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An extension of the Liouville theorem for Fourier multipliers to sub-exponentially growing solutions

David Berger, René L. Schilling, Eugene Shargorodsky, and Teo Sharia

ABSTRACT. We study the equation $m(D)f = 0$ in a large class of sub-exponentially growing functions. Under appropriate restrictions on $m \in C(\mathbb{R}^n)$, we show that every such solution can be analytically continued to a sub-exponentially growing entire function on \mathbb{C}^n if and only if $m(\xi) \neq 0$ for $\xi \neq 0$.

1. Introduction

The classical Liouville theorem for the Laplace operator $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on \mathbb{R}^n says that every bounded (polynomially bounded) solution of the equation $\Delta f = 0$ is in fact constant (is a polynomial). Recently, similar results have been obtained for solutions of more general equations of the form $m(D)f = 0$, where $m(D) := \mathcal{F}^{-1}m(\xi)\mathcal{F}$, and

$$\mathcal{F}\phi(\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi(x) dx \quad \text{and} \quad \mathcal{F}^{-1}u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} u(\xi) d\xi$$

are the Fourier and the inverse Fourier transforms (see [1], [2], [3], [11], and the references therein). Namely, it was shown that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, the implication

$$\begin{aligned} &f \text{ is bounded (polynomially bounded) and } m(D)f = 0 \\ \implies &f \text{ is constant (is a polynomial)} \end{aligned}$$

holds if and only if $m(\xi) \neq 0$ for $\xi \neq 0$. Much of this research has been motivated by applications to infinitesimal generators of Lévy processes.

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In this paper, we deal with solutions of $m(D)f = 0$ that can grow faster than any polynomial. Of course, one cannot expect such solutions to have simple structure, not even in the case of $\Delta f = 0$ in \mathbb{R}^2 (see, e.g., [21, Ch. I, §2]). We consider sub-exponentially growing solutions whose growth is controlled by a submultiplicative function (see (1)) satisfying the Beurling-Domar condition (3), and show that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, every such solution admits analytic continuation to a sub-exponentially growing entire function on \mathbb{C}^n if and only if $m(\xi) \neq 0$ for $\xi \neq 0$ (see Corollary 4.5). Results of this type have been obtained for solutions of partial differential equations with constant coefficients by A. Kaneko and G.E. Šilov (see [16], [17], [26], [7, Ch. 10, Sect. 2, Theorem 2], and Section 5 below).

Keeping in mind applications to infinitesimal generators of Lévy processes, we do not assume that m is the Fourier transform of a distribution with compact support, so our setting is different from that in, e.g., [6], [15, Ch. XVI].

The paper is organized as follows. In Chapter 2, we consider submultiplicative functions satisfying the Beurling-Domar condition and, for every such function g , introduce an auxiliary function S_g (see (14), (15)), which appears in our main estimates. Chapter 3 contains weighted L^p estimates for entire functions on \mathbb{C}^n , which are a key ingredient in the proof of our main results in Chapter 4. Another key ingredient is the Tauberian theorem 4.1, which is similar to [3, Theorem 7] and [23, Theorem 9.3]. The main difference is that the function f in Theorem 4.1 is not assumed to be polynomially bounded, and hence it might not be a tempered distribution. So, we avoid using the Fourier transform $\widehat{f} = \mathcal{F}f$ and its support (and non-quasianalytic type ultradistributions). Although we are mainly interested in the case $m(\xi) \neq 0$ for $\xi \neq 0$, we also prove a Liouville type result for m with compact zero set $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\}$ (see Theorem 4.4). Finally, we discuss in Section 5 A. Kaneko's Liouville type results for partial differential equations with constant coefficients ([16], [17]), which show that the Beurling-Domar condition is in a sense optimal in our setting.

2. Submultiplicative functions and the Beurling-Domar condition

Let $g : \mathbb{R}^n \rightarrow (0, \infty)$ be a locally bounded, measurable *submultiplicative* function, i.e. a locally bounded measurable function satisfying

$$g(x + y) \leq Cg(x)g(y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

where the constant $C \in [1, \infty)$ does not depend on x and y . Without loss of generality, we will always assume that $g \geq 1$, as otherwise one can replace g with $g + 1$. Also, replacing g with Cg , one can assume that

$$g(x + y) \leq g(x)g(y) \quad \text{for all } x, y \in \mathbb{R}^n. \quad (1)$$

A locally bounded submultiplicative function is exponentially bounded, i.e.

$$|g(x)| \leq Ce^{a|x|} \quad (2)$$

for suitable constants $C, a > 0$ (see [24, Section 25] or [13, Ch. VII]).

We will say that g satisfies the *Beurling-Domar* condition if

$$\sum_{l=1}^{\infty} \frac{\log g(lx)}{l^2} < \infty \quad \text{for all } x \in \mathbb{R}^n. \quad (3)$$

If g satisfies the Beurling-Domar condition, then it also satisfies the Gelfand-Raikov-Shilov condition

$$\lim_{l \rightarrow \infty} g(lx)^{1/l} = 1 \quad \text{for all } x \in \mathbb{R}^n,$$

while $g(x) = e^{|x|/\log(e+|x|)}$ satisfies the latter but not the former (see [9]). It is also easy to see that $g(x) = e^{|x|/\log^\gamma(e+|x|)}$ satisfies the Beurling-Domar condition if and only if $\gamma > 1$. The function

$$g(x) = e^{a|x|^b} (1 + |x|)^s (\log(e + |x|))^t$$

satisfies the Beurling-Domar condition for any $a, s, t \geq 0$ and $b \in [0, 1)$ (see [9]).

LEMMA 2.1. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition (3). Then for every $\varepsilon > 0$, there exists $R_\varepsilon \in (0, \infty)$ such that*

$$\int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau < \varepsilon \quad \text{for all } x \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : |y| = 1\}. \quad (4)$$

PROOF. Since $g \geq 1$ is locally bounded,

$$0 \leq M := \sup_{|y| \leq 1} \log g(y) < \infty. \quad (5)$$

Take any $x \in \mathbb{S}^{n-1}$. It follows from (1) that

$$\log g((l+1)x) - M \leq \log g(\tau x) \leq \log g(lx) + M \quad \text{for all } \tau \in [l, l+1].$$

Hence

$$\begin{aligned} \sum_{l=L}^{\infty} \frac{\log g((l+1)x) - M}{(l+1)^2} &\leq \sum_{l=L}^{\infty} \int_l^{l+1} \frac{\log g(\tau x)}{\tau^2} d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx) + M}{l^2} \\ \implies \sum_{l=L+1}^{\infty} \frac{\log g(lx)}{l^2} - \frac{M}{L} &\leq \int_L^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx)}{l^2} + \frac{M}{L-1} \end{aligned} \quad (6)$$

for $L \in \mathbb{N}$.

Let

$$\mathbf{e}_j := \underbrace{(0, \dots, 0)}_{j-1}, 1, 0, \dots, 0), \quad j = 1, \dots, n, \quad \mathbf{e}_0 := \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right), \quad (7)$$

$$Q := \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}^n : \frac{1}{2\sqrt{n}} < y_j < \frac{2}{\sqrt{n}}, j = 1, \dots, n \right\}.$$

For every $x \in \mathbb{S}^{n-1}$ there exists an orthogonal matrix $A_x \in O(n)$ such that $x = A_x \mathbf{e}_0$. Hence $\{AQ\}_{A \in O(n)}$ is an open cover of \mathbb{S}^{n-1} . Let $\{A_k Q\}_{k=1, \dots, K}$ be a finite subcover. Take an arbitrary $\varepsilon > 0$. It follows from (3) and (6) that there exists $R_\varepsilon > 0$ for which

$$\int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(\tau A_k \mathbf{e}_j)}{\tau^2} d\tau < \frac{\varepsilon}{2\sqrt{n}}, \quad k = 1, \dots, K, \quad j = 1, \dots, n.$$

For any $x \in \mathbb{S}^{n-1}$, there exist $k = 1, \dots, K$ and $a_j \in \left(\frac{1}{2\sqrt{n}}, \frac{2}{\sqrt{n}}\right)$, $j = 1, \dots, n$ such that

$$x = \sum_{j=1}^n a_j A_k \mathbf{e}_j.$$

Using (1), one gets

$$\begin{aligned} \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau &\leq \sum_{j=1}^n \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau a_j A_k \mathbf{e}_j)}{\tau^2} d\tau = \sum_{j=1}^n a_j \int_{a_j R_\varepsilon}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2} dr \\ &\leq \sum_{j=1}^n \frac{2}{\sqrt{n}} \int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2} dr < \sum_{j=1}^n \frac{2}{\sqrt{n}} \cdot \frac{\varepsilon}{2\sqrt{n}} = n \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

□

Let

$$\begin{aligned} I_{g,x}(r) &:= \int_{\max\{r,1\}}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau < \infty, \\ J_{g,x}(r) &:= \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x) d\tau < \infty, \\ S_{g,x}(r) &:= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2} d\tau \quad r \geq 0, \quad x \in \mathbb{S}^{n-1}. \end{aligned}$$

