known as *contractible contracts* (Peters and Szentes 2012), and introduce a novel class of *VCG contracts*. In contrast, in the incomplete information world, the problem of welfare maximization with multiple principals has remained largely open even for a

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single agent in the three decades since the common agency model was introduced by Bernheim and Whinston (1986a). Several works have considered the design of contracts with incomplete information from a computational point of view (see, e.g., Babaioff et al. 2006, Ho et al. 2016, Kleinberg and Kleinberg 2018, Kleinberg and Raghavan 2019, Dütting et al. 2020, Dütting et al. 2019); some of these works have multiple agents (e.g., Xiao et al. (2020), Alon et al. (2020)), or agent private types (e.g., Guruganesh et al. 2020), but none to our knowledge consider multiple principals with private types.

2. Model and Preliminaries

Common agency settings. We adopt the classic model of Bernheim and Whinston (1986a), in which there is a single *agent*, and n *principals*. The agent chooses an unobservable *action* that incurs a *cost* on the agent, and stochastically leads to an observable *outcome* that benefits the principals. This is formalized by a *common agency setting* $(\mathcal{A}, \psi, \mathcal{O}, \mathcal{V}, F)$ as we now describe: $\mathcal{A}$ is a set of q actions available to the agent. $\psi : \mathcal{A} \to \mathbb{R}_{\geq 0}$ maps an agent's action to its cost. We assume for simplicity that all actions have different costs. $\mathcal{O}$ is a set of m outcomes. For ease of notions, we assume outcomes are indexed, so every $o \in \mathcal{O}$ is associated with a number in $[m]$. F is a set of distributions, $F_{|a} \in \mathbb{R}_{\geq 0}^m \ \forall a \in \mathcal{A}$, such that the oth entry of $F_{|a}$ is the probability for the oth outcome given that the agent takes action a denoted by $p_{o \sim F_{|a}}(o)$. $\mathcal{V}$ is the tuple of principal's *valuations domains* $(\mathcal{V}^1, ..., \mathcal{V}^n)$ from which their valuations are drawn: A valuation *profile* $v \in \mathcal{V}$ is the tuple $(v^1, \ldots, v^n)$, where $v^\ell \in \mathcal{V}^\ell$ is the *valuation* function $v^\ell : \mathcal{O} \to \mathbb{R}_{\geq 0}$ of principal $\ell \in [n]$, which maps an outcome to a non-negative value. Note that the mapping v^ℓ can also be thought of as a vector in $\mathbb{R}_{\geq 0}^m$ (valuations in our model are thus succinct objects). For simplicity, we assume $\mathcal{V}^\ell$ is a convex, polynomially-described polytope; all our results extend to general convex domains. In the general case, the convex domains are represented by polynomial-time membership oracles. Convex valuation domains appear, e.g., as the main application in (Holmström 1979). Throughout, principals are indexed by ℓ, actions by j and outcomes by i. Towards the computational analysis, we assume that the representation size of $(\mathcal{A}, \psi, \mathcal{O}, \mathcal{V}, F)$ is polynomial in n, q and m.

Who knows what. The setting $(\mathcal{A}, \psi, \mathcal{O}, \mathcal{V}, F)$ is assumed to be public knowledge. The realized outcome i is publicly known, while the agent's chosen action j and the principals' valuations v^ℓ $\forall \ell \in [n]$ are privately known. This makes our model a *hidden action, hidden type* model (where the type is of the principals).

Contracts. A contract profile $t = (t^1, ..., t^n)$ (*contract* for short) includes a payment rule t^ℓ for each principal ℓ. We diverge from the classic model in that the payment rule t^ℓ depends not only on the reported values of principal ℓ (and the outcome of the agent's action), but also on the (reported) valuations of all principals: $t^\ell : \mathcal{V} \times \mathcal{O} \to \mathbb{R}$. Such contracts are known as *contractible*. Taken together, the n (publicly known) payment rules t^ℓ $\forall \ell \in [n]$ incentivize the agent when choosing an action.

The game. Fix a common agency setting. A contract t and valuation profile v define the following two-stage game. First, principals simultaneously report their values. We denote principal ℓ's report (a.k.a. *bid*) by $b^\ell \in \mathcal{V}^\ell$ (the report need not be truthful, i.e., b^ℓ is not necessarily v^ℓ). After perceiving the bids, the agent chooses a hidden action a, which imposes a distribution $F_{|a}$ over the outcomes. An outcome o is realized according to this distribution, and each principal ℓ pays the agent $t^\ell(b, o)$, where $b = (b^1, ..., b^n)$.

Utilities. We assume quasi-linear utilities and risk neutrality for all parties. The utility of principal ℓ's is her value minus payment, i.e., $v^\ell(o) - t^\ell(b, o)$. The agent's utility is the sum of payments minus the cost of chosen action, i.e., $\sum_{\ell \in [n]} t^\ell(b, o) - \psi(a)$.

Equilibrium. The agent's chosen action $x^*(b)$ given bid profile b maximizes his expected utility:

$$x^*(b) \in \arg\max_{a \in \mathcal{A}} \left\{ \sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a}} [t^\ell(b, o)] - \psi(a) \right\}.$$

As is standard, we assume the agent tie-breaks in favor of the design objective (for an example in the context of revenue see (Carroll 2015)). In our case the objective is the (declared) social welfare. If among the utility-maximizing actions there are several with highest (declared) welfare, we assume the agent chooses to invest maximum effort, i.e., takes the one with the maximum cost. (While we feel this kind of "eager" tie-breaking is in line with the standard tie-breaking assumption,

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our results do not rely on it and would hold for any consistent tie-breaking rule.) We assume the principals know the agent is rational and can anticipate his action $x^*(b)$ given bid profile b. Given the bids of all principals but ℓ, denoted by $b^{-\ell}$, principal ℓ's equilibrium bid b^ℓ maximizes her expected utility $\mathbb{E}_{o \sim F_{|x^*(b)}}[v^\ell(o) - t^\ell(b, o)]$ among all possible bids in $\mathcal{V}^\ell$, where $b = (b^{-\ell}, b^\ell)$ and $x^*(b)$ is the agent's anticipated action.

Consider now the game defined by contract t when the valuation profile v is not fixed. A *strategy* of principal ℓ in this game is a mapping from every possible valuation v^ℓ to a bid b^ℓ. A strategy is called *dominant* if it is guaranteed to maximize the principal's expected utility for every v^ℓ no matter what the other principals bid, i.e., for every $b^{-\ell} \in \mathcal{V}^{-\ell}$ (while still assuming the rationality of the agent). Principal ℓ is *truthful* as a dominant strategy if announcing her true valuation maximizes her expected utility, i.e., $v^\ell \in \arg\max_{b^\ell \in \mathcal{V}^\ell} \mathbb{E}_{o \sim F_{|x^*(b)}}[v^\ell(o) - t^\ell(b, o)]$ for every $v^\ell \in \mathcal{V}^\ell, b^{-\ell} \in \mathcal{V}^{-\ell}$. We say that contract t is *truthful* if every principal is truthful as a dominant strategy.

Design objectives. As mentioned above, our desiderata from a contract t concern (i) incentives, (ii) social welfare, and (iii) computational complexity. The first group of design objectives address incentives – principals' and agent's willingness to participate, and principals' willingness to provide private information. For the latter, we focus on truthful contracts, in which every principal's dominant strategy is to simply report her valuation. To ensure principals' participation in such contracts, we require that every truthful principal has non-negative expected utility no matter how the others bid – this is known as the *individual rationality (IR)* property for principals. Formally,

DEFINITION 1. A contract t satisfies *individual rationality (IR)* if the utility of every truthful principal is non-negative: $\mathbb{E}_{o \sim F_{|x^*(b^{-\ell}, v^\ell)}}[v^\ell(o) - t^\ell((b^{-\ell}, v^\ell), o)] \geq 0$ for every $\ell \in [n], b^{-\ell} \in \mathcal{V}^{-\ell}, v^\ell \in \mathcal{V}^\ell$.

We now turn to the agent's incentives, for which we need the following central definition:

DEFINITION 2. A contract t satisfies *limited liability (LL)* if every payment from a principal to the agent is non-negative: $t^\ell(b, o) \geq 0$ for every $\ell \in [n], b \in \mathcal{V}, o \in \mathcal{O}$.

The LL requirement guarantees that the agent never pays out-of-pocket. Under LL, to ensure the agent is guaranteed non-negative utility, we assume there exists an action with zero cost. We

mention that this assumption is for simplicity; the alternative of requiring IR for the agent would

not change our results.

