Review of Managerial Science
https://doi.org/10.1007/s11846-023-00715-z

ORIGINAL PAPER

Robust mean-to-CVaR optimization under ambiguity in distributions means and covariance

Somayyeh Lotfi[2] · **Stavros A. Zenios**[1,2,3,4]

Received: 27 February 2023 / Accepted: 25 October 2023

Abstract

We develop a robust mean-to-CVaR portfolio optimization model under interval ambiguity in returns means and covariance. The robust model satisfies second-order stochastic dominance consistency and is formulated as a semi-definite cone program. We use two controlled experiments to document the sensitivity of the optimal allocations to the ambiguity when asset correlation varies, and to the ambiguity intervals. We find that means ambiguity has a higher impact than covariance ambiguity. We apply the model to US equities data to corroborate works showing that ambiguity in mean returns induces a home bias; it can explain the puzzle in a two-country setting but not with three countries. We further establish that covariance ambiguity also induces bias, but with lower impact that can not explain the puzzle. Our results suggest what is needed for the ambiguity channel to provide a full explanation of the puzzle. The findings are robust to alternative model specifications and outliers.

Keywords Ambiguity · Conditional Value-at-Risk · International portfolios · Equity home bias puzzle

JEL Classification C61 · C69 · D81 · G11 · G12 · G15

✉ Stavros A. Zenios
zenios.stavros@ucy.ac.cy

Somayyeh Lotfi
lotfinoghabi.somayyeh@ucy.ac.cy

1 Durham University, Durham, UK

2 Department of Accounting and Finance, University of Cyprus, Nicosia, Cyprus

3 Cyprus Academy of Sciences, Letters, and Arts, Nicosia, Cyprus

4 Bruegel, Brussels, Belgium

Published online: 05 January 2024

1 Introduction

Data ambiguity has a well-documented impact on portfolio selection.[1] The problem has been studied in the mean-variance literature with several solutions proposed, including a *robust optimization* counterpart of the classical mean-variance model (Goldfarb and Iyengar 2003). Robust portfolio selection models have also been developed with higher order moments.[2] These works develop efficient frontiers that are robust to the data ambiguity. Tangency portfolios are used to identify a unique point on the frontier that maximizes a performance ratio such as Sharpe in mean-variance analysis (Sharpe 1994). In this paper, we advance the robust portfolio optimization literature (Mulvey et al. 1995) with a robust mean-to-CVaR (MtC) model for stable distributions (Farinelli et al. 2008; Rachev et al. 2008) with interval ambiguity in means and covariance following Ben-Tal et al. (2009).

The robust MtC model satisfies second-order stochastic dominance (SSD) consistency and, we formulate it as a semi-definite positive cone program (SDCP). Using controlled experiments, we show that ambiguity has a nuanced effect on optimal asset allocation. An ambiguous asset can be attractive depending on its expected return and diversification benefits against the less ambiguous assets and on the ambiguity interval.

SSD consistency allows us to use the model for a broad class of investors with concave and non-decreasing utility functions. We use it to study a potential explanation of the home equity bias puzzle (French and Poterba 1991) through the ambiguity channel, abstracting from investor risk preferences. The puzzle arises from the observed discrepancies between international investors' domestic equity holdings from the market portfolio. Ambiguity aversion has been suggested as a potential explanation of the puzzle by Dow et al. (1992) and Epstein and Wang (1994), but none of the testable models developed subsequently generate the observed home-biased asset allocations (Boyle et al. 2012; Uppal and Wang 2003; Epstein and Miao 2003; Easley and O'Hara 2009; Hara and Honda 2022). These works show that ambiguity induces bias, which is an important step in the right direction, but when put to the data, they do not explain the puzzle completely.

We test our model on US data to corroborate these earlier works and go further to explain the puzzle on a two-country example but not on three countries. We uncover two new important empirical facts. First, we show that covariance ambiguity also induces bias; this is a new result in the literature, but can not explain the puzzle. Second, we show that interval ambiguity, which ignores the correlation of the returns, also does not provide a complete explanation. This suggests that ambiguity in the means of correlated returns deserves investigation. In a follow-up paper, Lotfi and Zenios (2023) we develop a robust optimization model using ellipsoidal mean ambiguity sets for correlated returns and obtain allocations matching the observed home bias for a sample of 40 developed and emerging markets. Hence, the current paper

[1] See, e.g., Best and Grauer (1991), Broadie (1993), Chan et al. (1999), Chopra and Ziemba (1993), Hong and Liu (2009), Jagannathan and Ma (2003), Kaut et al. (2007), and Stoyanov et al. (2013).