One has, for $r > 1$ and any $\beta \in (0, 1)$,

$$\begin{aligned} J_{g,x}(r) &= \frac{1}{r^2} \int_0^r \log g(\tau x) d\tau = \frac{1}{r^2} \int_0^1 \log g(\tau x) d\tau \\ &\quad + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{r^{2\beta}} d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{r^2} d\tau \\ &\leq \frac{M}{r^2} + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{\tau^2} d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{\tau^2} d\tau \\ &\leq \frac{M}{r^2} + \frac{I_{g,x}(1)}{r^{2(1-\beta)}} + I_{g,x}(r^\beta) \end{aligned} \tag{8}$$

(see (5)). Further, if $r > 1$, then

$$\pi S_{g,x}(r) = \int_0^{\infty} \frac{\log g(\tau x)}{\tau^2 + r^2} d\tau + \int_0^{\infty} \frac{\log g(-\tau x)}{\tau^2 + r^2} d\tau$$

$$\begin{aligned}
 &\leq \int_0^r \frac{\log g(\tau x)}{r^2} d\tau + \int_r^\infty \frac{\log g(\tau x)}{\tau^2} d\tau \\
 &+ \int_0^r \frac{\log g(-\tau x)}{r^2} d\tau + \int_r^\infty \frac{\log g(-\tau x)}{\tau^2} d\tau \\
 &= I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r), \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 \pi S_{g,x}(r) &\geq \int_0^r \frac{\log g(\tau x)}{2r^2} d\tau + \int_r^\infty \frac{\log g(\tau x)}{2\tau^2} d\tau \\
 &+ \int_0^r \frac{\log g(-\tau x)}{2r^2} d\tau + \int_r^\infty \frac{\log g(-\tau x)}{2\tau^2} d\tau \\
 &= \frac{1}{2} (I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r)). \tag{10}
 \end{aligned}$$

Since g is locally bounded, it follows from Lemma 2.1 that I_g defined by

$$I_g(r) := \sup_{x \in \mathbb{S}^{n-1}} I_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \int_{\max\{r,1\}}^\infty \frac{\log g(\tau x)}{\tau^2} d\tau < \infty, \tag{11}$$

is a decreasing function such that

$$I_g(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{12}$$

Let

$$J_g(r) := \sup_{x \in \mathbb{S}^{n-1}} J_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x) d\tau, \tag{13}$$

$$S_g(r) := \sup_{x \in \mathbb{S}^{n-1}} S_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \frac{1}{\pi} \int_{-\infty}^\infty \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2} d\tau. \tag{14}$$

Then

$$\begin{aligned}
 J_g(r) &\leq \frac{M}{r^2} + \frac{I_g(1)}{r^{2(1-\beta)}} + I_g(r^\beta), \\
 \frac{1}{2\pi} \max\{I_g(r), J_g(r)\} &\leq S_g(r) \leq \frac{2}{\pi} (I_g(r) + J_g(r))
 \end{aligned}$$

(see (8), (9), (10)). So, $J_g(r) \rightarrow 0$, and

$$S_g(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty \tag{15}$$

(see (12)). It is clear that

$$S_g(r) = S_g(1) \quad \text{for } r \in [0, 1], \quad \text{and } S_g \text{ is a decreasing function.} \tag{16}$$

Examples.

1) If $g(x) = (1 + |x|)^s$, $s \geq 0$, then

$$\begin{aligned}
 S_g(r) &= \frac{1}{\pi} \int_{-\infty}^\infty \frac{s \log(1 + |\tau|)}{\tau^2 + r^2} d\tau = \frac{s}{\pi r} \int_{-\infty}^\infty \frac{\log(1 + r|\lambda|)}{\lambda^2 + 1} d\lambda \\
 &\leq \frac{s}{\pi r} \int_{-\infty}^\infty \frac{\log(1 + |\lambda|)}{\lambda^2 + 1} d\lambda + \frac{s \log(1 + r)}{\pi r} \int_{-\infty}^\infty \frac{1}{\lambda^2 + 1} d\lambda
 \end{aligned}$$

$$= \frac{c_1 s}{r} + \frac{s \log(1+r)}{r}, \quad r \geq 1, \quad (17)$$

where

$$c_1 := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1+|\lambda|)}{\lambda^2+1} d\lambda < \infty.$$

2) If $g(x) = (\log(e+|x|))^t$, $t \geq 0$, then using the obvious inequality

$$u+v \leq 2uv, \quad u, v \geq 1,$$

one gets

$$\begin{aligned} S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t \log \log(e+|\tau|)}{\tau^2+r^2} d\tau = \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log \log(e+r|\lambda|)}{\lambda^2+1} d\lambda \\ &\leq \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log(\log(e+|\lambda|) + \log(e+r))}{\lambda^2+1} d\lambda \\ &\leq \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log(2 \log(e+|\lambda|))}{\lambda^2+1} d\lambda + \frac{t \log \log(e+r)}{\pi r} \int_{-\infty}^{\infty} \frac{1}{\lambda^2+1} d\lambda \\ &= \frac{c_2 t}{r} + \frac{t \log \log(e+r)}{r}, \quad r \geq 1, \end{aligned} \quad (18)$$

where

$$c_2 := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(2 \log(e+|\lambda|))}{\lambda^2+1} d\lambda < \infty.$$

3) If $g(x) = e^{a|x|^b}$, $a \geq 0$, $b \in [0, 1)$, then

$$\begin{aligned} S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a|\tau|^b}{\tau^2+r^2} d\tau = \frac{ar^{b-1}}{\pi} \int_{-\infty}^{\infty} \frac{|\lambda|^b}{\lambda^2+1} d\lambda = \frac{2ar^{b-1}}{\pi} \int_0^{\infty} \frac{t^b}{t^2+1} dt \\ &= \frac{ar^{b-1}}{\pi} \int_0^{\infty} \frac{s^{\frac{b-1}{2}}}{s+1} ds = \frac{ar^{b-1}}{\sin(\frac{1-b}{2}\pi)}, \quad r \geq 1 \end{aligned} \quad (19)$$

(see, e.g., [4, Ch. V, Example 2.12]).

4) Finally, let $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma > 1$. Since

$$\frac{\tau(e+\tau)}{\tau^2+r^2} = \frac{1+\frac{e}{\tau}}{1+\frac{r^2}{\tau^2}} \leq 1 + \frac{e}{\tau} \leq 1 + \frac{e}{r} \quad \text{for } \tau \geq r,$$

then for any $\beta \in (0, 1)$,

$$\begin{aligned} S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\tau|}{(\tau^2+r^2) \log^\gamma(e+|\tau|)} d\tau = \frac{2}{\pi} \int_0^{\infty} \frac{\tau}{(\tau^2+r^2) \log^\gamma(e+\tau)} d\tau \\ &= \frac{2}{\pi} \int_0^{r^\beta} + \int_{r^\beta}^r + \int_r^{\infty} \frac{\tau}{(\tau^2+r^2) \log^\gamma(e+\tau)} d\tau \\ &\leq \frac{2}{\pi} \int_0^{r^\beta} \frac{\tau}{\tau^2+r^2} d\tau + \frac{2}{\pi \log^\gamma(e+r^\beta)} \int_{r^\beta}^r \frac{\tau}{\tau^2+r^2} d\tau \\ &\quad + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \int_r^{\infty} \frac{1}{(e+\tau) \log^\gamma(e+\tau)} d\tau \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \log(\tau^2 + r^2) \Big|_0^{r^\beta} + \frac{1}{\pi \log^\gamma(e + r^\beta)} \log(\tau^2 + r^2) \Big|_{r^\beta}^r \\
 &\quad + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{1 - \gamma} \log^{1-\gamma}(e + \tau) \Big|_r^\infty \\
 &\leq \frac{1}{\pi} \log(1 + r^{2(\beta-1)}) + \frac{\log 2}{\pi \log^\gamma(e + r^\beta)} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma - 1} \log^{1-\gamma}(e + r) \\
 &\leq \frac{r^{2(\beta-1)}}{\pi} + \frac{\log 2}{\pi \log^\gamma(e + r^\beta)} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma - 1} \log^{1-\gamma}(e + r), \quad r \geq 1.
 \end{aligned}$$

Since

$$\lim_{r \rightarrow \infty} \frac{r^{2(\beta-1)} + (\log 2) \log^{-\gamma}(e + r^\beta)}{\log^{-\gamma}(e + r)} = \frac{\log 2}{\beta^\gamma} \quad \text{for all } \beta \in (0, 1),$$

one gets, upon taking $\beta \in ((\log 2)^{1/\gamma}, 1)$, the following estimate

$$S_g(r) \leq \frac{\log^{-\gamma}(e + r)}{\pi} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma - 1} \log^{1-\gamma}(e + r) \quad (20)$$

for sufficiently large r .