The second design objective is social welfare. Given a valuation profile v, the (expected) *welfare*

of action a, denoted by $\text{Wel}^a(v)$, is the total expected value minus cost, and $a^*(v)$ is a welfare-

maximizing action:

$$\text{Wel}^a(v) = \sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)] - \psi(a); \quad a^*(v) \in \arg\max_{a \in \mathcal{A}} \text{Wel}^a(v).$$

We assume tie-breakings are consistent and aligned. Thus, as mentioned above, if there are several

welfare maximizing actions, $a^*(v)$ is the one with maximum effort.

We seek for *welfare maximizing* contracts such that for every bid profile b, the agent's action

choice $x^*(b)$ maximizes the declared welfare, i.e., $x^*(b) = a^*(b)$. Since we focus on truthful contracts,

when playing according to their dominant strategies the principals report $b = v$, and the agent's

choice $a^*(v)$ is indeed socially efficient.

The last design objective is computational complexity: given a bid profile b and an outcome o,

the payments $t^\ell(b,o)$ should be computable in polynomial time for every $\ell \in [n]$. To summarize

our desiderata, we seek truthful, welfare maximizing, IR and LL contracts with payments that are

computable in polynomial time.

3. Impossibility

In this section we present our impossibility result. We show here that while there exists truthful

and welfare maximizing contract for all common agency settings (see Section 3.1), there exists no

such contract that satisfy both LL and IR:

THEOREM 1. *For any number of principals $n > 0$, there exists a common agency setting for which*

no contract satisfies truthfulness, welfare maximization, limited liability and individual rationality.

The proof, and the positive results in the next sections rely on the following characterization

truthful and welfare-maximizing contracts (see Proposition 2).

3.1. A Characterization of Truthful and Welfare Maximizing Contracts

Here, we characterize the family of truthful and welfare maximizing contracts. Specifically, we define the family of *Incomplete Information VCG (IIVCG)* contracts (see Definition 3), and show that a contract is truthful and welfare maximizing if and only if it is an IIVCG contract (see Proposition 2). Furthermore, we show here that for every common agency setting there exists an IIVCG contract (see Proposition 1).

Recall that compared to the classic VCG mechanism where there is a social planner, here, the agent needs to be *incentivized* by the contract to take the welfare maximizing choice. To deduce the exact payments that incentivize the agent to maximize welfare, we will make use of the following definition.

$$\mathcal{L}_a = \left\{ w \in \mathbb{R}^m_{\geq 0} \ \Big| \ a \in \arg\max_{a'}\{\mathbb{E}_{o\sim F_{|a'}}[w(o)] - \psi(a')\} \right\} \quad a \in \mathcal{A},$$

where $w(o)$ for $o \in [m]$ is the oth entry of vector w. $\mathcal{L}_a$ can be interpreted as follows: It is the set of all non-negative payment vectors $w \in \mathbb{R}^m_{\geq 0}$ such that action a maximizes the agent's expected utility given w. Indeed, if the principals were cooperating to incentivize the agent to take an action, e.g., $a^*(b)$, they could have jointly pay the agent $w(o)$ for every outcome o, for some $w \in \mathcal{L}_{a^*(b)}$.

We use the definition of $\mathcal{L}_{a^*(b)}$ to define the following class of contracts which is (uniquely) truthful and welfare-maximizing.

DEFINITION 3 (IIVCG CONTRACTS). A contract $t = (t^1, ..., t^n)$ is called *Incomplete Information VCG (IIVCG)* if there exist functions $h^1, \ldots, h^n$ where $h^\ell : \mathcal{V}^{-\ell} \to \mathbb{R}$, and $w^1, ..., w^n \in \mathbb{R}^m_{\geq 0}$ where $\sum_{\ell \in [n]} w^\ell \in \mathcal{L}_{a^*(b)}$ such that

$$t^\ell(b, o) = h^\ell(b^{-\ell}) - \text{Wel}^{a^*(b)}(b^{-\ell}, w^\ell) + w^\ell(o) \quad \forall \ell \in [n], o \in \mathcal{O}. \tag{1}$$

PROPOSITION 1 **(Existence)**. *For every common agency setting, there exists an IIVCG contract.*

The proof is deferred to Appendix B where we give two different instantiations of the IIVCG family of contracts for all settings. One of the instantiations satisfies LL, while the other one satisfies IR.

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PROPOSITION 2 (**Characterization**). *A contract is truthful and welfare maximizing if and only if it is an IIVCG contract.*

We prove here the backward direction – given an IIVCG contract t, we show it is truthful and welfare maximizing. The forward direction is deferred to Appendix A.

Proof of Proposition 2 (backward direction). We first show that the agent is incentivized to maximize social welfare, i.e., that $x^*(b) = a^*(b) \ \forall b \in \mathcal{V}$. Consider a bid profile b, and the corresponding welfare maximizing action $a^*(b)$. Recall that $x^*(b) \in \arg\max_{a' \in \mathcal{A}} \{ \sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a'}}[t^\ell(b, o)] - \psi(a') \}$ by definition. Since the only term in $t^\ell(b, o)$ that depends on the agent's choice of action is $w^\ell(o)$, and since summing up this term over the principals gives vector $\sum_{\ell \in [n]} w^\ell \in \mathcal{L}_{a^*(b)}$, it follows that $x^*(b) = a^*(b)$.

We prove that t satisfies dominant strategy truthfulness. We show that reporting truthfully maximizes principal ℓ's expected utility for every principal ℓ and other principals' bids $b^{-\ell}$. Observe that since $x^*(b) = a^*(b) \ \forall b \in \mathcal{V}$, the principal's expected utility for reporting b^ℓ is equal to $\mathbb{E}_{o \sim F_{|a^*(b)}}[v^\ell(o)] - h^\ell(b^{-\ell}) + \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0})$, which, by summing up the first and the last term is equal to $\text{Wel}^{a^*(b)}(b^{-\ell}, v^\ell) - h^\ell(b^{-\ell})$. Since h^ℓ is independent of b^ℓ, this is maximized when the first term is maximized. That is, by the definitions of $\text{Wel}^a(\cdot)$ and $a^*(\cdot)$, when the principal reports truthfully, i.e., when $b^\ell = v^\ell$.

3.2. Proof of Theorem 1

In this section we leverage the characterization of truthful welfare maximizing contracts from the previous section to prove the impossibility in Theorem 1.

Let $n > 0$ be the number of principals, Example 1 presents a common agency setting with n principals for which there exists no truthful, welfare maximizing that is both LL and IR.

EXAMPLE 1. There are two actions a_1, a_2 and two possible outcomes o_1, o_2. The distributions imposed by actions are: $p_{o \sim F_{|a_1}}(o_1) = \frac{1}{2}$, $p_{o \sim F_{|a_2}}(o_1) = 1$. There are n principals with valuation domains $\mathcal{V}^\ell = \mathbb{R}^2_{\geq 0} \ \forall \ell \in [n]$. The cost for each action is: $\psi(a_1) = 0$, $\psi(a_2) = \epsilon$, where $\epsilon > 0$.