[2] Ceria and Stubbs (2006), El Ghaoui et al. (2003), Gao et al. (2017), Lotfi and Zenios (2018), Paç and Pınar (2014), Tütüncü and Koenig (2004), Ye et al. (2012), and Zhu and Fukushima (2009).

sets the direction for a complete explanation but also, importantly, provides a significant supplementary test. Specifically, our ellipsoidal ambiguity assumes known correlations, and here we show this assumption not to be limiting.[3]

We proceed in three steps. First, we derive the robust counterpart to MtC optimization, establish that it is SSD consistent, and develop its SDCP formulation.

Second, we study the impact of ambiguity on the optimal allocations through two controlled experiments to test for the impact of varying correlation and ambiguity intervals on the optimal allocations. We find that the ambiguity effect dominates the diversification benefits when means are ambiguous, while ambiguity in covariance is less important, and diversification benefits dominate the ambiguity effect as correlation decreases. The ambiguity interval location (interval of higher returns or interval of lower risks) and size (smaller interval) can affect the optimal allocations and an ambiguous asset may become attractive. That is, the effects of ambiguity depend on the return, risk, and ambiguity of an asset.

Third, we test the model for US equity investors where home bias is well documented. We consider portfolios comprised of the US and the rest of the world. With mean ambiguity, the model explains the puzzle by generating a home-biased allocation that matches the observed allocation for ambiguity intervals derived from market data. Ambiguity in covariance also induces home bias but alone does not match the actual allocations and can not explain the puzzle. When using a broader set of assets in emerging markets, the model allocations remain home-biased but do not match the observed one. Using interval ambiguity without modeling the returns correlation in the ambiguity sets appears limiting when the correlations and ambiguities of multiple foreign markets enter the portfolio selection problem.

Finally, we show that our findings hold with alternative models. Specifically, we test with a deviation risk measure or for different CVaR quantiles to rule out outlier effects.

The paper is organized as follows: The robust MtC portfolio optimization model is given in Sect. 2. Section 3 discusses the controlled experiments, and Sect. 4 describes our data and the application to the US equity home bias puzzle, including tests with alternative models. Section 5 concludes.

2 Robust MtC portfolio selection model

2.1 Preliminaries

Portfolio return $\tilde{r}_p = \tilde{r}^\top x$ is a function of random vector $\tilde{r} \in \mathbb{R}^n$ of asset returns with mean $\bar{r}$ and covariance Σ, and the vector of portfolio weights $x \in \mathbb{X}$ with

[3] Other attempts to explain the puzzle make assumptions on ambiguity-aversion or risk-aversion or utility functional forms (Bossaerts et al. 2010; Cao et al. 2005; Cooper et al. 2012; Easley and O'Hara 2009; Epstein and Miao 2003; Gilboa and Schmeidler 1989; Klibanoff et al. 2005; Peijnenburg 2018; Uppal and Wang 2003). Our model assumes worst-case ambiguity aversion and only requires that the utility function is concave and non-decreasing.

$$\mathbb{X} = \left\{ x \in \mathbb{R}^n \mid x \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}. \tag{1}$$

$\mathbb{X}$ is the set of feasible portfolios with no short sales. We assume a risk-free asset with return r_f.

We account for higher-order moments using CVaR as the risk criterion:

Definition 2.1 (*Conditional Value-at-Risk,* CVaR) The conditional Value-at-Risk at confidence level $\alpha \in (0, 1)$, for a random variable $\tilde{r}_p$ is

$$\mathrm{CVaR}_\alpha\left(\tilde{r}_p\right) = -\mathbb{E}\left[\tilde{r}_p \mid \tilde{r}_p \leq \zeta\right], \tag{2}$$

where $\mathbb{E}$ is the expectation operator and $\zeta \in \mathbb{R}$ is the value-at-risk i.e., the $(1-\alpha)$ -quantile of $\tilde{r}_p$ given by the highest γ such that $\tilde{r}_p$ will not exceed γ with probability $1-\alpha$,

$$\mathrm{VaR}_\alpha\left(\tilde{r}_p\right) \doteq \zeta = \max\left\{\gamma \in \mathbb{R} \mid \mathrm{Prob}\left(\tilde{r}_p \leq \gamma\right) \leq 1-\alpha\right\}. \tag{3}$$

CVaR coincides with the coherent tail VaR of Artzner et al. (1999) for the case of continuous distributions. The definition for general distributions, including the discrete distributions we consider in this paper, is due to Rockafellar and Uryasev (2002).

For a given α, CVaR is a function of x. CVaR is defined as the negative of the expected value of excess returns below a threshold; this measures losses. Minimizing CVaR for different target expected returns $\bar{r}^\top x \geq \mu$, we obtain efficient frontiers in mean-CVaR space, and this can be achieved using linear programming for the case of discrete distributions (Rockafellar and Uryasev 2002). We use the following result in the model formulation.