3. Estimates for entire functions

Let $1 \leq p \leq \infty$ and let $\omega : \mathbb{R}^n \rightarrow [0, \infty)$ be a measurable function such that $\omega > 0$ Lebesgue almost everywhere. We set

$$\begin{aligned}
 \|f\|_{L_\omega^p} &:= \|\omega f\|_{L^p} \quad \text{and} \\
 L_\omega^p(\mathbb{R}^n) &:= \{f : \mathbb{R}^n \rightarrow \mathbb{C} \mid f \text{ measurable, } \|f\|_{L_\omega^p} < \infty\}.
 \end{aligned} \quad (21)$$

LEMMA 3.1. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition (3). Let φ be a measurable function such that for almost every $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, $\varphi(z_1, x')$ is analytic in z_1 for $\text{Im } z_1 > 0$ and continuous up to \mathbb{R} . Suppose also that $\log |\varphi(z_1, x')| = O(|z_1|)$ for $|z_1|$ large, $\text{Im } z_1 \geq 0$, and that the restriction of φ to \mathbb{R}^n belongs to $L_{g^{\pm 1}}^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Finally, suppose that*

$$k_\varphi := \text{ess sup}_{x' \in \mathbb{R}^{n-1}} \left(\limsup_{0 < y_1 \rightarrow \infty} \frac{\log |\varphi(iy_1, x')|}{y_1} \right) < \infty. \quad (22)$$

Then

$$\|\varphi(\cdot + iy_1, \cdot)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_g e^{(k_\varphi + S_g(y_1))y_1} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y_1 > 0 \quad (23)$$

(see (14), (15)), where the constant $C_g < \infty$ depends only on g .

PROOF. Let $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. It follows from (1) that

$$\begin{aligned}
 &\int_{-\infty}^{\infty} \frac{\log^+(g^{\mp 1}(t, x'))}{1 + t^2} dt \leq \int_{-\infty}^{\infty} \frac{\log(g(t, x'))}{1 + t^2} dt \\
 &\leq \int_{-\infty}^{\infty} \frac{\log(g(t, 0)) + \log(g(0, x'))}{1 + t^2} dt \leq \pi ((S_g(1) + \log(g(0, x')))) < +\infty.
 \end{aligned}$$

Since $g^{\pm 1}\varphi \in L^p(\mathbb{R}^n)$, Fubini's theorem implies that

$$g^{\pm 1}(\cdot, x')\varphi(\cdot, x') \in L^p(\mathbb{R})$$

for almost all $x' \in \mathbb{R}^{n-1}$. For such $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\log^+ |\varphi(t, x')|}{1+t^2} dt \\ & \leq \int_{-\infty}^{\infty} \frac{\log^+ (g^{\pm 1}(t, x')|\varphi(t, x')|)}{1+t^2} dt + \int_{-\infty}^{\infty} \frac{\log^+ (g^{\mp 1}(t, x'))}{1+t^2} dt < \infty. \end{aligned}$$

Then

$$\log |\varphi(x_1 + iy_1, x')| \leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')|}{(t-x_1)^2 + y_1^2} dt, \quad x_1 \in \mathbb{R}, y_1 > 0$$

([19, Ch. III, G, 2], see also [21, Ch. V, Theorems 5 and 7]).

Applying (1) again, one gets

$$\begin{aligned} \log g(x) & \leq \log g(t, x') + \log g(x_1 - t, 0), \\ \log g(t, x') & \leq \log g(x) + \log g(t - x_1, 0) \quad \text{for all } x = (x_1, x') \in \mathbb{R}^n, t \in \mathbb{R}. \end{aligned}$$

The latter inequality can be rewritten as follows

$$\log g^{-1}(x) \leq \log g^{-1}(t, x') + \log g(t - x_1, 0).$$

Hence

$$\log g^{\pm 1}(x) \leq \log g^{\pm 1}(t, x') + \log g(\pm(x_1 - t), 0) \quad \text{for all } x = (x_1, x') \in \mathbb{R}^n, t \in \mathbb{R},$$

and

$$\begin{aligned} \log (|\varphi(x_1 + iy_1, x')|g^{\pm 1}(x)) & \leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')|}{(t-x_1)^2 + y_1^2} dt + \log g^{\pm 1}(x) \\ & = k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')| + \log g^{\pm 1}(x)}{(t-x_1)^2 + y_1^2} dt \\ & \leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log (|\varphi(t, x')|g^{\pm 1}(t, x'))}{(t-x_1)^2 + y_1^2} dt + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\pm(x_1 - t), 0)}{(t-x_1)^2 + y_1^2} dt \\ & = k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log (|\varphi(t, x')|g^{\pm 1}(t, x'))}{(t-x_1)^2 + y_1^2} dt + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau. \end{aligned}$$

If $y_1 \in [0, 1]$, then

$$\begin{aligned} \frac{y_1}{\pi} \int_0^\infty \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau & \leq M \frac{y_1}{\pi} \int_0^1 \frac{1}{\tau^2 + y_1^2} d\tau + \frac{y_1}{\pi} \int_1^\infty \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau \\ & \leq M \frac{y_1}{\pi} \int_{\mathbb{R}} \frac{1}{\tau^2 + y_1^2} d\tau + \frac{1}{\pi} \int_1^\infty \frac{\log g(\tau, 0)}{\tau^2} d\tau \leq M + \frac{I_g(1)}{\pi}. \end{aligned} \quad (24)$$

It follows from (14) that for $y_1 > 1$,

$$\frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau \leq y_1 S_g(y_1).$$

So,

$$\begin{aligned} \log (|\varphi(x_1 + iy_1, x')|g^{\pm 1}(x)) &\leq c_g + (k_\varphi + S_g(y_1))y_1 \\ &\quad + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log (|\varphi(t, x')|g^{\pm 1}(t, x'))}{(t - x_1)^2 + y_1^2} dt, \end{aligned}$$

where $c_g := M + \frac{I_g(1)}{\pi}$. Using Jensen's inequality, one gets

$$|\varphi(x_1 + iy_1, x')|g^{\pm 1}(x) \leq C_g e^{(k_\varphi + S_g(y_1))y_1} \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{|\varphi(t, x')|g^{\pm 1}(t, x')}{(t - x_1)^2 + y_1^2} dt,$$

where

$$C_g := e^{M + \frac{I_g(1)}{\pi}}. \quad (25)$$

Estimate (23) now follows from Young's convolution inequality and (21). \square

REMARK 3.2. Let $n = 1$, $g : \mathbb{R} \rightarrow [1, \infty)$ be a Hölder continuous submultiplicative function satisfying the Beurling-Domar condition, and let

$$\begin{aligned} w(x + iy) &:= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log g(t)}{(t - x)^2 + y^2} dt \\ &\quad + \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{x - t}{(t - x)^2 + y^2} + \frac{t}{t^2 + 1} \right) \log g(t) dt, \quad x \in \mathbb{R}, y > 0. \end{aligned}$$

Then $\varphi(z) := e^{w(z)}$ is analytic in z for $\text{Im } z > 0$ and continuous up to \mathbb{R} ,

$$|\varphi(x)| = e^{\text{Re}(w(x))} = e^{\log g(x)} = g(x), \quad x \in \mathbb{R}$$

(see, e.g., [8, Ch. III, §1]), and

$$|\varphi(iy)| = e^{\text{Re}(w(iy))} = \exp \left(\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log g(t)}{t^2 + y^2} dt \right) = e^{S_g(y)y}, \quad y > 0.$$

So,

$$k_\varphi = \limsup_{0 < y \rightarrow \infty} \frac{\log |\varphi(iy)|}{y} = \limsup_{y \rightarrow \infty} S_g(y) = 0$$

(see (15)), and

$$\begin{aligned} \|\varphi(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R})} &\geq \frac{|\varphi(iy)|}{g(0)} \geq |\varphi(iy)| = e^{S_g(y)y} = e^{S_g(y)y} \|1\|_{L^\infty(\mathbb{R})} \\ &= e^{S_g(y)y} \|g^{-1}\varphi\|_{L^\infty(\mathbb{R})} = e^{S_g(y)y} \|\varphi\|_{L_{g^{-1}}^\infty(\mathbb{R})}, \end{aligned}$$

which shows that the factor $e^{S_g(y_1)y_1}$ in the right-hand side of (23) is optimal in this case.

Clearly,

$$S_{\check{g}} = S_g, \quad C_{\check{g}} = C_g, \quad (26)$$

where $\check{g}(x) := g(Ax)$ and $A \in O(n)$ is an arbitrary orthogonal matrix (see (14), (25) and (5)).

THEOREM 3.3. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition (3). Let $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$ be an entire function such that $\log |\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and that the restriction of φ to \mathbb{R}^n belongs to $L_{g^{\pm 1}}^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then for every multi-index $\alpha \in \mathbb{Z}_+^n$,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_\alpha e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (27)$$

where

$$\kappa_\varphi(\omega) := \sup_{x \in \mathbb{R}^n} \left(\limsup_{0 < t \rightarrow \infty} \frac{\log |\varphi(x + it\omega)|}{t} \right) < \infty, \quad \omega \in \mathbb{S}^{n-1}, \quad (28)$$

and the constant $C_\alpha \in (0, \infty)$ depends only on α and g .