Consider the common agency setting in Example 1 and the valuation profile $v^1 = (3\epsilon, 0)$ and $v^\ell = (0,0) \ \forall \ell \in [n] \setminus \{1\}$. Note that $\text{Wel}^{a_1}(v) = 1.5\epsilon$, and $\text{Wel}^{a_2}(v) = 2\epsilon$. Thus, the socially efficient action is a_2. Suppose toward a contradiction that there exists a truthful welfare maximizing contract t that satisfies LL and IR. Since t is an IIVCG contract (see Proposition 2), in a dominant strategy equilibrium the principal is truthful, and the agent takes the socially efficient action, i.e., a_2. Since the agent maximizes the sum of payments minus cost, it must hold that $\sum_{\ell \in [n]} t^\ell(b, o_1) - \epsilon \geq \sum_{o \in \mathcal{O}} \frac{1}{2} \cdot \sum_{\ell \in [n]} t^\ell(b, o)$. Thus, $\sum_{\ell \in [n]} t^\ell(b, o_1) - t^\ell(b, o_2) \geq 2\epsilon$. Further, since t satisfies IR and LL $t^\ell(b, o) = 0$ $\forall \ell \in [n] \setminus \{1\}, o \in \mathcal{O}$ in a dominant strategy equilibrium. Therefore, $t^1(b, o_1) - t^1(b, o_2) \geq 2\epsilon$. By the assumption that t satisfies LL we have that $t^1(b, o_2) \geq 0$, so $t^1(b, o_1) \geq 2\epsilon$. That is, $t^1(b, o_1) = \mathbb{E}_{o \sim F_{|a_2}}[t^1(b, o)] = h^1(b^{-1}) - \text{Wel}^{a_2}(b^{-1}, \mathbf{0}) \geq 2\epsilon$. Since $\text{Wel}^{a_2}(b^{-1}, \mathbf{0}) = -\epsilon$, we conclude that $h^1(b^{-1}) \geq \epsilon$. Now, consider a different valuation profile $\tilde{v}^\ell = (0,0) \ \forall \ell \in [n]$. Again, in a dominant strategy equilibrium the principal is truthful $\tilde{b} = \tilde{v}$ and the agent takes the socially efficient action, which is now a_1. Since t satisfies IR, it holds that $\mathbb{E}_{o \sim F_{|a_2}}[\tilde{v}^1(o)] \geq \mathbb{E}_{o \sim F_{|a_1}}[t^1(\tilde{b}, o)]$. That is, $0 \geq h^1(\tilde{b}^{-1}) - \text{Wel}^{a_1}(\tilde{b}^{-1}, \mathbf{0})$, and since $\text{Wel}^{a_1}(\tilde{b}^{-1}, \mathbf{0}) = 0$, we have that $0 \geq h^1(\tilde{b}^{-1})$. Note that by definition of IIVCG, h^1 is independent of b^1. Thus, $h^1(\tilde{b}^{-1}) = h^1(b^{-1})$, which implies that $0 \geq h^1(\tilde{b}^{-1}) \geq \epsilon$. We have reached a contradiction.

REMARK 1. With two or more principals, Theorem 1 is related to the classic result of Myerson and Satterthwaite, by which there do not exist socially-efficient and budget-balanced mechanisms (see, e.g., Myerson and Satterthwaite (1983)). Indeed, the agent in our setting can be viewed as the mechanism designer in the classic result, and the principals in our setting can be viewed as the players. However, the impossibility result of Myerson and Satterthwaite crucially relies on the existence of two players, whereas Theorem 1 applies even to settings with a single principal.

4. Applicable Settings

The impossibility result (Theorem 1) motivates the following definition.

DEFINITION 4. A common agency setting is *applicable* if there exists a truthful, welfare maximizing contract that satisfies LL and IR for the setting.

Here, we characterize the non-empty set of applicable settings (see Lemma 1 and Observation 1), and provide a polynomial time algorithm to determine whether or not a setting is applicable (see Theorem 3). Our characterization is constructive, yielding an IIVCG contract that satisfies both LL and IR whenever such a contract exists. The contract we provide is conceptually different from classic contract mapping – expressed using a mathematical formula. We introduce an algorithmic approach, in which the contract mapping is algorithm-based. Specifically, we design a polynomial time algorithm that upon receiving reports $b = (b_1, ..., b_n)$ and an outcome o, it computes $t^\ell(b, o)$ for every principal ℓ in order to determine the contractual payments. Altogether, the following two theorems constitute our main results:

THEOREM 2. *For every applicable setting there is a truthful welfare maximizing contract t that satisfies LL and IR and that its payment profile $t(b, o)$ is computable by a polynomial time algorithm for every bid profile b and every outcome o.*

THEOREM 3. *There is a polynomial time algorithm that determines whether or not a given common agency setting is applicable.*

The proofs of both theorems are given in Sections 4.1 and 4.2, respectively.

4.1. Proof of Theorem 2

Here we prove Theorem 2. We will present Algorithm 1 and show that the payments it computes form a truthful welfare maximizing contract that satisfies LL and IR for every applicable setting. The proof relies on Definition 5, Lemma 1 and Lemma 2.

DEFINITION 5. Consider a common agency setting. Define:

1. $m^\ell(b) = \min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \mathrm{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) - \mathrm{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \ \forall \ell \in [n], b \in \mathcal{V}$.

2. $k_a = \min_{w \in \mathcal{L}_a} \mathbb{E}_{o \sim F_{|a}}[w(o)] \ \forall a$.

LEMMA 1. *Consider a common agency setting. If the setting is applicable, $k_{a^*(b)} \leq \sum_{\ell \in [n]} m^\ell(b)$ $\forall b \in \mathcal{V}$. Conversely, if $k_{a^*(b)} \leq \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}$, an IIVCG contract that satisfies LL and IR may be constructed by using Definition 3 and*

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1. $h^\ell(b^{-\ell}) = \min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) \ \forall \ell \in [n], b^{-\ell} \in \mathcal{V}^{-\ell}$.

2. *For every* $b \in \mathcal{V}$, *take* $w^b \in \arg\min_{w \in \mathcal{L}_{a^*(b)}} \mathbb{E}_{o \sim F_{|a^*(b)}}[w(o)]$ *and choose* $w^1, ..., w^n$ *as follows:*
$w^\ell = x_\ell^b \cdot w^b$ *where* $0 \le x_\ell^b$, $x_\ell^b \cdot k_{a^*(b)} \le m^\ell(b) \ \forall \ell \in [n]$, *and* $\sum_{\ell \in [n]} x_\ell^b = 1$.

Proof of Lemma 1. For the forward direction, we will show that $k_{a^*(b)} \le \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}$ for every applicable setting. Consider an applicable setting and a bid profile $b \in \mathcal{V}$. By Definition 4, there exists an IIVCG contract t that satisfies LL and IR. Define $w \in \mathbb{R}^m$ as $w(o) = \sum_{\ell \in [n]} t^\ell(b, o)$ $\forall o \in [m]$. Since t satisfies limited liability, w is non-negative, i.e., $w \in \mathbb{R}_{\ge 0}^m$. Furthermore, by Proposition 2, the agent maximizes social welfare. Thus, $a^*(b) \in \arg\max_{a'} \{\mathbb{E}_{o \sim F_{|a'}}[w(o)] - \psi(a')\}$. This implies that $w \in \mathcal{L}_{a^*(b)}$ by definition of $\mathcal{L}_a \ \forall a \in \mathcal{A}$. Therefore, by the definition of $k_{a^*(b)}$, we have that

$$k_{a^*(b)} \le \sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)]. \tag{2}$$

To complete the forward direction, we show that $\sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] \le \sum_{\ell \in [n]} m^\ell(b)$. We do so by showing that an IIVCG contract t satisfies IR if and only if $\mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] \le m^\ell(b) \ \forall \ell \in [n], b \in \mathcal{V}$. By definition, t satisfies IR if and only if each principal's expected payment when she bids truthfully is at most her expected value. That is, $\mathbb{E}_{o \sim F_{|a^*(b^{-\ell}, v^\ell)}}[t^\ell((b^{-\ell}, v^\ell), o)] \le \mathbb{E}_{o \sim F_{|a^*(b^{-\ell}, v^\ell)}}[v^\ell(o)]$ $\forall \ell \in [n], b \in \mathcal{V}, v^\ell \in \mathcal{V}^\ell$. By definition of IIVCG (Definition 3), the above holds if and only if $h^\ell(b^{-\ell}) - \text{Wel}^{a^*(b^{-\ell}, v^\ell)}(b^{-\ell}, \mathbf{0}) \le \mathbb{E}_{o \sim F_{|a^*(b^{-\ell}, v^\ell)}}[v^\ell(o)] \ \forall \ell \in [n], b \in \mathcal{V}, v^\ell \in \mathcal{V}^\ell$. By adding $\text{Wel}^{a^*(b^{-\ell}, v^\ell)}(b^{-\ell}, \mathbf{0}) - \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0})$ to both sides of the inequality, we have that t satisfies IR if and only if $h^\ell(b^{-\ell}) - \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \le \text{Wel}^{a^*(b^{-\ell}, v^\ell)}(b^{-\ell}, v^\ell) - \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \ \forall \ell \in [n], b \in \mathcal{V}, v^\ell \in \mathcal{V}^\ell$. Since the left-hand side is independent of v^ℓ, the above is equivalent to the following condition (obtained by minimizing the right-hand side over all $v^\ell \in \mathcal{V}^\ell$ and dropping the $\forall v^\ell \in \mathcal{V}^\ell$ quantifier): $h^\ell(b^{-\ell}) - \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \le \min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) - \text{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \ \forall \ell \in [n], b \in \mathcal{V}$. That is, by definition of IIVCG, and by the definition of $m^\ell(b)$, if and only if $\mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] \le m^\ell(b) \ \forall \ell \in [n], b \in \mathcal{V}$. By summing up the above over all principals $\ell \in [n]$, and using (2) we conclude that if an IIVCG contract t satisfies LL and IR then $k_{a^*(b)} \le \sum_{\ell \in [n]} \mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] \le \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}$.