Theorem 2.1 (Fundamental minimization formula, Rockafellar and Uryasev 2002) *As a function of* $\gamma \in \mathbb{R}$, *the auxiliary function*

$$F_\alpha\left(\tilde{r}_p, \gamma\right) = \gamma + \frac{1}{1-\alpha}\mathbb{E}\left[\max\left\{-\tilde{r}_p - \gamma, 0\right\}\right]$$

is finite and convex, *with*

$$\mathrm{CVaR}_\alpha\left(\tilde{r}_p\right) = \min_{\gamma \in \mathbb{R}} F_\alpha\left(\tilde{r}_p, \gamma\right).$$

Dropping the parameter α for simplicity, we define the *mean-to-CVaR ratio*[4]:

$$\mathrm{MtC} = \frac{\mathbb{E}\left(\tilde{r}_p - r_f\right)}{\mathrm{CVaR}\left(\tilde{r}_p - r_f\right)}. \tag{4}$$

[4] We set $\alpha = 0.95$ in our tests and carry out a sensitivity analysis in Sect. 4.3.2.

This is a reward ratio, like Sharpe, in the sense that it measures the expected excess return per unit of risk (Farinelli et al. 2008; Rachev et al. 2008). The tangency portfolio that maximizes the reward ratio is obtained by solving

$$\mathrm{MtC}^* = \max_{x\in\mathbb{X}} \frac{\mathbb{E}\left(\tilde{r}_p - r_f\right)}{\mathrm{CVaR}\left(\tilde{r}_p - r_f\right)}. \tag{5}$$

See Pagliardi et al. (2021, Appendix A.1) for the linear programming formulation.

Sharpe ratio is the slope of the tangency portfolio using variance as the risk measure, whereas MtC* uses CVaR. CVaR under normality is given by $\mathrm{CVaR}_\alpha(\tilde{r}_p) = -\bar{r}_p + \kappa_{1-\alpha}\sigma_{\tilde{r}_p}$ where $\bar{r}_p$ and $\sigma_{\tilde{r}_p}$ are the mean and standard deviation of $\tilde{r}_p$, and $\kappa_{1-\alpha} = \frac{1}{1-\alpha}\phi(\Psi^{-1}(1-\alpha))$ with ϕ and Ψ the normal density and cumulative distribution functions, respectively. Therefore, when the normality assumption holds, the Sharpe ratio portfolio is a solution of (5). Beyond the nice properties of MtC optimization (SSD consistency and a tractable linear programming model), Pagliardi et al. (2021) also demonstrate that the mean-CVaR model optimal weights can be more robust than a mean-variance model, that MtC portfolios are more positively skewed than Sharpe portfolios and that the model can hedge the systematic component of the highly-skewed global political risk factor (P-factor) of Gala et al. (2023).

2.2 Robust model formulation

Model (5) depends on the joint distributions of the assets. If this information is known, we have no ambiguity in the model data. In practice, however, we usually assume a discrete empirical distribution for returns with values obtained from a finite set of historical observations with equiprobable probabilities, but these are just but one observation of some underlying distribution and hence ambiguous. Likewise, if the returns are estimated from a linear factor model, the estimated regression coefficients are only known within some confidence interval (Goldfarb and Iyengar 2003).

To incorporate ambiguity in the model, we apply *robust optimization* (Mulvey et al. 1995) using the max-min formulation with *ambiguity sets* of Ben-Tal et al. (2009). For example, when the return distribution is ambiguous, the probabilities and moments are only known to the extent that they belong to an ambiguity set. We obtain the robust counterpart to the MtC model allowing for ambiguous means and covariance. We show that the model can be cast as an SDCP, which can be solved using the interior-point method (Grant and Boyd 2014).

We assume that the joint probability distribution of returns (π) is ambiguous and belongs to the class of all distributions with mean $\bar{r}$ and covariance Σ in some interval ambiguity sets.

Definition 2.2 (*Ambiguity in distribution*) The random variable $\tilde{r}$ assumes a distribution from

$$\mathbb{D} = \left\{ \pi \mid \mathbb{E}_\pi[\tilde{r}] = \bar{r},\ \mathrm{Cov}_\pi[\tilde{r}] = \Sigma \succ 0 \right\},$$

where $\bar{r}$ and Σ are given and $\Sigma \succ 0$ indicates Σ is a positive definite matrix.

Definition 2.2, assumes ambiguous return distributions, specified only to the extent that their first and second central moments are known to be equal to $\bar{r}$ and Σ, respectively. We next define the ambiguity of the means and covariance.

Definition 2.3 (*Interval ambiguity in means and covariance*) Means and covariance of returns belong to the interval set:

$$U_I = \{(\bar{r}, \Sigma) \in \mathbb{R}^n \times \mathbb{S}^n \mid \bar{r}_- \le \bar{r} \le \bar{r}_+,\ \Sigma_- \le \Sigma \le \Sigma_+\},$$

where $\bar{r}_-, \bar{r}_+, \Sigma_-, \Sigma_+$ are given vectors and matrices, and the inequalities are component-wise, and $\mathbb{S}^n$ denotes the cone of positive semi-definite matrices.