PROOF. (Cf. the proof of Lemma 9.29 in [20].) Take any $y \in \mathbb{R}^n \setminus \{0\}$. There exist an orthogonal matrix $A \in O(n)$ such that $A\mathbf{e}_1 = \omega := y/|y|$ (see (7)). Let $\check{\varphi}(z) := \varphi(Az)$, $z \in \mathbb{C}^n$, and $\check{g}(x) := g(Ax)$, $x \in \mathbb{R}^n$. Then $\check{\varphi} : \mathbb{C}^n \rightarrow \mathbb{C}$ is an entire function, and one can apply to it Lemma 3.1 with \check{g} in place of g (see (26)).

For any $x \in \mathbb{R}^n$, one has $\varphi(x + iy) = \check{\varphi}(\tilde{x} + i|y|\mathbf{e}_1) = \check{\varphi}(\tilde{x}_1 + i|y|, \tilde{x}_2, \dots, \tilde{x}_n)$, where $\tilde{x} := A^{-1}x$. Hence

$$\begin{aligned} \|\varphi(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} &= \|\check{\varphi}(\cdot + i|y|\cdot)\|_{L_{\check{g}^{\pm 1}}^p(\mathbb{R}^n)} \leq C_{\check{g}} e^{(k_{\check{\varphi}} + S_{\check{g}}(|y|))|y|} \|\check{\varphi}\|_{L_{\check{g}^{\pm 1}}^p(\mathbb{R}^n)} \\ &\leq C_g e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\check{\varphi}\|_{L_{\check{g}^{\pm 1}}^p(\mathbb{R}^n)} = C_g e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \end{aligned}$$

(see (26)), which proves (27) for $\alpha = 0$ and $y \neq 0$. This estimate is trivial for $\alpha = 0$ and $y = 0$.

Iterating the standard Cauchy integral formula for one complex variable, one gets

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)} \left(\prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \cdots d\theta_n, \\ \zeta &\in \Delta(z) := \{\eta \in \mathbb{C}^n : |\eta_k - z_k| < 1, k = 1, \dots, n\}, \quad z \in \mathbb{C}^n \end{aligned}$$

(cf. [20, Ch. 1, §1]), which implies

$$\partial^\alpha \varphi(\zeta) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)^{\alpha_k + 1}} \left(\prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \cdots d\theta_n.$$

Hence

$$\partial^\alpha \varphi(z) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n e^{i\alpha_k \theta_k}} d\theta_1 \cdots d\theta_n,$$

and

$$|\partial^\alpha \varphi(z)| \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})| d\theta_1 \cdots d\theta_n. \quad (29)$$

Since $g \geq 1$ is locally bounded,

$$1 \leq M_1 := \sup_{|s_k| \leq 1, k=1, \dots, n} g(s) < \infty.$$

Then it follows from (1) that

$$g^{\pm 1}(x_1 - \cos \theta_1, \dots, x_n - \cos \theta_n) \leq M_1 g^{\pm 1}(x). \quad (30)$$

According to the conditions of the theorem, there exists a constant $c_\varphi \in (0, \infty)$ such that $\log |\varphi(\zeta)| \leq c_\varphi |\zeta|$ for $|\zeta|$ large. Then $\kappa_\varphi(\omega) \leq c_\varphi$ (see (28)). Let $\varphi_y := \varphi(\cdot + iy)$, $y = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$. Then, similarly to the above inequality, $\kappa_{\varphi_y}(\omega) \leq c_\varphi$. Applying (27) with $\alpha = 0$ to the function φ_y in place of φ and using (16), (30), one derives from (29)

$$\begin{aligned} & \|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \|\varphi(\cdot + iy_1 + e^{i\theta_1}, \dots, \cdot + iy_n + e^{i\theta_n})\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} d\theta_1 \cdots d\theta_n \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 \|\varphi(\cdot + iy_1 + i \sin \theta_1, \dots, \cdot + iy_n + i \sin \theta_n)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} d\theta_1 \cdots d\theta_n \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} d\theta_1 \cdots d\theta_n \\ & = \alpha! M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}. \end{aligned}$$

Applying (27) with $\alpha = 0$ again, one gets

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq \alpha! M_1 C_0^2 e^{(c_\varphi + S_g(1))\sqrt{n}} e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}.$$

□

COROLLARY 3.4. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition (3). Let $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$ be an entire function such that $\log |\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and that the restriction of φ to \mathbb{R}^n belongs to $L_{g^{\pm 1}}^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then for every multi-index $\alpha \in \mathbb{Z}_+^n$ and every $\varepsilon > 0$,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_{\alpha, \varepsilon} e^{(\kappa_\varphi(y/|y|) + \varepsilon)|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (31)$$

where κ_φ is defined by (28), and the constant $C_{\alpha, \varepsilon} \in (0, \infty)$ depends only on α , ε , and g .

PROOF. It follows from (15) that for every $\varepsilon > 0$, there exists c_ε such that

$$S_g(|y|)|y| \leq c_\varepsilon + \varepsilon|y| \quad \text{for all } y \in \mathbb{R}^n.$$

Hence (27) implies (31). □

4. Main results

We will use the notation $\tilde{g}(x) := g(-x)$, $x \in \mathbb{R}^n$.

Taking $y - x$ in place of y in (1) and rearranging, one gets

$$\frac{1}{g(x)} \leq \frac{g(y-x)}{g(y)}. \quad (32)$$

Using this inequality, one can easily show that $f * u \in L_{g^{-1}}^\infty(\mathbb{R}^n)$ for every $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$ and $u \in L_g^1(\mathbb{R}^n)$. The Fubini-Tonelli theorem implies that

$$f * (v * u) = (f * v) * u \quad \text{for all } f \in L_{g^{-1}}^\infty(\mathbb{R}^n) \quad \text{and } v, u \in L_g^1(\mathbb{R}^n). \quad (33)$$

THEOREM 4.1. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition (3), $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$, and Y be a linear subspace of $L_g^1(\mathbb{R}^n)$ such that*

$$f * v = 0 \quad \text{for every } v \in Y. \quad (34)$$

Suppose the set

$$Z(Y) := \bigcap_{v \in Y} \{\xi \in \mathbb{R}^n \mid \widehat{v}(\xi) = 0\} \quad (35)$$

is bounded, and $u \in L_g^1(\mathbb{R}^n)$ is such that $\widehat{u} = 1$ in a neighbourhood of $Z(Y)$. Then $f = f * u$.

PROOF. It is sufficient to show that

$$\langle f, h \rangle = \langle f * u, h \rangle \quad \text{for every } h \in L_g^1(\mathbb{R}^n). \quad (36)$$

Since the set of functions h with compactly supported Fourier transforms \widehat{h} is dense in $L_g^1(\mathbb{R}^n)$ (see [5, Theorems 1.52 and 2.11]), it is sufficient to prove (36) for such h . Further,

$$\langle f, h \rangle = (f * \widehat{h})(0).$$

So, it is sufficient to show that

$$f * w = f * u * w \quad (37)$$

for every $w \in L_g^1(\mathbb{R}^n)$ with compactly supported Fourier transform \widehat{w} . Take any such w and choose $R > 0$ such that the support of \widehat{w} lies in $B_R := \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$. It is clear that \tilde{g} satisfies the Beurling-Domar condition. Then there exists $u_R \in L_{\tilde{g}}^1(\mathbb{R}^n)$ such that $0 \leq \widehat{u}_R \leq 1$, $\widehat{u}_R(\xi) = 1$ for $|\xi| \leq R$, and $\widehat{u}_R(\xi) = 0$ for $|\xi| \geq R + 1$ (see [5, Lemma 1.24]).

Let V be an open neighbourhood of $Z(Y)$ such that $\widehat{u} = 1$ in V . Similarly to the above, there exists $u_0 \in L_g^1(\mathbb{R}^n)$ such that $0 \leq \widehat{u}_0 \leq 1$, $\widehat{u}_0 = 1$ in a neighbourhood $V_0 \subset V$ of $Z(Y)$, and $\widehat{u}_0 = 0$ outside V (see [5, Lemma 1.24]).

Since Y is a linear subspace, for every $\eta \in B_{R+1} \setminus V_0 \subset \mathbb{R}^n \setminus Z(Y)$, there exists $v_\eta \in Y$ such that $\widehat{v}_\eta(\eta) = 1$. Since $v_\eta \in L^1(\mathbb{R}^n)$, \widehat{v}_η is continuous, and there is a neighbourhood V_η of η such that $|\widehat{v}_\eta(\xi) - 1| < 1/2$ for all $\xi \in V_\eta$.

Similarly to the above, there exists $u_\eta \in L_g^1(\mathbb{R}^n)$ such that $\operatorname{Re}(\widehat{v}_\eta \widehat{u}_\eta) \geq 0$, and $\operatorname{Re}(\widehat{v}_\eta \widehat{u}_\eta) > \frac{1}{2}$ in a neighbourhood $V_\eta^0 \subset V_\eta$ of η .