For the backwards direction, consider a setting that satisfies $k_{a^*(b)} \leq \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}$, we show that the construction in Lemma 1 forms an IIVCG contract that satisfies LL and IR. We first show that this contract is in IIVCG. This is by definition of IIVCG since $\sum_{\ell \in [n]} w^\ell = w^b \in \mathcal{L}_{a^*(b)} \ \forall b \in \mathcal{V}$, and since h^ℓ is indeed independent of b^ℓ. To show LL, we must show that

$$t^\ell(b, o) = h^\ell(b^{-\ell}) - \mathrm{Wel}^{a^*(b)}(b^{-\ell}, w^\ell) + w^\ell(o) \geq 0 \ \forall \ell \in [n], b \in \mathcal{V}, o \in \mathcal{O}.$$ Since $w^\ell(o)$ is non-negative, the last inequality holds if $h^\ell(b^{-\ell}) - \mathrm{Wel}^{a^*(b)}(b^{-\ell}, w^\ell) \geq 0 \ \forall \ell \in [n], b \in \mathcal{V}$. By adding $\mathbb{E}_{o \sim F_{|a^*(b)}}[w^\ell(o)]$ to both sides of the inequality, the above inequality is equal to $h^\ell(b^{-\ell}) - \mathrm{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) \geq \mathbb{E}_{o \sim F_{|a^*(b)}}[w^\ell(o)] \ \forall \ell \in [n], b \in \mathcal{V}$. By definition of $m^\ell(b)$, $k_{a^*(b)}$, and by our choice of $h^\ell(b^{-\ell}) = \min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \mathrm{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell)$ and of $w^\ell = x_\ell^b \cdot \arg\min_{w \in \mathcal{L}_{a^*(b)}} \mathbb{E}_{o \sim F_{|a^*(b)}}[w(o)]$, this is equivalent to $m^\ell(b) \geq x_\ell^b \cdot k_{a^*(b)} \ \forall \ell \in [n], b \in \mathcal{V}$, which holds by construction. LL is therefore satisfied. IR follows from the arguments in the proof of the forward direction, where we show that an IIVCG contract t satisfies IR if and only if $\mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] \leq m^\ell(b) \ \forall \ell \in [n], b \in \mathcal{V}$. By the choice of $h^\ell(b^{-\ell})$, the letter holds with equality since $\mathbb{E}_{o \sim F_{|a^*(b)}}[t^\ell(b, o)] = \min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \mathrm{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) - \mathrm{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0})$.

OBSERVATION 1. *The set of applicable settings in non empty. For example, it can be verified that the setting in Example 1 where the principals' valuation profile is $\mathcal{V}^\ell = \mathbb{R}^2_{\geq 2\epsilon} \ \forall \ell \in [n]$ is applicable.*

REMARK 2. One may consider relaxing the LL constraint, by requiring that only the *aggregate* payment from all principals to the agent is non-negative. The proof of Lemma 1 shows that requiring the payments from each individual principal to the agent to be non-negative is without loss of generality.

LEMMA 2. *Consider an applicable common agency setting. Algorithm 1 computes the payments in Lemma 1 in polynomial time for every bid profile $b \in \mathcal{V}$ and every outcome $o \in \mathcal{O}$.*

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ALGORITHM 1: Computes the payments of LL and IR IIVCG contract.

Input: An applicable setting $(\mathcal{A}, \mathcal{O}, \mathcal{V}, F, \psi)$, a bid profile $b \in \mathcal{V}$, and an outcome $o \in \mathcal{O}$.

Output: The payments $\{t^\ell(b, o)\}_{\ell \in [n]}$ of an IIVCG contract that satisfies LL and IR.

`// Compute` $m^\ell(b) \ \forall \ell \in [n]$ `(see Definition 5)`

1 for $\ell \in [n]$ **do**

2 $\quad \underset{\tilde{v}^\ell \in \mathcal{V}^\ell, h^\ell \in \mathbb{R}}{\text{minimize}} \ h^\ell$

$\qquad$ subject to $F_{|a_j} \cdot (b^{-\ell} + \tilde{v}^\ell) - \psi(a_j) \leq h^\ell \ \forall j \in [q]$, where $b^{-\ell}(o) = \sum_{\ell \neq \ell' \in [n]} b^{\ell'}(o) \ \forall o \in \mathcal{O}$;

3 $\quad m^\ell \leftarrow h^\ell - \{F_{|a^*(b)} \cdot b^{-\ell} - \psi(a^*(b))\}$;

`// Compute` $k_{a^*(b)}$ `(see Definition 5 below)`

4 $\underset{w \in \mathbb{R}^m}{\text{minimize}} \ F_{a^*(b)} \cdot w$

$\quad$ subject to $F_{|a_j} \cdot w - \psi(a_j) \leq F_{|a^*(b)} \cdot w - \psi(a^*(b)) \ \forall j \in [q]$,

$\qquad\qquad w_i \geq 0 \ \forall i \in [m]$;

5 $k_{a^*(b)} \leftarrow F_{|a^*(b)} \cdot w$

`// Compute contractual payments using Lemma 2 and Definition 3`

6 $\underset{x^b \in \mathbb{R}^n}{\text{maximize}} \ \sum_{\ell \in [n]} x_\ell^b$

$\quad$ subject to $1 \geq \sum_{\ell \in [n]} x_\ell^b$

$\qquad\qquad x_\ell^b \cdot k_{a^*(b)} \leq m^\ell \ \forall \ell \in [n]$

$\qquad\qquad 0 \leq x_\ell^b \ \forall \ell \in [n]$;

7 for $\ell \in [n]$ **do**

8 $\quad t^\ell(b, o) \leftarrow h^\ell - \{F_{|a^*(b)} \cdot (b^{-\ell} + x_\ell^b \cdot w^b) - \psi(a^*(b))\} + x_\ell^b \cdot w^b(o)$;

return $\{t^\ell(b, o)\}_{\ell \in [n]}$;

Proof of Lemma 2. To compute the construction in Lemma 1 for a given bid profile $b \in \mathcal{V}$, we first compute $m^\ell(b) \ \forall \ell \in [n]$. The first loop computes this term for every ℓ by computing $\min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell)$ using the linear program in line 2 as we shortly explain and subtracting $\text{Wel}^{a^*(b)}(b^{-\ell}, 0) = F_{|a^*(b)} \cdot b^{-\ell} - \psi(a^*(b))$ from its solution. As for the linear program in line 2, the jth constraint ensure that the welfare from action a_j given $(b^{-\ell}, \tilde{v}^\ell)$, i.e., $\text{Wel}^{a_j}(b^{-\ell}, \tilde{v}^\ell) = F_{|a_j} \cdot (b^{-\ell} +$

$\tilde{v}^\ell) - \psi(a_j)$ is at most h^ℓ. Thus, the optimal welfare given $(b^{-\ell}, \tilde{v}^\ell)$, i.e., $\text{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) \leq h^\ell$ at

the feasible region by. Minimizing h^ℓ over all $\tilde{v}^\ell \in \mathcal{V}^\ell$ gives $\min_{\tilde{v}^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell)$, which is

the first term in $m^\ell(b)$. The linear program is solvable in polynomial time since it consists of $O(m)$

variables: $0 \leq h^\ell, \tilde{v}^\ell \in \mathcal{V}^\ell \subseteq \mathbb{R}^m$; and polynomially-many constraints: $O(q)$ constraints in addition to

the polynomially-many constraints ensuring that $v \in \mathcal{V}$ (recall that $\mathcal{V}^\ell$ is a convex, polynomially-

described polytope for every ℓ). Next, we compute $k_{a^*(b)}$ by solving the linear program in line

4. The first set of constraints ensures that $a^*(b) \in \arg\max_{a'}\{\mathbb{E}_{o \sim F_{|a'}}[w(o)] - \psi(a')\}$. The second

set of constrains ensures that $w \in \mathbb{R}^m_{\geq 0}$. Altogether, $w \in \mathcal{L}_{a^*(b)}$ at the feasible region. Minimiz-

ing $F_{a^*(b)} \cdot w = \mathbb{E}_{o \sim F_{|a^*(b)}}[w(o)]$ computes $k_{a^*(b)}$. This linear program is also solvable in polynomial

time since it consists of $O(m)$ variables: $w \in \mathbb{R}^m$, and $O(q+m)$ constraints. Then, we use the linear

program in line 6 to compute x from Lemma 1 (2). This also runs in polynomial time because that

there are $O(n)$ constraints and $O(n)$ variables. The last loop uses all the above to compute the

exact payments according to The definition of IIVCG.