The robust counterpart of MtC for ambiguity sets $\mathbb{D}$ and U_I is as follows:

$$\max_{x \in \mathbb{X}} \min_{(\bar{r},\Sigma) \in U_I, \pi \in \mathbb{D}} \frac{\mathbb{E}\left(\tilde{r}_p - r_f\right)}{\mathrm{CVaR}\left(\tilde{r}_p - r_f\right)}. \tag{6}$$

The following theorem provides the SDCP formulation.

Theorem 2.2 *Assuming positive worst-case CVaR on excess returns of the optimal portfolio of the model* (6), *the robust MtC portfolio optimization model can be cast as*:

$$\begin{aligned}
\max_{v'_+, v'_- \in \mathbb{R}^n,\, v \in \mathbb{R},\, \Lambda, \Lambda_+, \Lambda_- \in \mathbb{R}^{n \times n}} \quad & \left(\bar{r}_- - r_f e\right)^\top v'_- - \left(\bar{r}_+ - r_f e\right)^\top v'_+ \\
s.t. \quad & \\
& \left(\bar{r}_+ - r_f e\right)^\top v'_+ - \left(\bar{r}_- - r_f e\right)^\top v'_- + \\
& \frac{\alpha}{1-\alpha} v + tr\left(\Lambda_+ \Sigma_+\right) - tr\left(\Lambda_- \Sigma_-\right) \le 1 \\
& \begin{bmatrix} \Lambda & \frac{v'_- - v'_+}{2} \\ \frac{(v'_- - v'_+)^\top}{2} & v \end{bmatrix} \succeq 0 \\
& \Lambda \le \Lambda_+ - \Lambda_- \\
& v'_- - v'_+ \ge 0 \\
& e^\top (v'_- - v'_+) > 0 \\
& v'_+, v'_- \ge 0,\ \Lambda_+,\ \Lambda_- \ge 0,
\end{aligned} \tag{7}$$

where tr denotes the trace operator. Given the optimal solutions of (7) v'^*_+ *and* v'^*_-, *we obtain the optimal solution of* (6) *as* $x^* = \frac{1}{e^\top v' *} v' *$ *where* $v' * = v'^*_- - v'^*_+$.

For the proof see Appendix 1.[5,6]

Robustification preserves SSD consistency (Lotfi and Zenios 2023), and SSD consistency of MtC follows from (Pagliardi et al. 2021). We can use the model to draw inferences without any assumptions on the utility function beyond being concave and non-decreasing, and we apply it to the equity home bias puzzle.

3 Controlled experiments

We study the optimal portfolio allocations for different interval ambiguity sets using two controlled experiments on a portfolio of two hypothetical risky assets, A and B. We consider an ambiguity set specified by a *maximum interval* and control the magnitude of the ambiguity using a *shrinkage factor* in [0, 1], where 0 shrinks the ambiguity to a single point — i.e., the certain mean — and 1 is the maximum.

The use of a shrinkage factor is convenient for our empirical work with the US data when we obtain an estimate of the ambiguity interval from market data. Shrinking the interval, we test if we can explain the puzzle with the ambiguity of the data (shrinkage 1) or if we must assume lower ambiguity (< 1). We first use the shrinkage factor to control the ambiguity of the hypothetical assets. When the shrinkage of A is 0 and of B is 1, only the latter asset is ambiguous; with both factors zero, we use the original MtC model.

For the controlled tests, we solve the robust MtC with varying relative ambiguity for combinations of shrinkage factors and display the optimal allocations to asset A. The x- and y-axes denote the shrinkage factors of A and B. The z-axis is the allocation to A.[7] We study the impact of ambiguity in means or standard deviations on asset allocations. When we consider ambiguity in the means, we solve model (7) where $\Sigma_- = \Sigma_+$ and equal to the unambiguous covariance. For ambiguity in standard deviations, we solve the model with $\bar{r}_- = \bar{r}_+$ and equal to the unambiguous mean returns.

3.1 Varying correlation

We investigate the impact of asset correlation on optimal allocations for assets with identical ambiguity intervals. We consider two assets with mean excess returns of 7% in the interval [− 13%, 27%], with a standard deviation of 20% in the interval [15%, 25%]. We control for correlation with increasing diversification benefits of

[5] Note that the model is formulated with a strict positivity constraint on $e^\top(v'_- - v'_+)$, following from reasonable assumption of positive CVaR in the MtC optimization model (5), given it signifies losses and for sufficiently large α. This can be implemented using standard optimization algorithms by setting the constraint to be greater or equal to a small constant ϵ, which we set at 10^{-6}. In practice, the assumption is satisfied trivially since all CVaR in Table 1 are positive, as they are for the sample of 40 developed and emerging markets in (Data Appendix Lotfi and Zenios 2023).