Since $B_{R+1} \setminus V_0$ is compact, its open cover $\{V_\eta^0\}_{\eta \in B_{R+1} \setminus V_0}$ has a finite subcover. So, there exist functions $v_j \in Y$ and $u_j \in L_g^1(\mathbb{R}^n)$, $j = 1, \dots, N$ such that

$$\operatorname{Re}(\sigma) > \frac{1}{2}, \quad \text{where} \quad \sigma := \widehat{u}_0 + \sum_{j=1}^N \widehat{v}_j \widehat{u}_j + 1 - \widehat{u}_R.$$

Then there exists $v \in L_g^1(\mathbb{R}^n)$ such that $\widehat{v} = 1/\sigma$ (see [5, Theorem 1.53]).

Since $\widehat{u}_0(1 - \widehat{u}) = 0$ and $(1 - \widehat{u}_R)\widehat{w} = 0$, one has

$$\begin{aligned} \left(\widehat{u} + \sum_{j=1}^N \widehat{v}_j \widehat{u}_j \widehat{v} (1 - \widehat{u}) \right) \widehat{w} &= (\widehat{u} + (\sigma - (\widehat{u}_0 + 1 - \widehat{u}_R)) \widehat{v} (1 - \widehat{u})) \widehat{w} \\ &= (\widehat{u} + (1 - \widehat{u}) - (\widehat{u}_0 + 1 - \widehat{u}_R) \widehat{v} (1 - \widehat{u})) \widehat{w} = (1 - (1 - \widehat{u}_R) \widehat{v} (1 - \widehat{u})) \widehat{w} \\ &= \widehat{w} - (1 - \widehat{u}_R) \widehat{w} \widehat{v} (1 - \widehat{u}) = \widehat{w}. \end{aligned}$$

It now follows from (33) and (34) that

$$\begin{aligned} f * w &= f * \left(u + \sum_{j=1}^N v_j * u_j * (v - v * u) \right) * w \\ &= f * u * w + f * \left(\sum_{j=1}^N v_j * u_j * (v - v * u) \right) * w \\ &= f * u * w + \sum_{j=1}^N (f * v_j) * u_j * (v - v * u) * w = f * u * w. \end{aligned}$$

□

COROLLARY 4.2. *If $Z(Y) = \emptyset$ in Theorem 4.1, then $f = 0$.*

PROOF. It is sufficient to show that

$$f * w = 0$$

for every $w \in L_g^1(\mathbb{R}^n)$ with compactly supported Fourier transform \widehat{w} (see the beginning of the proof of Theorem 4.1). Take $u \in L_g^1(\mathbb{R}^n)$ is such that $\widehat{u} = 1$ in an open set, and the support of \widehat{u} does not intersect that of \widehat{w} . The latter condition implies that $u * w = 0$. Since $Z(Y) = \emptyset$, it follows from Theorem 4.1 that $f = f * u$. Hence,

$$f * w = (f * u) * w = f * (u * w) = f * 0 = 0$$

(see (33)).

□

For a bounded set $E \subset \mathbb{R}^n$, let $\text{conv}(E)$ denote its closed convex hull and H_E denote its support function:

$$H_E(y) := \sup_{\xi \in E} y \cdot \xi = \sup_{\xi \in \text{conv}(E)} y \cdot \xi, \quad y \in \mathbb{R}^n.$$

Clearly, H_E is positively homogeneous and convex:

$$H_E(\tau y) = \tau H_E(y), \quad H_E(y + x) \leq H_E(y) + H_E(x) \\ \text{for all } y, x \in \mathbb{R}^n, \tau \geq 0.$$

For every positively homogeneous convex function H ,

$$K := \{\xi \in \mathbb{R}^n \mid y \cdot \xi \leq H(y) \text{ for all } y \in \mathbb{R}^n\} \quad (38)$$

is the unique convex compact set such that $H_K = H$ (see, e.g., [14, Theorem 4.3.2]).

THEOREM 4.3. *Let g , f , and Y satisfy the conditions of Theorem 4.1, and let*

$$\mathcal{H}_Y(y) := H_{Z(Y)}(-y) = \sup_{\xi \in Z(Y)} (-y) \cdot \xi = - \inf_{\xi \in Z(Y)} y \cdot \xi, \quad y \in \mathbb{R}^n. \quad (39)$$

Then f admits analytic continuation to an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{\mathcal{H}_Y(y) + S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n \quad (40)$$

(see (14), (15)), where the constant $C_\alpha \in (0, \infty)$ depends only on α and g .

PROOF. Take any $\varepsilon > 0$. There exists $u \in L_g^1(\mathbb{R}^n)$ such that $\widehat{u} = 1$ in a neighbourhood of $Z(Y)$, and $\widehat{u} = 0$ outside the $\frac{\varepsilon}{2}$ -neighbourhood of $Z(Y)$ (see [5, Lemma 1.24]). It follows from the Paley–Wiener–Schwartz theorem (see, e.g., [14, Theorem 7.3.1]) that $u = \mathcal{F}^{-1}\widehat{u}$ admits analytic continuation to an entire function $u : \mathbb{C}^n \rightarrow \mathbb{C}$ satisfying the estimate

$$|u(x + iy)| \leq c_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|/2} \quad \text{for all } x, y \in \mathbb{R}^n$$

with some constant $c_\varepsilon \in (0, \infty)$. So, u satisfies the conditions of Corollary 3.4 with \tilde{g} in place of g , and

$$\|u(\cdot + iy)\|_{L_{\tilde{g}}^1(\mathbb{R}^n)} \leq C_{0, \varepsilon/2} e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|u\|_{L_{\tilde{g}}^1(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n. \quad (41)$$

Since

$$f(x) = \int_{\mathbb{R}^n} u(x - s) f(s) ds$$

(see Theorem 4.1), f admits analytic continuation

$$f(x + iy) := \int_{\mathbb{R}^n} u(x + iy - s) f(s) ds$$

(see Corollary 3.4), and

$$\|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq \|u(\cdot + iy)\|_{L_{\tilde{g}}^1(\mathbb{R}^n)} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \\ \leq C_{0, \varepsilon/2} e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|u\|_{L_{\tilde{g}}^1(\mathbb{R}^n)} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} =: M_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}$$

(see (32)). Since

$$\frac{|f(x+iy)|}{g(x)} \leq M_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)},$$

one has $\log |f(x+iy)| = O(|x+iy|)$ for $|x+iy|$ large (see (2)), and

$$\begin{aligned} \limsup_{0 < t \rightarrow \infty} \frac{\log |f(x+it\omega)|}{t} &\leq \limsup_{0 < t \rightarrow \infty} \frac{\log \left(M_\varepsilon g(x) \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \right) + t\mathcal{H}_Y(\omega) + \varepsilon t}{t} \\ &= \mathcal{H}_Y(\omega) + \varepsilon. \end{aligned}$$

Hence,

$$\kappa_f(\omega) := \sup_{x \in \mathbb{R}^n} \left(\limsup_{0 < t \rightarrow \infty} \frac{\log |f(x+it\omega)|}{t} \right) \leq \mathcal{H}_Y(\omega) + \varepsilon$$

for every $\varepsilon > 0$, i.e.

$$\kappa_f(\omega) \leq \mathcal{H}_Y(\omega).$$

So, (40) follows from Theorem 3.3. \square

THEOREM 4.4. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable sub-multiplicative function satisfying the Beurling-Domar condition (3) and let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \tilde{m}(D)\phi := \mathcal{F}^{-1}(\tilde{m}\widehat{\phi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L_g^1(\mathbb{R}^n)$. Suppose $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e.

$$\langle f, \tilde{m}(D)\phi \rangle = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n). \quad (42)$$

If $K := \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$ is compact, then f admits analytic continuation to an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{H(y)+S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n \quad (43)$$

(see (14), (15)), where where $H(y) := H_K(-y)$, and the constant $C_\alpha \in (0, \infty)$ depends only on α and g .

Conversely, if every $f \in L^\infty(\mathbb{R}^n)$ satisfying (42) admits analytic continuation to an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that

$$\|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq M_\varepsilon e^{H(y)+\varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (44)$$

holds for every $\varepsilon > 0$ with a constant $M_\varepsilon \in (0, \infty)$ that depends only on ε , m , and g , then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq K$, where K is the unique convex compact set such that $H_K(y) = H(-y)$ (cf. (38)).