Proof of Theorem 2. Consider an applicable setting. By lemma 1, there exists an IIVCG con-

tract that satisfies LL and IR. By lemma 2, the payment profile of such contract is computable in

polynomial time.

4.2. Proof of Theorem 3.

To use the contract provided in the previous section, one must determine whether or not a setting

is applicable. Lemma 1 suggests that a setting is applicable if and only if

$$k_{a^*(b)} \leq \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}.$$

In this section, we present Algorithm 2 that computes this predicate in polynomial time. The key

observation for Algorithm 2 is that in order to test the above predicate $\forall b \in \mathcal{V}$, it suffices to test

in only for $O(q)$ bid profiles, where q is the number of actions.

OBSERVATION 2. Define $\mathcal{V}_a = \{v \in \mathcal{V} \mid a = a^*(v)\} \ \forall a$. Then $k_{a^*(b)} \leq \sum_{\ell \in [n]} m^\ell(b) \ \forall b \in \mathcal{V}$ if and only

if $k_a \leq \min_{b \in \mathcal{V}_a} \sum_{\ell \in [n]} m^\ell(b) \ \forall a$.

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We complete the proof of Theorem 3 by showing that the above predicate can be computed in polynomial time using Algorithm 2.

ALGORITHM 2: Determines if a setting is applicable.

Input: A common agency setting $(\mathcal{A}, \mathcal{O}, \mathcal{V}, F, \psi)$.

Output: APPLICABLE if the setting is applicable, and NOT-APPLICABLE otherwise.

`// Compute the predicate in Lemma 1 using Observation 2`

for $a \in \mathcal{A}$ **do**

    `// Compute` $\min_{b \in \mathcal{V}_a} \sum_{\ell \in [n]} m^\ell(b)$ `(see Definition 5)`

1 $\displaystyle\operatorname*{minimize}_{b \in \mathcal{V}_a, \tilde{v} \in \mathcal{V}, h \in \mathbb{R}^n} \ \sum_{\ell \in [n]} \{h^\ell - (F_{|a} \cdot b^{-\ell} - \psi(a))\}$

       subject to $F_{|a_j} \cdot (b^{-\ell} + \tilde{v}^\ell) - \psi(a_j) \le h^\ell \ \forall j \in [q], \ell \in [n].1.$

    `// Compute` k_a `(see Definition 5)`

2 $\displaystyle\operatorname*{minimize}_{w \in \mathbb{R}^m} \ F_a \cdot w$

    subject to $F_{|a_j} \cdot w - \psi(a_j) \le F_{|a} \cdot w - \psi(a) \ \forall j \in [q],$

           $w_i \ge 0 \ \forall i \in [m];$

3   **if** $k_a > \min_{b \in \mathcal{V}_a} \sum_{\ell \in [n]} m^\ell(b)$

    then
      | **return** NOT-APPLICABLE*;*

4 return APPLICABLE*;*

Proof of Theorem 3. Consider a common agency setting. According to Observation 2, it suffices to test if $k_a \le \min_{b \in \mathcal{V}_a} \sum_{\ell \in [n]} m^\ell(b) \ \forall a$ in order to determine whether the setting is applicable. This is done in Algorithm 2. For each action a, the linear program in line 1 computes $\arg\min_{b \in \mathcal{V}_a} \sum_{\ell \in [n]} m^\ell(b)$. This linear program is similar to the linear program in line 2 in Algorithm 1, which computes the first term in $m^\ell(b)$, denoted by h^ℓ. We use the same set of constraints for each principal to ensure that $\mathrm{Wel}^{a^*(b^{-\ell}, \tilde{v}^\ell)}(b^{-\ell}, \tilde{v}^\ell) \le h^\ell \ \forall \ell \in [n]$ at the feasible region. However, the search space and the objective are different. For the objecting, instead of computing the first term in $m^\ell(b)$, we subtract $\mathrm{Wel}^{a^*(b)}(b^{-\ell}, 0) = F_{|a^*(b)} \cdot b^{-\ell} - \psi(a^*(b))$ and sum over all principals to compute $\sum_{\ell \in [n]} m^\ell(b)$. We can do so since $\tilde{v}^\ell$s are independent. For the search space, we consider

all b such that $a^*(b) = a$, i.e., $b \in \mathcal{V}_a$ on top of all $\tilde{v} \in \mathcal{V}$ to find $\min \sum_{\ell \in [n]} m^\ell(b)$. Then, in line 2 we compute k_a using a similar linear program to line 4 in Algorithm 1. If the predicate in line 3 is never satisfied then the algorithm returns that the setting is applicable.

REMARK 3. When the principals have no private information (i.e., their types are fixed and publicly known so V_i is a singleton), there are always contracts in IIVCG that satisfy LL and IR. That is, our Algorithm 2 will return APPLICABLE and our Algorithm 1 will find the best such contract for the principals. As a concrete example, one LL and IR IIVCG contract is as follows. Take $h^\ell = \text{Wel}^{a^*(v)}(v)$, and $w^\ell = v^\ell$ in Definition 3 of the IIVCG family. One can verify that a principal's payments in this contract are exactly her values. LL holds since the principal's values are non-negative, and IR holds since the principal's utility is zero. We mention that this contract favors the agent's utility over that of the principals'. At the other extreme, one can use Algorithmic-IIVCG. Our LP approach to this class assures that the minimum payment required to incentivize the agent is charged from the principals, thus favoring their utility over the agent's.

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Appendix A: IIVCG Contracts: Missing Proofs from Section 3

In this section we prove the backwards direction of Proposition 2 – given a welfare maximizing and dominant strategy truthful contract t, it is an IIVCG contract. To do so, we establish the following lemma.

LEMMA 3. *For every welfare maximizing and dominant strategy truthful contract t, it holds that*

$$\mathbb{E}_{o\sim F_{|a^*(b)}}[t^\ell(b,o)] = h^\ell(b^{-\ell}) - \text{Wel}^{a^*(b)}(b^{-\ell},\mathbf{0}) \quad \forall \ell \in [n]. \tag{3}$$

This result is reminiscent of the well-known result by Holmström (1979), establishing uniqueness of VCG *auctions* for convex domains.2. In fact, our proof makes use of the following lemma from Holmström (1979), in addition to the convexity of the valuation domains.

LEMMA 4 (**Lemma 1 in Holmström (1979)**). *Let $f:[0,1]^2 \to \mathbb{R}$ and $g:[0,1] \to \mathbb{R}$ satisfy:*

1. $y \in \arg\max_{m\in[0,1]} f(m,y)$,
2. $y \in \arg\max_{m\in[0,1]} f(m,y) + g(m)$,
3. $\left|\frac{\partial f(m,y)}{\partial y}\right| \le K < \infty$ *for all $m,y \in [0,1]$.*

Then $g(m) = $ constant on $[0,1]$.

Proof of Lemma 3. Let t be a welfare maximizing and dominant strategy truthful contract. Define $h^\ell : \mathcal{V} \to \mathbb{R}$ as follows: $h^\ell(b) = \mathbb{E}_{o\sim F_{|a^*(b)}}[t^\ell(b,o)] + \text{Wel}^{a^*(b)}(b^{-\ell},\mathbf{0}) \ \forall b \in \mathcal{V}$ and fix an arbitrary $b^{-\ell} \in \mathcal{V}^{-\ell}$. To prove Lemma 3, it suffices to show that $h^\ell(b^{-\ell},\cdot)$ is constant on b^ℓ. We make the following observations: First, bidding truthfully is socially efficient, i.e.,

$$v^\ell \in \arg\max_{b^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell},b^\ell)}(b^{-\ell},v^\ell) \quad \forall v^\ell \in \mathcal{V}^\ell. \tag{4}$$

Second, since principal ℓ is truthful, $v^\ell \in \arg\max_{b^\ell \in \mathcal{V}^\ell} \mathbb{E}_{o\sim F_{|a^*(b)}}[v^\ell(o) - t^\ell(b,o)] \ \forall v^\ell \in \mathcal{V}^\ell$. Replacing $\mathbb{E}_{o\sim F_{|a^*(b)}}[t^\ell(b,o)]$ with $h^\ell(b) - \text{Wel}^{a^*(b)}(b^{-\ell},\mathbf{0})$, it holds that

$$v^\ell \in \arg\max_{b^\ell \in \mathcal{V}^\ell} \text{Wel}^{a^*(b^{-\ell},b^\ell)}(b^{-\ell},v^\ell) - h^\ell(b) \ \forall v^\ell \in \mathcal{V}^\ell. \tag{5}$$