[6] Analytical solutions for two assets can be obtained following Lotfi and Zenios (2023); see, e.g., the solutions of other CVaR-based robust models by Gotoh and Takano (2007) and Pang and Karan (2018).

[7] When both shrinkage factors are zero, we solve the non-robust MtC model (5) of Pagliardi et al. (2021) with a matrix of return scenarios with the given unambiguous means and covariance. For this corner point, we generate 120 scenarios assuming a multivariate normal distribution.

(i) 0.60, (ii) 0, and (iii) − 0.60. Note that ambiguity in standard deviations implies covariance ambiguity even if the correlation is fixed. All data are annualized.[8] We solve robust MtC for varying shrinkage factors and display the allocation to asset A in Fig. 1. Panel A (column figures) shows results when the mean returns are ambiguous, with the standard deviations fixed at 20%. Panel B shows results with ambiguous standard deviations, with mean returns at 7%.

From these figures, we observe the following.

In all cases, there is a monotonic relation between the level of ambiguity of asset A relative to B and the optimal allocation to A. The higher the ambiguity of asset B compared to A (shrinkage of B greater than A), the higher the allocation to A. Ceteris paribus, optimal allocations are tilted towards less ambiguous assets. Moreover, when the shrinkage factors of the two assets are equal, the allocation is evenly split. By construction of the two assets (identical mean returns and standard deviations), the model is indifferent between the two if they have identical ambiguity intervals.

When mean returns are ambiguous (Panel A, column figures), the lower correlation does not impact the optimal allocation, and the shape of the curve remains virtually unchanged. Even though a lower correlation should increase diversification benefits by shifting toward more diversified portfolios, the impact of mean ambiguity dominates the diversification benefit, and the allocation to the less-ambiguous asset persists. However, this situation changes when standard deviations are ambiguous (Panel B). As correlation decreases, the diversification effect dominates the ambiguity effect. Notice that for the large negative correlation of − 0.60, ambiguity has a minor effect, and the optimal allocation is about evenly split between the two assets to achieve the highest diversification benefit.

Hence, mean returns ambiguity is a significant determinant for tilting allocations toward less ambiguous assets. Importantly, the effect of ambiguity persists even for large negative asset correlations so that ambiguity dominates the diversification benefits. Standard deviation ambiguity has a smaller tilt effect towards less ambiguous assets. If the assets are highly correlated, the ambiguity has a noticeable and monotonic effect, just like ambiguity in the means. If the assets are negatively correlated, the tilt toward the less-ambiguous asset is muted, and the diversification benefit dominates.

3.2 Varying ambiguity interval

We now change the maximum ambiguity interval of asset B relative to A. In particular, we shift the returns of asset B such that (i) for mean return ambiguity, the interval contains higher values, or (ii) for standard deviation ambiguity, the interval contains lower values. The size of the ambiguity interval is the same for both assets.

We assume a correlation of 0.60, solve the robust MtC model for ambiguities with varying shrinkage factors, and display the allocation to asset A in Fig. 2. Panel A (column figures) shows results when the mean returns are ambiguous, with standard deviations fixed at 20%. The mean return of asset A is equal to 7% in the interval

[8] Whenever any of the covariance matrices are near-singular, we add a very small positive number to diagonal elements to avoid an ill-posed system of equations in the procedure of an interior-point method.