PROOF. Let

$$(T_v\phi)(x) := \phi(x-v), \quad x, v \in \mathbb{R}^n.$$

Since $T_v\phi \in C_c^\infty(\mathbb{R}^n)$ for every $\phi \in C_c^\infty(\mathbb{R}^n)$ and all $v \in \mathbb{R}^n$, it follows from (42) that

$$\left(f * \widetilde{\tilde{m}(D)\phi} \right) (v) = \langle f, T_v\tilde{m}(D)\phi \rangle = \langle f, \tilde{m}(D)(T_v\phi) \rangle = 0 \quad \text{for all } v \in \mathbb{R}^n.$$

Hence

$$f * \widetilde{m}(D)\phi = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

It is easy to see that

$$\begin{aligned} & \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid \widetilde{m}(D)\phi(\eta) = 0 \right\} = \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid \widetilde{m}(D)\phi(-\eta) = 0 \right\} \\ & = \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid m(\eta)\widehat{\phi}(-\eta) = 0 \right\} = \{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} = K. \end{aligned}$$

Applying Theorem 4.3 with

$$Y := \left\{ \widetilde{m}(D)\phi \mid \phi \in C_c^\infty(\mathbb{R}^n) \right\} \subset L_g^1(\mathbb{R}^n)$$

and $Z(Y) = K$, one gets (43).

For the converse direction, we assume the contrary, i.e. that the zero-set $\{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \}$ contains some $\gamma \notin K$ (see (38)). Then there exists $y_0 \in \mathbb{R}^n \setminus \{0\}$ such that $y_0 \cdot \gamma > H_K(y_0) = H(-y_0)$. It is easy to see that $f(x) := e^{ix \cdot \gamma}$ satisfies $m(D)e^{ix \cdot \gamma} = e^{ix \cdot \gamma}m(\gamma) = 0$ for all $x \in \mathbb{R}^n$. Take $\varepsilon < (y_0 \cdot \gamma - H(-y_0))/|y_0|$. Clearly, $f \in L^\infty(\mathbb{R}^n)$, and

$$\begin{aligned} & \frac{\|f(\cdot - i\tau y_0)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}}{e^{H(-\tau y_0) + \varepsilon|\tau y_0|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}} = \frac{e^{\tau(y_0 \cdot \gamma)}}{e^{\tau(H(-y_0) + \varepsilon|y_0|)}} \\ & = e^{\tau(y_0 \cdot \gamma - H(-y_0) - \varepsilon|y_0|)} \rightarrow \infty \quad \text{as } \tau \rightarrow \infty. \end{aligned}$$

So, f does not satisfy (44). \square

COROLLARY 4.5. *Let $g : \mathbb{R}^n \rightarrow [1, \infty)$ be a locally bounded, measurable sub-multiplicative function satisfying the Beurling-Domar condition (3) and let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \widetilde{m}(D)\phi := \mathcal{F}^{-1}(\widetilde{m}\widehat{\phi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L_g^1(\mathbb{R}^n)$. Suppose $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e. (42) holds. If $\{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} = \{0\}$, then f admits analytic continuation to an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (45)$$

where the constant $C_\alpha \in (0, \infty)$ depends only on α and g . If $\{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} = \emptyset$, then $f = 0$.

Conversely, if every $f \in L^\infty(\mathbb{R}^n)$ satisfying (42) admits analytic continuation to an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that

$$\|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq M_\varepsilon e^{\varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (46)$$

holds for every $\varepsilon > 0$ with a constant $M_\varepsilon \in (0, \infty)$ that depends only on ε , m , and g , then $\{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} \subseteq \{0\}$.

PROOF. The only part that does not follow immediately from Theorem 4.4 is that $f = 0$ in the case $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$. In this case, one can take the same Y as in the proof of Theorem 4.4, note that $Z(Y) = \emptyset$ and apply Corollary 4.2 to conclude that $f = 0$. (It is instructive to compare this result to [17, Proposition 2.2].) \square

REMARK 4.6. The condition that $\tilde{m}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ to $L_g^1(\mathbb{R}^n)$ is satisfied if m is a linear combination of terms of the form ab , where $a = F\mu$, μ is a finite complex Borel measure on \mathbb{R}^n such that

$$\int_{\mathbb{R}^n} \tilde{g}(y) |\mu|(dy) < \infty,$$

and b is the Fourier transform of a compactly supported distribution. Indeed, it is easy to see that $\tilde{b}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into itself, while the convolution operator $\phi \mapsto \tilde{\mu} * \phi$ maps $C_c^\infty(\mathbb{R}^n)$ to $L_g^1(\mathbb{R}^n)$.

REMARK 4.7. We are mostly interested in super-polynomially growing weights here as polynomially growing ones have been dealt with in our previous paper [3]. Nevertheless, it is instructive to look at the behaviour of the factor $e^{S_g(|y|)|y|}$ for typical super-polynomially, polynomially, and sub-polynomially growing weights.

It follows from (20) that if $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma > 1$, then there exists a constant C_γ such that

$$\begin{aligned} e^{S_g(|y|)|y|} &\leq C_\gamma e^{\frac{1}{\pi}|y|\log^{-\gamma}(e+|y|)(1+\frac{2}{\gamma-1}\log(e+|y|))} \\ &= C_\gamma \left(e^{|y|/\log^\gamma(e+|y|)} \right)^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))} = C_\gamma (g(y))^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))}. \end{aligned}$$

Similarly, if $g(x) = e^{a|x|^b}$, $a \geq 0$, $b \in [0, 1)$, then (19) implies

$$e^{S_g(|y|)|y|} = e^{a|y|^b(\sin(\frac{1-b}{2}\pi))^{-1}} = (g(y))^{(\sin(\frac{1-b}{2}\pi))^{-1}}. \quad (47)$$

If $g(x) = (1 + |x|)^s$, $s \geq 0$, then (17) implies

$$e^{S_g(|y|)|y|} \leq e^{c_1 s + s \log(1+|y|)} = C_s (1 + |y|)^s = C_s g(y). \quad (48)$$

Finally, if $g(x) = (\log(e + |x|))^t$, $t \geq 0$, then (18) implies

$$e^{S_g(|y|)|y|} \leq e^{c_2 t + t \log \log(e+|y|)} = C_t (\log(e + |y|))^t = C_t g(y).$$

REMARK 4.8. If g is polynomially bounded in Corollary 4.5, then it follows from (45) and (48) that f is a polynomially bounded entire function on \mathbb{C}^n and hence a polynomial (see, e.g., [20, Corollary 1.7]). The fact that f is a polynomial in this case was established in [3] and [11].

REMARK 4.9. Let $n = 2$, $g(x) := (1 + |x|)^k$, $k \in \mathbb{N}$, $f(x_1, x_2) := (x_1 + ix_2)^k$ (or $f(x_1, x_2) := (x_1 + ix_2)^k + (x_1 - ix_2)^k$ if one prefers to have a real-valued f). Then $f \in L_{g^{-1}}^\infty(\mathbb{R}^2)$, $\Delta f = 0$, $f(x + iy_1 \mathbf{e}_1) = (x_1 + iy_1 + ix_2)^k$ for any $y_1 \in \mathbb{R}$ (see (7)), and

$$\frac{\|f(\cdot + iy_1 \mathbf{e}_1)\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)}}{g(y_1 \mathbf{e}_1)} \geq \frac{|y_1|^k}{(1 + |y_1|)^k} \rightarrow 1 = \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)} \quad \text{as } |y_1| \rightarrow \infty.$$

So, the factor $e^{S_g(|y|)|y|} \leq C_k g(y)$ (see (48)) is optimal in (45) in this case.

The case $g(x) = e^{a|x|^b}$, $a > 0$, $b \in [0, 1]$ is perhaps more interesting. Let us take $b = \frac{1}{2}$. Then it follows from (47) that $e^{S_g(|y|)|y|} = (g(y))^{\sqrt{2}}$. Let us show that one cannot replace this factor in (45) with $(g(y))^{\sqrt{2}(1-\varepsilon)}$, $\varepsilon > 0$. Take any $\varepsilon > 0$. Since

$$\sqrt[4]{1 + \tau^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau}\right) \rightarrow \frac{1}{\sqrt{2}} \quad \text{as } \tau \rightarrow 0, \tau > 0,$$

there exists $\tau_\varepsilon > 0$ such that

$$\sqrt[4]{1 + \tau_\varepsilon^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \leq \frac{1 + \varepsilon}{\sqrt{2}}.$$

Let us estimate $\operatorname{Re} \sqrt{x_1 + i\kappa x_2}$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $\kappa > 0$ is a constant to be chosen later, and $\sqrt{\cdot}$ is the branch of the square root that is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive on $(0, +\infty)$. If $x_1 \geq \tau_\varepsilon \kappa |x_2|$, then

$$\begin{aligned} \operatorname{Re} \sqrt{x_1 + i\kappa x_2} &\leq |\sqrt{x_1 + i\kappa x_2}| = \sqrt[4]{x_1^2 + \kappa^2 x_2^2} \leq \sqrt[4]{\left(1 + \frac{1}{\tau_\varepsilon^2}\right) x_1^2} \\ &\leq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{x_1} \leq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{|x|}. \end{aligned}$$

If $0 < x_1 < \tau_\varepsilon \kappa |x_2|$, then

$$\begin{aligned} \operatorname{Re} \sqrt{x_1 + i\kappa x_2} &\leq |\sqrt{x_1 + i\kappa x_2}| \cos\left(\frac{1}{2} \arctan \frac{\kappa |x_2|}{x_1}\right) \\ &\leq \left|\sqrt{\tau_\varepsilon \kappa |x_2| + i\kappa x_2}\right| \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \\ &= \kappa^{1/2} |x_2|^{1/2} \sqrt[4]{1 + \tau_\varepsilon^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \leq \frac{1 + \varepsilon}{\sqrt{2}} \kappa^{1/2} |x|^{1/2}. \end{aligned}$$