After establishing (4) and (5), we show that $h^\ell(b^{-\ell},\cdot)$ is constant on b^ℓ. Equivalently, we show that for every pair of bid profiles $b^\ell, \tilde{b}^\ell \in \mathcal{V}^\ell \ h^\ell(b^{-\ell},b^\ell) = h^\ell(b^{-\ell},\tilde{b}^\ell)$. To do so, define $\mathcal{V}^\ell(b^\ell,\tilde{b}^\ell) = \{v^\ell(y) \mid y \in [0,1]\}$, where $v^\ell(y) : \mathcal{O} \to \mathbb{R}$ as follows $v^\ell(y;o) = yb^\ell(o) + (1-y)\tilde{b}^\ell(o) \ \forall o \in \mathcal{O}, y \in [0,1]$. Note that $v^\ell(1) = b^\ell$ and that $v^\ell(0) = \tilde{b}^\ell$. Thus, it suffices to show that $h^\ell(b^{-\ell},v^\ell(1)) = h^\ell(b^{-\ell},v^\ell(0))$. By the assumption that $\mathcal{V}$ is convex, $\mathcal{V}^\ell(b^\ell,\tilde{b}^\ell)$ is a sub domain of $\mathcal{V}^\ell$. Hence, using the notation of $a(m) = a^*(b^{-\ell},v^\ell(m))$, it follows from (4) and (5) that,

$$y \in \arg\max_{m\in[0,1]} \text{Wel}^{a(m)}(b^{-\ell},v^\ell(y)) \quad \forall y \in [0,1]. \tag{6}$$

$$y \in \underset{m \in [0,1]}{\arg\max} \operatorname{Wel}^{a(m)}(b^{-\ell}, v^{\ell}(y)) - h^{\ell}(b^{-\ell}, v^{\ell}(m)) \quad \forall y \in [0,1]. \tag{7}$$

Define $f(m,y) = \operatorname{Wel}^{a(m)}(b^{-\ell}, v^{\ell}(y))$ and $g(m) = -h^{\ell}(b^{-\ell}, v^{\ell}(m))$ $\forall m, y \in [0,1]$. With these notations, note that (6), (7) are exactly 1,2 in Lemma 4 (respectively). To apply Lemma 4 we show that 3 holds. Define $K = \arg\max_{a \in \mathcal{A}} \left| \mathbb{E}_{o \sim F_{|a}} [b^{\ell}(o) - \tilde{b}^{\ell}(o)] \right|$. Since $\mathcal{A}$ is a finite set, $K < \infty$. We show that $\left| \frac{\partial f(m,y)}{\partial y} \right| \leq K$ for all $m, y \in [0,1]$ as follows. By definition of f and of the social welfare $\operatorname{Wel}(\cdot)$,

$$\left| \frac{\partial f(m,y)}{\partial y} \right| = \left| \frac{\partial \operatorname{Wel}^{a(m)}(b^{-\ell}, v^{\ell}(y))}{\partial y} \right| = \left| \frac{\partial \mathbb{E}_{o \sim F_{|a(m)}}[v^{\ell}(y,o)]}{\partial y} \right| \tag{8}$$

By definition of $v^{\ell}(y,o)$,

$$\mathbb{E}_{o \sim F_{|a(m)}}[v^{\ell}(y,o)] = y \cdot \mathbb{E}_{o \sim F_{|a(m)}}[b^{\ell}(o)] + (1-y) \cdot \mathbb{E}_{o \sim F_{|a(m)}}[\tilde{b}^{\ell}(o)]. \tag{9}$$

According to (8) and (9) and definition of K,

$$\left| \frac{\partial \mathbb{E}_{o \sim F_{|a(m)}}[v^{\ell}(y,o)]}{\partial y} \right| = \left| \mathbb{E}_{o \sim F_{|a^*(m)}}[b^{\ell}(o) - \tilde{b}^{\ell}(o)] \right| \leq K.$$

By Lemma 4, $g(m)$ is constant on m. That is, $h^{\ell}(b^{-\ell}, v^{\ell}(m))$ is constant on m. We conclude that $h^{\ell}(b^{-\ell}, v^{\ell}(1)) = h^{\ell}(b^{-\ell}, v^{\ell}(0))$ and the proof is complete.

Proof of Proposition 2 (backwards direction). Consider a dominant strategy truthful and welfare maximizing contract t. Note that $t^{\ell} : \mathcal{V} \times \mathcal{O} \to \mathbb{R}$ can be re-composed as the sum of two functions; $c^{\ell} : \mathcal{V} \to \mathbb{R}$, that depends only on b, and $g^{\ell} : \mathcal{V} \times \mathcal{O} \to \mathbb{R}$ that depends on b as well as on the realised outcome, i.e.,

$$t^{\ell}(b,o) = c^{\ell}(b) + g^{\ell}(b,o). \tag{10}$$

With these notations, for every principal ℓ and every bid profile b,

$$\mathbb{E}_{o \sim F_{|a^*(b)}}[t^{\ell}(b,o)] = c^{\ell}(b) + \mathbb{E}_{o \sim F_{|a^*(b)}}[g^{\ell}(b,o)]. \tag{11}$$

Equalizing (11), and the expected payment in (3) it holds that there exists $\{h^{\ell}\}_{\ell \in [n]}$ such that $c^{\ell}(b) = h^{\ell}(b^{-\ell}) - \operatorname{Wel}^{a^*(b)}(b^{-\ell}, \mathbf{0}) - \mathbb{E}_{o \sim F_{|a^*(b)}}[g^{\ell}(b,o)]$. That is, $c^{\ell}(b) = h^{\ell}(b^{-\ell}) - \operatorname{Wel}^{a^*(b)}(b^{-\ell}, g^{\ell}(b,o))$. Therefore, by (10), $t^{\ell}(b,o) = h^{\ell}(b^{-\ell}) - \operatorname{Wel}^{a^*(b)}(b^{-\ell}, g^{\ell}(b,o)) + g^{\ell}(b,o)$. Since the agent maximizes the (declared) social welfare, $a^*(b) \in \arg\max_{a \in \mathcal{A}} \{ \mathbb{E}_{o \sim F_{|a}}[\sum_{\ell \in [n]} t^{\ell}(b,o)] - \psi(a) \}$. Note that in $t^{\ell}(b,o)$, the only outcome-depended term in is $g^{\ell}(b,o)$.3. Hence,

$$a^*(b) \in \underset{a \in \mathcal{A}}{\arg\max} \mathbb{E}_{o \sim F_{|a}}[\sum_{\ell \in [n]} g^{\ell}(b,o)] - \psi(a). \tag{12}$$

Define $w^{\ell} : \mathcal{O} \to \mathbb{R}_{\geq 0}^m$ as follows: $w^{\ell}(o) = g^{\ell}(b,o) - \min_{o \in \mathcal{O}} g^{\ell}(b,o)$ $\forall o \in \mathcal{O}$. Hence, $t^{\ell}(b,o) = h^{\ell}(b^{-\ell}) - \operatorname{Wel}^{a^*(b)}(b^{\ell}, w^{\ell}) + w^{\ell}(o)$, where $\sum_{\ell \in [n]} w^{\ell} \in \mathcal{L}_{a^*(b)}$ by (12) and the definition of $\mathcal{L}_a$.

Appendix B: IIVCG Instantiations

We describe two different contracts for the case where the setting is not applicable; First, in Section B.1 is an IIVCG contract that satisfies IR and not necessarily LL. Second, in Section B.2 is an IIVCG contract that satisfies LL. Note that providing a contract that is LL and IR but not truthful welfare maximizing is trivial.

B.1. IIVCG Instantiation: Auction Inspired IIVCG

In this section we give an instantiation of the IIVCG family of contracts that satisfies IR. The instantiation is closely related to the VCG contracts of Lavi and Shamash (2019) as well as to VCG *auctions*, so we refer to the resulting contracts as "Auction-Inspired IIVCG".

DEFINITION 6. Contract t is an *Auction-Inspired IIVCG* contract if

$$t^\ell(b,o) = \text{Wel}^{a^*(b^{-\ell},\mathbf{0})}(b^{-\ell},\mathbf{0}) - \text{Wel}^{a^*(b)}(b) + b^\ell(o) \quad \forall \ell \in [n], b \in \mathcal{V}, o \in \mathcal{O}.$$

OBSERVATION 3. Every Auction-Inspired IIVCG contract belongs to the IIVCG family of contracts by using Definition 3 and $h^\ell(b^{-\ell}) = \text{Wel}^{a^*(b^{-\ell},\mathbf{0})}(b^{-\ell},\mathbf{0})$, $w^\ell = b^\ell$ $\forall b \in \mathcal{V}$.