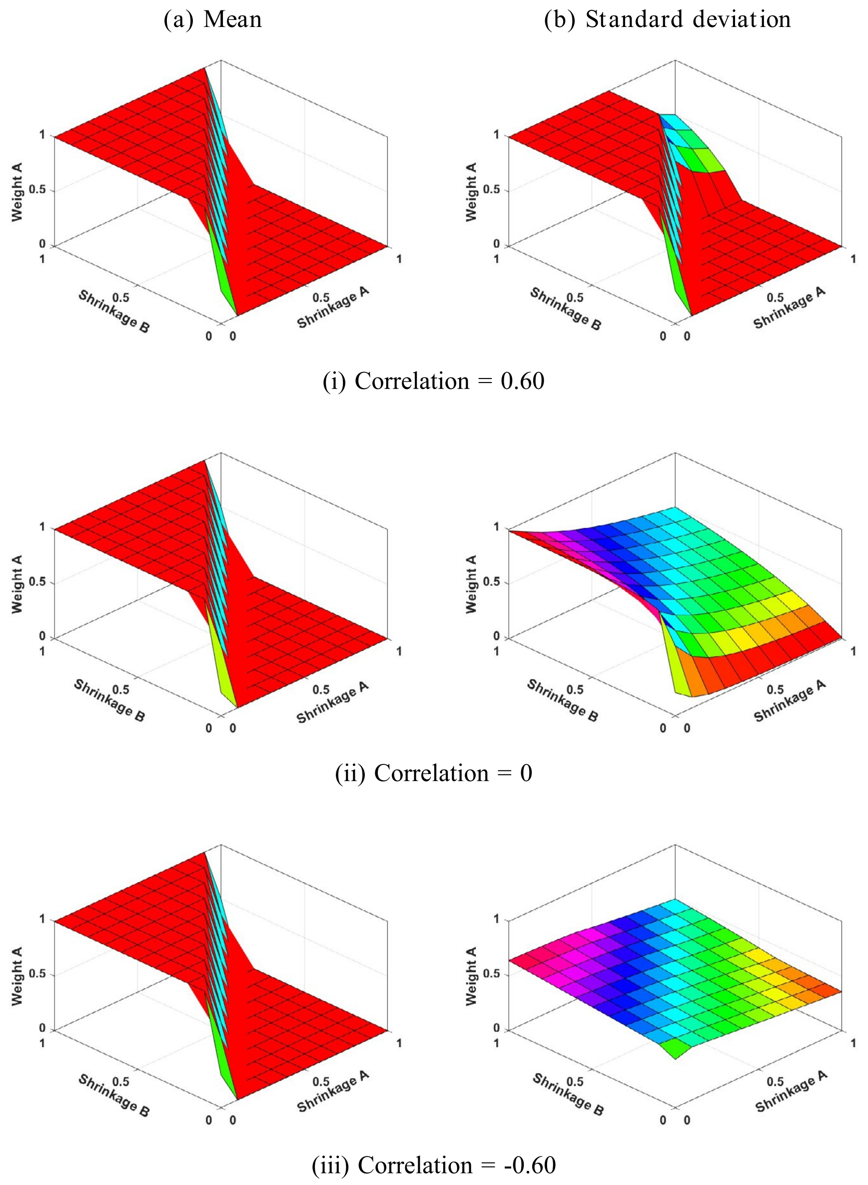

Fig. 1 Optimal allocations under ambiguity with varying correlation. This figure illustrates the optimal allocation to asset A when the correlation between assets A and B varies from (i) 0.60, to (ii) 0, and (iii) − 0.60, and the ambiguity interval shrinkage factors of the two assets vary between 0 and 1. Panels **A** and **B** show optimal allocations when means and standard deviations are ambiguous, respectively. Both assets in the portfolio have identical mean return and standard deviation ambiguity sets. The mean return is equal to 7% in the interval [− 13%, 27%], and the standard deviation is equal to 20% in the interval [15%, 25%]. All statistics are annualized

[− 13%, 27%], while for the mean return of B we consider three cases: (i) 11%, in the interval [− 9%, 31%], (ii) 15%, in the interval [− 5%, 35%], and (iii) 19%, in the interval [− 1%, 39%]. Panel B shows results with ambiguous standard deviations, with mean returns fixed at 7%. The standard deviation of asset A is equal to 20% in the interval [15%, 25%], while for the standard deviation of asset B we test three cases: (i) 15%, in the interval [10%, 20%], (ii) 10%, in the interval [5%, 15%], and (iii) 5%, in the interval [0%, 10%].

From the figures, we observe the following.

When both assets are unambiguous, the optimal allocation is fully in asset B since this asset has either a higher mean return (Panel A) or lower risk (Panel B). However, when means are ambiguous (Panel A), as we shift the location of the ambiguity interval of asset B this asset weighs in more in the optimal portfolio even when it is more ambiguous than A (i.e., shrinkage of B greater than A). This is because the maximum ambiguity interval of B includes higher mean returns than A. So, with mean return ambiguity, the optimal investment tilts toward the asset with a favorable ambiguity interval. The same observation holds for standard deviation ambiguity (Panel B). As the maximum ambiguity interval of the standard deviation of B shifts from [10%, 20%] to [0%, 10%], the optimal allocations tilt toward B even when it is more ambiguous than A.[9]

To summarize, the relative ambiguity of the two assets, i.e., their minimum and maximum values and the size of the ambiguity intervals, are significant determinants of the optimal allocations and can tilt the allocations toward more ambiguous assets. The effect of ambiguity is nuanced and the portfolios are not always tilted towards the less ambiguous assets.

4 An application to the equity home bias puzzle

From the second controlled experiment, it follows that it is not a foregone conclusion that ambiguity per se creates a home bias. To establish ambiguity as an explanation of the puzzle, one needs to conduct a full empirical study putting a model with ambiguity to the data of a broad set of countries, using different methods to estimate the ambiguity. We test whether the interval ambiguity model could be successful in such an empirical investigation; the answer is no, not completely. We also test whether covariance ambiguity could be a potential explanation of the puzzle and the answer is, again, not completely.