Now, take $\kappa_\varepsilon \geq 1$ such that

$$\frac{1 + \varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} \geq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4}.$$

Then

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leq \frac{1 + \varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2} \quad (49)$$

for $x_1 > 0$. If $x_1 \leq 0$, then the argument of $\sqrt{x_1 + i\kappa_\varepsilon x_2}$ belongs to $\pm[\pi/4, \pi/2]$, depending on the sign of x_2 . Hence

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leq |\sqrt{x_1 + i\kappa_\varepsilon x_2}| \cos \frac{\pi}{4} \leq \frac{1}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2},$$

and (49) holds for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Since the Taylor series of $\cos w$ contains only even powers of w , $\cos(i\sqrt{z})$ is an analytic function of $z \in \mathbb{C}$. So, $\cos(i\sqrt{x_1 + ix_2})$ is a harmonic function of

$x = (x_1, x_2) \in \mathbb{R}^2$. Hence $f(x_1, x_2) := \cos(i\sqrt{x_1 + i\kappa_\varepsilon x_2})$ is a solution of the elliptic partial differential equation

$$\left(\partial_{x_1}^2 + \frac{1}{\kappa_\varepsilon^2} \partial_{x_2}^2 \right) f(x_1, x_2) = 0.$$

It follows from (49) that

$$|f(x_1, x_2)| \leq \frac{1}{2} \left(1 + e^{\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2}} \right) \leq e^{\frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2}}.$$

So, $f \in L_{g^{-1}}^\infty(\mathbb{R}^2)$, where $g(x) = e^{a|x|^{1/2}}$ with $a = \frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2}$. Clearly, the analytic continuation of f to \mathbb{C}^2 is given by the formula

$$f(x_1 + iy_1, x_2 + iy_2) = \cos \left(i\sqrt{x_1 + iy_1 + i\kappa_\varepsilon(x_2 + iy_2)} \right).$$

Then (see (7))

$$\begin{aligned} \frac{\|f(\cdot + iy_2 \mathbf{e}_2)\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)}}{(g(y_2 \mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} &\geq \frac{|f(0 + iy_2 \mathbf{e}_2)|}{g(0)(g(y_2 \mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} = \frac{|\cos(i\sqrt{-\kappa_\varepsilon y_2})|}{e^{\sqrt{2}(1-\varepsilon)\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} \\ &\geq \frac{e^{\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2e^{(1-\varepsilon^2)\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} = \frac{e^{\varepsilon^2 \kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2} \rightarrow \infty \quad \text{as } y_2 \rightarrow -\infty. \end{aligned}$$

5. Concluding remarks

Corollary 4.5 shows that sub-exponentially growing solutions of $m(D)f = 0$ admit analytic continuation to entire functions on \mathbb{C}^n . It is well known that no growth restrictions are necessary in the case when $m(D)$ is an elliptic partial differential operator with constant coefficients, and every solution of $m(D)f = 0$ in \mathbb{R}^n admits analytic continuation to an entire function on \mathbb{C}^n (see [22], [6]).

REMARK 5.1. The latter result has a local version similar to Hayman's theorem on harmonic functions (see [12, Theorem 1]) : for every elliptic partial differential operator $m(D)$ with constant coefficients there exists a constant $c_m \in (0, 1)$ such that every solution of $m(D)f = 0$ in the ball $\{x \in \mathbb{R}^n : |x| < R\}$ of any radius $R > 0$ admits continuation to an analytic function in the ball $\{x \in \mathbb{C}^n : |x| < c_m R\}$. Indeed, let $m_0(D) = \sum_{|\alpha|=N} a_\alpha D^\alpha$ be the principal part of $m(D) = \sum_{|\alpha| \leq N} a_\alpha D^\alpha$. There exists $C_m > 0$ such that

$$\sum_{|\alpha|=N} a_\alpha (a + ib)^\alpha = 0, \quad a, b \in \mathbb{R}^n \quad \implies \quad |a| \geq C_m |b|$$

(see, e.g., [25, §7]). Then the same argument as in the proof of [18, Corollary 8.2] shows that f admits continuation to an analytic function in the ball $\{x \in \mathbb{C}^n : |x| < (1 + C_m^{-2})^{-1/2} R\}$. Note that in the case of the Laplacian, one can take $C_m = 1$ and $c_m = (1 + C_m^{-2})^{-1/2} = \frac{1}{\sqrt{2}}$, which is the optimal constant for harmonic functions (see [12]).

Let us return to equations in \mathbb{R}^n . Below, $m(\xi)$ will always denote a polynomial with $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} \subseteq \{0\}$. For non-elliptic partial differential operators $m(D)$, one needs to place growth restrictions on solutions of $m(D)f = 0$ to make sure that they admit analytic continuation to entire functions on \mathbb{C}^n .

We say that a function f defined on \mathbb{R}^n (or \mathbb{C}^n) is of *infra-exponential* growth if for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(z)| \leq C_\varepsilon e^{\varepsilon|z|} \quad \text{for all } z \in \mathbb{R}^n \text{ (} z \in \mathbb{C}^n \text{)}.$$

Let $\mu : [0, \infty) \rightarrow [0, \infty)$ be an increasing to infinity function such that

$$\mu(t) \leq At + B, \quad t \geq 0$$

for some $A, B > 0$, and

$$\int_1^\infty \frac{\mu(t)}{t^2} dt < \infty. \quad (50)$$

Suppose $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} = \{0\}$. Then, under additional restrictions on μ , every solution f of $m(D)f = 0$ that has growth $O(e^{\varepsilon\mu(|x|)})$ for every $\varepsilon > 0$ admits analytic continuation to an entire function of infra-exponential growth on \mathbb{C}^n (see [17]). It is easy to see that (50) is equivalent to the Beurling-Domar condition (3) for $g(x) := e^{\mu(|x|)}$.

One cannot replace $O(e^{\varepsilon\mu(|x|)})$ with $O(e^{\varepsilon|x|})$ in the above result without placing a restriction on the complex zeros of m . If there exists $\delta > 0$ such that $m(\zeta)$ has no complex zeros in

$$|\operatorname{Im} \zeta| < \delta, \quad |\operatorname{Re} \zeta| > \delta^{-1}, \quad (51)$$

then every solution of $m(D)f = 0$ that, together with its partial derivatives up to the order of $m(D)$, is of infra-exponential growth on \mathbb{R}^n , admits analytic continuation to an entire function of infra-exponential growth on \mathbb{C}^n (see [16], [17]).

On the other hand, if for every $\delta > 0$, (51) contains complex zeros of $m(\zeta)$, then $m(D)f = 0$ has a solution in C^∞ all of whose derivatives are of infra-exponential growth, but which is not entire infra-exponential in \mathbb{C}^n . The proof of the latter result in [16], [17] is not constructive, and the author writes: “Unfortunately we cannot present concrete examples of such” solutions. However, it is not difficult to construct, for any $\varepsilon > 0$, a solution in C^∞ all of whose derivatives have growth $O(e^{\varepsilon|x|})$, but which is not real-analytic. Indeed, according to the assumption, there exist complex zeros

$$\zeta_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}^n, \quad k \in \mathbb{N}$$

of $m(\zeta)$ such that

$$|\eta_k| < k^{-1}, \quad |\xi_k| > k. \quad (52)$$

Choosing a subsequence, we can assume that $\omega_k := |\xi_k|^{-1}\xi_k$ converge to a point $\omega_0 \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ as $k \rightarrow \infty$, and that $|\omega_k - \omega_0| < 1$ for all

$k \in \mathbb{N}$. Then

$$\omega_k \cdot \omega_0 = \frac{|\omega_k|^2 + |\omega_0|^2 - |\omega_k - \omega_0|^2}{2} > \frac{1 + 1 - 1}{2} = \frac{1}{2}, \quad k \in \mathbb{N}. \quad (53)$$

Consider

$$f(x) := \sum_{k > \varepsilon^{-1}} \frac{e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} = \sum_{k > \varepsilon^{-1}} \frac{e^{i\xi_k \cdot x - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}}, \quad x \in \mathbb{R}^n. \quad (54)$$

Then, for every multi-index α ,

$$\begin{aligned} |\partial^\alpha f(x)| &= \left| \sum_{k > \varepsilon^{-1}} \frac{(i\zeta_k)^\alpha e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} \right| \leq \sum_{k > \varepsilon^{-1}} \frac{(|\xi_k| + 1)^{|\alpha|} e^{|\eta_k||x|}}{e^{|\xi_k|^{1/2}}} \\ &\leq e^{\varepsilon|x|} \sum_{k > \varepsilon^{-1}} \frac{(|\xi_k| + 1)^{|\alpha|}}{e^{|\xi_k|^{1/2}}} =: C_\alpha e^{\varepsilon|x|}, \quad x \in \mathbb{R}^n \end{aligned}$$

(see (52)). Further,

$$m(D)f(x) = \sum_{k > \varepsilon^{-1}} \frac{m(\zeta_k) e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} = 0.$$

On the other hand, f is not real-analytic. Before we prove this, note that formally putting $x - it\omega_0$, $t > 0$ in place of x in the right-hand side of (54), one gets a divergent series. Indeed, its terms can be estimated as follows

$$\left| \frac{e^{i\xi_k \cdot x + t\xi_k \cdot \omega_0 - \eta_k \cdot x + it\eta_k \cdot \omega_0}}{e^{|\xi_k|^{1/2}}} \right| = \frac{e^{t|\xi_k|\omega_k \cdot \omega_0 - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}} \geq e^{-\varepsilon|x|} \frac{e^{t|\xi_k|/2}}{e^{|\xi_k|^{1/2}}} \rightarrow \infty$$

as $k \rightarrow \infty$ (see (52), (53)).