It is not hard to see that in Auction-Inspired IIVCG contracts, the expected payment of principal ℓ given bid profile b is her *externality* on the other principals: the difference between their (declared) expected welfare had she not participated, i.e., $\text{Wel}^{a^*(b^{-\ell},\mathbf{0})}(b^{-\ell},\mathbf{0})$, and their (declared) expected welfare with her participation, i.e., $\text{Wel}^{a^*(b)}(\mathbf{0},b^\ell)$. This is exactly as in VCG auctions with the Clarke pivot rule.

OBSERVATION 4. Every Auction-Inspired IIVCG contract is IR.

Proof. It can be verified that principal ℓ's expected utility when she bids truthfully $b^\ell = v^\ell$ and when others bid $b^{-\ell}$ is $\max_{a \in \mathcal{A}} \text{Wel}^a(b^{-\ell},v^\ell) - \max_{a \in \mathcal{A}} \text{Wel}^a(b^{-\ell},\mathbf{0})$, which is clearly non-negative.

PROPOSITION 3. *There exist common agency settings for which Auction-Inspired IIVCG contracts do not satisfy LL.*

Proof Sketch. Consider a common agency setting with a valuation profile such that no principal has a positive externality on the others. Assume the principals are truthful (LL should hold for any bid profile, including truthful ones). As mentioned above, in any Auction-Inspired IIVCG, principal ℓ's expected payment is her externality. The principals' expected payments are therefore all zero. The LL requirement that the agent is never paid a negative amount then means that he cannot be paid a positive amount either. When all payments are zero, the agent will not take any action with nonzero cost, in contradiction to the requirement that the agent maximizes welfare.

A specific setting that illustrates Proposition 3 appears in Section 3.2.
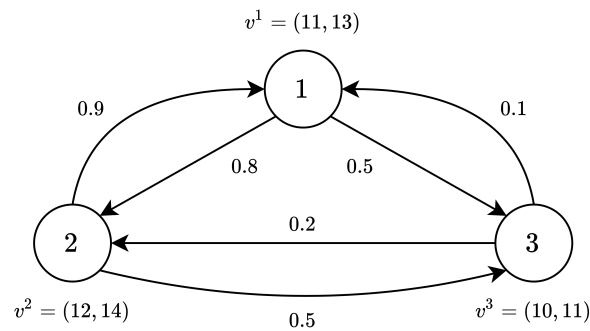
Figure 1 A possible correlation graph of the common agency setting in Example 2.

B.2. IIVCG Instantiation: Weighted IIVCG Contracts

In this section we introduce a class within IIVCG which guarantees LL but not necessarily IR, and give a condition on common agency settings ("*G*-correlation") that suffices for the existence of IR as well. The high-level idea is as follows. From our discussion at the end of Section 3.1 it follows that in order to satisfy LL, principals' expected payment should be more than their externality on others. While increasing principals' payments may damage IR, a type of contract which we term *Weighted IIVCG* succeeds in providing IR for a wide family of common agency settings. The central idea is to approximate each principal's valuation according to other principals using the concept of a *correlation graph*, and use this to set the payments. Correlation graphs and *G*-correlated settings are defined in Section B.2.1, and in Section B.2.2 we use these to define Weighted IIVCG.

B.2.1. Correlation Graphs and *G*-Correlated Settings A correlation graph (Definition 7) describes how principals' valuations are correlated. (As one interpretation, principals' valuations can be intuitively thought of as reflecting their socioeconomic status, and the correlation graph as a natural way to describe it.)

DEFINITION 7. A *correlation graph G* is a weighted directed graph with vertex set $[n]$ and weights on edges $d : [n]^2 \to [0,1]$ such that $\sum_{k \in [n]} d(k, \ell) = 1$, and $d(\ell, \ell) = 0 \ \forall \ell \in [n]$.

We introduce the following "running example" for this section to demonstrate our definitions. Figure 1 depicts a correlation graph of the setting in the following example.

EXAMPLE 2. There are 3 principals, 2 actions and 2 outcomes. The valuation domain is $\mathcal{V}^\ell = [10, 15]^2 \ \forall \ell \in [3]$. The costs are: $\psi(a_1) = 0$, and $\psi(a_2) = 1$. The probabilities over outcomes are: $p_{o \sim F_{|a_1}}(o_1) = 1$, and $p_{o \sim F_{|a_2}}(o_1) = 0.25$, and principals' valuations are: $v^1 = (11, 13)$, $v^2 = (12, 14)$, and $v^3 = (10, 11)$.

We mention that a correlation graph may consist of several connected components. (In our intuitive interpretation, think of a state with multiple cities, within each city individuals are socioeconomically correlated.)

Given a correlation graph G, we define principals' approximate valuations, referred to as their G-weighted valuations, as follows.

DEFINITION 8. Consider a correlation graph G, and a valuation profile v. The *G-weighted valuation of principal ℓ is*: $\hat{v}^\ell_G = \sum_{k \in [n]} d(\ell, k) \cdot \hat{v}^k$, where $\hat{v}^k = v^k(o) - \min_{o'} v^k(o') \; \forall o \in \mathcal{O}, k \in [n]$.

In words, the G-weighted valuation of principal ℓ is a weighted average on other principals valuations, when their valuations are shifted downwards. We will demonstrate this for Example 2 and the correlation graph G in Figure 1: The G-weighted valuation of principal 1 is $\hat{v}^1_G = d(1,2) \cdot \hat{v}^2 + d(1,3) \cdot \hat{v}^3$. It can be verified that $\hat{v}^2 = (0,2)$, $\hat{v}^3 = (0,1)$, and since $d(1,2) = 0.8$, and $d(1,3) = 0.5$, the G-weighted valuation of 1 is $\hat{v}^1_G = (0,1.6) + (0,0.5) = (0,2.1)$.

DEFINITION 9. Given a correlation graph G, a setting is *G-correlated* if $\text{Wel}^{a^*(v^{-\ell}, \hat{v}^\ell_G)}(v^{-\ell}, \hat{v}^\ell_G) \leq \text{Wel}^{a^*(v)}(v) \; \forall \ell \in [n], v \in \mathcal{V}$.

That is, a setting is G-correlated if the welfare when principal ℓ's valuation is replaced with her G-weighted valuation, is bounded by the actual welfare (for every valuation profile). We demonstrate this condition for the setting and the valuation profile in Example 2, and the correlation graph in Figure 1. Recall that $\hat{v}^1_G = (0, 2.1)$. It can thus be verified that $\text{Wel}^{a_1}(v) = 33$, $\text{Wel}^{a_2}(v) = 35.75$ and that $\text{Wel}^{a_1}(v^{-\ell}, \hat{v}^\ell_G) = 22$, $\text{Wel}^{a_2}(v^{-\ell}, \hat{v}^\ell_G) = 24.825$. Thus, the welfare when principal ℓ's valuation is replaced with her G-weighted valuation is bounded by the actual welfare (for v).

Examples of G-correlated settings. The following claims describe natural special cases for G-correlated settings.

CLAIM 1. *Consider a common agency setting. If principals share the same expected value for each action, there exists a correlation graph G for which the setting is G-correlated.4.*

Proof. Take G as any graph on $[n]$ vertices that has a weighted in-degree and out-degree of 1 for all vertices.5. Consider a valuation profile v and a principal ℓ. To satisfy the condition for G-correlation it suffices to have $\mathbb{E}_{o \sim F_{|a}}[\hat{v}^\ell_G(o)] \leq \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)] \; \forall a$. Since it implies that $\text{Wel}^{a^*(v^{-\ell}, \hat{v}^\ell_G)}(v^{-\ell}, \hat{v}^\ell_G) \leq \text{Wel}^{a^*(v^{-\ell}, \hat{v}^\ell_G)}(v^{-\ell}, v^\ell) \leq \text{Wel}^{a^*(v)}(v)$. By the definition of $\hat{v}^\ell$ and of $\hat{v}^k$, $\mathbb{E}_{o \sim F_{|a}}[\hat{v}^\ell_G(o)] = \mathbb{E}_{o \sim F_{|a}}[\sum_{k \in [n]} d(\ell, k) \cdot \hat{v}^k(o)] \leq \mathbb{E}_{o \sim F_{|a}}[\sum_{k \in [n]} d(\ell, k) \cdot v^k(o)]$. Since principal's share the same expected valuation, $\mathbb{E}_{o \sim F_{|a}}[\sum_{k \in [n]} d(\ell, k) \cdot v^k(o)] = \sum_{k \in [n]} d(\ell, k) \cdot \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)]$, and since $\sum_{k \in [n]} d(\ell, k) = 1$, the last term is $\mathbb{E}_{o \sim F_{|a}}[v^\ell(o)]$. We conclude that $\mathbb{E}_{o \sim F_{|a}}[\hat{v}^\ell_G(o)] \leq \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)] \; \forall a$ as desired. This completes the proof.