4.1 Data

We consider the US market, which is well documented to exhibit equity home bias. We use the US equity market index as the home. As foreign, we take the rest of the

[9] An alternative test would be to reduce the size of the ambiguity set by shrinking the maximum interval of mean return or standard deviation of B relative to A while keeping the centers the same. We performed this test, reaching the same conclusion.

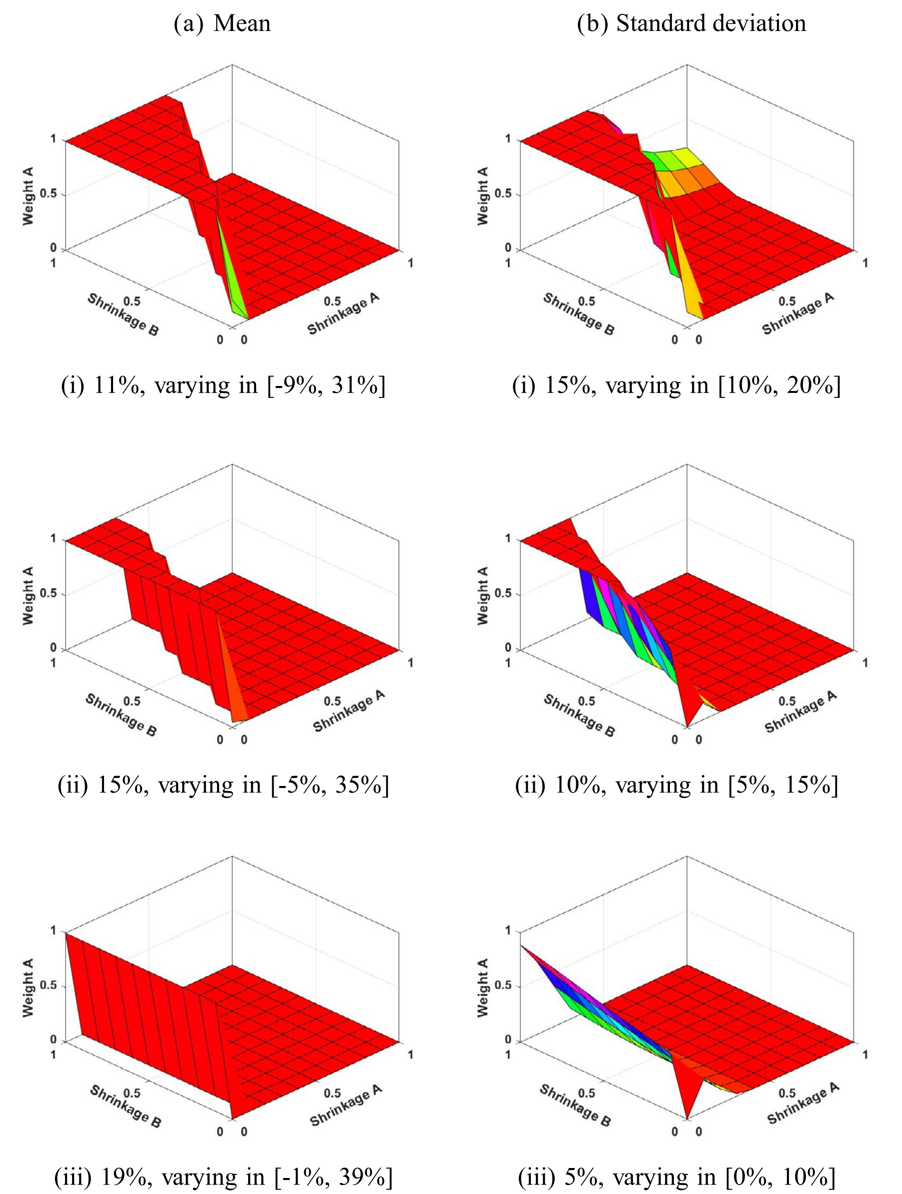

Fig. 2 Optimal allocations with varying ambiguity intervals. This figure illustrates the optimal allocation to asset A when the maximum ambiguity interval of asset B changes and shrinkage factors of assets A and B vary between 0 and 1. Panels **A** and **B** show optimal allocation when means and standard deviations are ambiguous, respectively. In panel **A**, the standard deviation of both assets equals 20%, the mean return of asset A equals 7%, varying in the interval [− 13%, 27%], and the mean return of asset B is equal to (i) 11%, varying in the interval [− 9%, 31%], (ii) 15%, varying in the interval [− 5%, 35%], and (iii) 19%, varying in the interval [− 1%, 39%]. In panel **B**, the mean return of both assets is equal to 7%, the standard deviation of asset A equal to 20%, varying in the interval [15%, 25%], and the standard deviation of asset B is equal to (i) 15%, varying in the interval [10%, 20%], (ii) 10%, varying in the interval [5%, 15%], and (iii) 5%, varying in the interval [0%, 10%]. The correlation between the two assets is fixed at 0.60. All statistics are annualized

world (RoW) of an equally-weighted portfolio of 40 countries excluding the US.[10] The monthly data span from January 1999 to December 2019. We also construct two indices for a second example, using equally-weighted returns of an MSCI subset of emerging markets. We construct equally-weighted portfolios EME_1 of Asian markets, and EME_2 of European and Latin American markets.[11]