For any $j > \varepsilon^{-1}$, there exists $\ell_j \in \mathbb{N}$ such that

$$\ell_j \leq |\xi_j|^{1/2} < \ell_j + 1. \quad (55)$$

It is clear that $\ell_j \rightarrow \infty$ as $j \rightarrow \infty$ (see (52)). Note that

$$|\arg(\omega_0 \cdot \zeta_k)| \leq \frac{|\omega_0 \cdot \eta_k|}{|\omega_0 \cdot \xi_k|} \leq \frac{2}{k|\xi_k|}.$$

If $|\xi_k| \geq \frac{6\ell_j}{\pi k}$, then

$$|\arg(\omega_0 \cdot \zeta_k)^{\ell_j}| \leq \frac{2\ell_j}{k|\xi_k|} \leq \frac{\pi}{3},$$

and

$$\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j} \geq \frac{1}{2} |\omega_0 \cdot \zeta_k|^{\ell_j} \geq \frac{1}{2^{\ell_j+1}} |\xi_k|^{\ell_j}.$$

Clearly, $|\xi_j| \geq \frac{6\ell_j}{\pi j}$ for sufficiently large j (see (55)). Hence, one has the following estimate for the directional derivative ∂_{ω_0}

$$|((-i\partial_{\omega_0})^{\ell_j} f)(0)| \geq \sum_{k > \varepsilon^{-1}} \frac{\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j}}{e^{|\xi_k|^{1/2}}}$$

$$\begin{aligned}
&\geq - \sum_{k>\varepsilon^{-1}, |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{|\zeta_k|^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \sum_{k>\varepsilon^{-1}, |\xi_k|\geq\frac{6\ell_j}{\pi k}} \frac{|\xi_k|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_k|^{1/2}}} \\
&\geq - \sum_{k>\varepsilon^{-1}, |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{(|\xi_k| + \frac{1}{k})^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \frac{|\xi_j|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_j|^{1/2}}} \\
&\geq - \sum_{k>\varepsilon^{-1}, |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{1}{e^{|\xi_k|^{1/2}}} \left(\frac{10\ell_j}{\pi k}\right)^{\ell_j} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{(\ell_j^2+1)^{1/2}}} \\
&\geq -(10\ell_j)^{\ell_j} \sum_{k=1}^{\infty} \frac{1}{e^{|\xi_k|^{1/2}} k^2} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{\ell_j+1}} = -C(10\ell_j)^{\ell_j} + (2e)^{-(\ell_j+1)} \ell_j^{2\ell_j}.
\end{aligned}$$

Hence

$$|((-i\partial_{\omega_0})^{\ell_j} f)(0)| \geq \ell_j^{\frac{3}{2}\ell_j}$$

for all sufficiently large j , which means that f is not real-analytic in a neighbourhood of 0.

The operator $m(D)$ in the previous example is not hypoelliptic. If $m(D)$ is hypoelliptic, then every solution of $m(D)f = 0$, such that $|f(x)| \leq Ae^{a|x|}$, $x \in \mathbb{R}^n$, for some constants $A, a > 0$, admits analytic continuation to an entire function of order one on \mathbb{C}^n (see [10, §4, Corollary 2]). For elliptic operators, this result can be strengthened: every solution of $m(D)f = 0$, such that $|f(x)| \leq Ae^{a|x|^\beta}$, $x \in \mathbb{R}^n$, for $\beta \geq 1$ and some constants $A, a > 0$, admits analytic continuation to an entire function of order β on \mathbb{C}^n (see [10, §4, Corollary 3]). Let us show that for every $\beta > 1$ there exists a semi-elliptic operator $m(D)$ (see [15, Theorem 11.1.11]) and a C^∞ solution of $m(D)f = 0$, all of whose derivatives have growth $O(e^{a|x|^\beta})$, but which does not admit analytic continuation to an entire function on \mathbb{C}^n .

A simple example of such a semi-elliptic operator is $\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}$ with $\ell \in \mathbb{N}$ satisfying $1 + \frac{1}{2\ell} \leq \beta$, i.e. $\ell \geq \frac{1}{2(\beta-1)}$.

Let

$$f(x_1, x_2) := \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}x_1+kx_2}}{e^{k^{2\ell+1}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

If $x_2 > 0$, then the function $t \mapsto tx_2 - t^{2\ell+1}$ achieves maximum at $t = \left(\frac{x_2}{2\ell+1}\right)^{\frac{1}{2\ell}}$, and this maximum is equal to

$$2\ell \left(\frac{1}{2\ell+1}\right)^{1+\frac{1}{2\ell}} x_2^{1+\frac{1}{2\ell}} =: c_\ell x_2^{1+\frac{1}{2\ell}}.$$

Hence, for every multi-index α ,

$$|\partial^\alpha f(x_1, x_2)| \leq \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{kx_2 - k^{2\ell+1}}$$

$$\begin{aligned}
 &= \sum_{k=1}^{\left[\frac{1}{x_2^{2\ell}}\right]+1} k^{(2\ell+1)|\alpha|} e^{kx_2 - k^{2\ell+1}} + \sum_{k=\left[\frac{1}{x_2^{2\ell}}\right]+2}^{\infty} k^{(2\ell+1)|\alpha|} e^{k(x_2 - k^{2\ell})} \\
 &\leq \left(\left[\frac{1}{x_2^{2\ell}}\right] + 1\right)^{(2\ell+1)|\alpha|+1} e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{-k} \\
 &\leq 2^{(2\ell+1)|\alpha|+1} \left(x_2^{2|\alpha|+1} + 1\right) e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + c_{\ell,\alpha} \leq C_{\ell,\alpha} e^{(c_\ell+1)x_2^{1+\frac{1}{2\ell}}}.
 \end{aligned}$$

If $x_2 \leq 0$, then

$$|\partial^\alpha f(x_1, x_2)| \leq \sum_{k=1}^{\infty} \frac{k^{(2\ell+1)|\alpha|}}{e^{k^{2\ell+1}}} < \sum_{j=1}^{\infty} \frac{j^{|\alpha|}}{e^j} =: C_\alpha < \infty.$$

So, $f \in C^\infty(\mathbb{R}^2)$, and $\partial^\alpha f(x_1, x_2) = O\left(e^{(c_\ell+1)|x_2|^{1+\frac{1}{2\ell}}}\right) = O\left(e^{(c_\ell+1)|x|^{1+\frac{1}{2\ell}}}\right)$. It is easy to see that $(\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2})f(x_1, x_2) = 0$.

The function f admits analytic continuation to the set

$$\Pi_1 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im } z_1 < 1\}.$$

Indeed, let

$$\begin{aligned}
 f(z_1, z_2) &= f(x_1 + iy_1, x_2 + iy_2) = \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}(x_1+iy_1)+k(x_2+iy_2)}}{e^{k^{2\ell+1}}} \\
 &= \sum_{k=1}^{\infty} e^{i(ky_2 - k^{2\ell+1}x_1)} e^{k^{2\ell+1}(y_1-1)+kx_2}.
 \end{aligned}$$

It is easy to see that the last series is uniformly convergent on compact subsets of Π_1 . So, f admits analytic continuation to Π_1 . On the other hand, $f(iy_1, 0) \rightarrow \infty$ as $y_1 \rightarrow 1 - 0$. Indeed,

$$f(iy_1, 0) = \sum_{k=1}^{\infty} e^{k^{2\ell+1}(y_1-1)}.$$

Take any $N \in \mathbb{N}$. If $y_1 > 1 - N^{-(2\ell+1)}$, then

$$f(iy_1, 0) > \sum_{k=1}^{\infty} e^{-k^{2\ell+1}N^{-(2\ell+1)}} > \sum_{k=1}^N e^{-k^{2\ell+1}N^{-(2\ell+1)}} \geq \sum_{k=1}^N e^{-1} = \frac{N}{e}.$$

So, $f(iy_1, 0) \rightarrow \infty$ as $y_1 \rightarrow 1 - 0$.

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