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CLAIM 2. *Consider a common agency setting. If principals' valuation domain is $[a,b]^m$ where $b - a \leq a$, there exists a correlation graph G for which the setting is G-correlated.*

We mention that in this case, first price contracts may fail.

Proof. The proof proceeds as in Claim 1, but requires a different argument for proving that $\mathbb{E}_{o \sim F_{|a}}[\hat{v}_G^\ell(o)] \leq \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)] \ \forall a$. By definition, $\mathbb{E}_{o \sim F_{|a}}[\hat{v}_G^\ell(o)] = \mathbb{E}_{o \sim F_{|a}}[\sum_{k \in [n]} d(\ell, k) \cdot \hat{v}^k(o)]$. Note that since $v^k(o) \leq b$, and $a \leq \min_{o' \in \mathcal{O}} v^k(o')$, we have that $\hat{v}^k(o) \leq b - a \leq a \ \forall k \in [n], o \in \mathcal{O}$. This implies that, $\mathbb{E}_{o \sim F_{|a}}[\sum_{k \in [n]} d(\ell, k) \cdot \hat{v}^k(o)] \leq \sum_{k \in [n]} d(\ell, k) \cdot a$, and since $\sum_{k \in [n]} d(\ell, k) = 1$ the last term is a. Recall that principal's valuations are at least a. Thus, $a \leq \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)]$. We conclude that $\mathbb{E}_{o \sim F_{|a}}[\hat{v}_G^\ell(o)] \leq \mathbb{E}_{o \sim F_{|a}}[v^\ell(o)] \ \forall a$. This completes the proof.

B.2.2. G-Weighted IIVCG Contracts We can now use the concepts introduced in the previous section to define Weighted IIVCG contracts and analyze their properties.

DEFINITION 10. Consider a correlation graph G. A *G-Weighted IIVCG Contract* is given by

$$t^\ell(b, o) = \text{Wel}^{a^*(b^{-\ell}, \hat{b}_G^\ell)}(b^{-\ell}, \hat{b}_G^\ell) - \text{Wel}^{a^*(b)}(b^{-\ell}, \hat{b}_G^\ell) + \hat{b}_G^\ell(o) \ \ \forall \ell \in [n], b \in \mathcal{V}, o \in \mathcal{O}.$$

PROPOSITION 4. *Consider a correlation graph G. Every G-Weighted IIVCG contract is in IIVCG.*

Proof. The proof relies on Definition 3 when $h^\ell(b^{-\ell}) = \text{Wel}^{a^*(b^{-\ell}, \hat{b}_G^\ell)}(b^{-\ell}, \hat{b}_G^\ell)$ and $w^\ell = \hat{b}_G^\ell(o)$. We prove that the conditions of Definition 3 are met. First, h^ℓ is independent of b^ℓ, since $\hat{b}_G^\ell$ is independent of b^ℓ ($d(\ell, \ell) = 0 \ \forall \ell \in [n]$). Second, we show that $\sum_{\ell \in [n]} w^\ell \in \mathcal{L}_{a^*(b)}$. To do so, we establish that $\sum_{\ell \in [n]} w^\ell = \sum_{\ell \in [n]} \hat{b}^\ell$ using the following chain of equalities:

$$\sum_{\ell \in [n]} w^\ell \overset{(1)}{=} \sum_{\ell \in [n]} \sum_{k \in [n]} d(\ell, k) \hat{b}^k \overset{(2)}{=} \sum_{k \in [n]} \sum_{\ell \in [n]} d(\ell, k) \hat{b}^k \overset{(3)}{=} \sum_{k \in [n]} \hat{b}^k.$$

Equality (1) follows from the choice of w^ℓs above, and the definition of $\hat{b}_G^\ell$ (Definition 8). Equality (2) changes the order of summands, and Equality (3) follows from the fact that $\sum_{k \in [n]} d(k, \ell) = 1$ $\forall \ell \in [n]$ (Definition 9). Thus, $\sum_{\ell \in [n]} w^\ell = \sum_{\ell \in [n]} \hat{b}^\ell$, and it suffices to show that $\sum_{\ell \in [n]} \hat{b}^\ell \in \mathcal{L}_{a^*(b)}$. To do so, we show that $\sum_{\ell \in [n]} \hat{b}^\ell \in \mathbb{R}_{\geq 0}^m$ and that $a^*(b) \in \arg\max_{a'}\{\mathbb{E}_{o \sim F_{|a'}}[\sum_{\ell \in [n]} \hat{b}^\ell(o)] - \psi(a')\}$. First, by definition $\hat{b}^\ell(o) \geq 0 \ \forall o \in \mathcal{O}, \ell \in [n]$. Thus, $\sum_{\ell \in [n]} \hat{b}^\ell \in \mathbb{R}_{\geq 0}^m$. Second, by definition of $a^*(b)$, it holds that $a^*(b) \in \arg\max_{a'}\{\mathbb{E}_{o \sim F_{|a'}}[\sum_{\ell \in [n]} b^\ell(o)] - \psi(a')\}$. Thus $a^*(b) \in \arg\max_{a'}\{\mathbb{E}_{o \sim F_{|a'}}[\sum_{\ell \in [n]} b^\ell(o) - y] - \psi(a')\} \ \forall y \in \mathbb{R}$, and specifically for $y = \sum_{\ell \in [n]} \min_{o \in \mathcal{O}} b^\ell(o)$. We conclude that $a^*(b) \in \arg\max_{a'}\{\mathbb{E}_{o \sim F_{|a'}}[\sum_{\ell \in [n]} \hat{b}^\ell(o)] - \psi(a')\}$. This completes the proof.

PROPOSITION 5. *Consider a correlation graph G. Every G-Weighted IIVCG contract satisfies LL. Furthermore, IR is satisfied if and only if the setting is G-correlated.*

Proof. Truthfulness and social efficiency are immediate since G-Weighted IIVCG is in IIVCG. LL follows from the fact that the first term in the payment formula , $\mathrm{Wel}^{a^*(b^{-\ell},\hat{b}_G^\ell)}(b^{-\ell},\hat{b}_G^\ell)$, is (weakly) higher than the second term, $\mathrm{Wel}^{a^*(b)}(b^{-\ell},\hat{b}_G^\ell)$, and since $\hat{b}_G^\ell(o) \geq 0$ $\forall o$. We prove IR if and only if the setting is G-correlated. By definition, IR holds if and only if each principal ℓ's expected payment when she bids truthfully is at most her expected value. That is, by the definition of G-Weighted IIVCG IR holds, if and only if

$$\mathrm{Wel}^{a^*(b^{-\ell},\hat{b}_G^\ell)}(b^{-\ell},\hat{b}_G^\ell) - \mathrm{Wel}^{a^*(b^{-\ell},v^\ell)}(b^{-\ell},\mathbf{0}) \leq \mathbb{E}_{o \sim F_{|a^*(b^{-\ell},v^\ell)}}[v^\ell(o)], \quad \forall b^{-\ell} \in \mathcal{V}^{-\ell}, v^\ell \in \mathcal{V}^\ell, \ell \in [n].$$

By adding $\mathrm{Wel}^{a^*(b^{-\ell},v^\ell)}(b^{-\ell},\mathbf{0})$ to both sides of the inequality, we get

$$\mathrm{Wel}^{a^*(b^{-\ell},\hat{b}_G^\ell)}(b^{-\ell},\hat{b}_G^\ell) \leq \mathrm{Wel}^{a^*(b^{-\ell},v^\ell)}(b^{-\ell},v^\ell), \quad \forall b^{-\ell} \in \mathcal{V}^{-\ell}, v^\ell \in \mathcal{V}^\ell, \ell \in [n].$$

Denoting $b = (b^{-\ell}, v^\ell)$, this condition is equivalent to Definition 9 of the G-correlated setting. This completes the proof.

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