We use the MSCI Investable indices to avoid positive biases when ignoring investability frictions, such as illiquidity risk and index replicability. We calculate the excess return using the 1-month USA Treasury Bill rate. Risk-free rates are from Refinitive Eikon and equity market indices are from Datastream.

We report in Table 1 statistics of the excess return of the four indices during the sample period. The return distributions are negatively skewed with considerable tail risk. The US index has a lower mean return and standard deviation than RoW, and the two emerging market indices, with positive and quite high correlations.

The IMF Coordinated Portfolio Investment Surveys (CPIS), available annually from 2001 to 2019, provides equity holdings (in USD) across countries. We use EQ_i to represent the holdings of domestic investors in country i equity market and TEQ_i as the value of the total equity holding for country i. The *actual home allocation* of holdings of domestic equities in the total equity holdings is $\mathrm{a}_i = \mathrm{EQ}_i/\mathrm{TEQ}_i$. The US actual allocation averaged over the sample period is 0.84, and we use this value as the benchmark for the model to match under the ambiguity estimated from market data.

Assuming that the International Capital Asset Pricing Model (ICAPM) holds, the optimal allocations are each country's market capitalization (MC). That is, the optimal weights under ICAPM are given by $\mathrm{w}_i = \mathrm{MC}_i / \sum_{j=1}^{n} \mathrm{MC}_j$ where n is the number of countries in the world portfolio. The average weight of the US home over the sample period is estimated as 0.38. From the actual US home allocations and market-cap weights, we obtain the equity home bias as $\mathrm{EHB}_i = \frac{\mathrm{a}_i - \mathrm{w}_i}{1 - \mathrm{w}_i}$. When EHB for country i is one, there is a complete equity home bias, and for zero, the portfolio is optimally diversified according to ICAPM. The EHB index for the US is 0.74. This large discrepancy has been characterized as a puzzle; the data above shows that the puzzle has persisted since first observed by French and Poterba (1991). We ask if the model can explain this discrepancy.

We obtain the maximum ambiguity intervals of means and covariance from market data as follows: First, we use block bootstrapping (Paparoditis and Politis 2003) of the time series of the US and RoW returns to generate 5000 samples.[12] Then, we calculate the mean returns and covariance matrix for each sampled time series of returns and obtain the maximum ambiguity interval between the minimum and maximum values. In Table 2, we report the minimum, maximum, and size of the ambiguity intervals of the means (Panel A) and covariance (Panel B) for the US and RoW.

[10] Our sample consists of Australia, Austria, Belgium, Brazil, Canada, Chile, China, Colombia, Czech Republic, Denmark, Egypt, Finland, France, Germany, Greece, Hong Kong, Hungary, India, Israel, Italy, Japan, Korea, Malaysia, Mexico, Netherlands, New Zealand, Norway, Peru, Philippines, Poland, Portugal, Russia, South Africa, Spain, Sweden, Switzerland, Thailand, Turkey, UK, and the US.

[11] We use, respectively, China, India, Israel, Korea, Malaysia, Philippines, Poland, Russia, Thailand, Turkey, and Brazil, Chile, Colombia, Czech Republic, Hungary, Mexico, Peru, Poland.

[12] Tests in Pagliardi et al. (2021) suggest an efficient block size of about six.

Table 1 Descriptive statistics

Country	Mean	Std	Skew	Kurt	SR	CVaR	MtC	Corr
Home	0.06	0.15	− 0.64	1.02	0.40	0.10	0.050	–
RoW	0.09	0.19	− 0.64	2.69	0.47	0.12	0.063	0.82
EME_1	0.12	0.21	− 0.23	2.01	0.57	0.13	0.077	0.75
EME_2	0.12	0.22	− 0.53	2.42	0.55	0.13	0.077	0.70

This table reports descriptive statistics of the excess return for the home (US), rest of world (RoW) market, and two emerging market indices (EME_1 and EME_2). The RoW is constructed as an equally-weighted portfolio of 40 countries excluding the US. The EME_1 is constructed as an equally-weighted portfolio of Asian emerging market indices, and EME_2 is constructed as an equally-weighted portfolio of European and Latin American emerging market indices in our sample. The statistics are mean, standard deviation (std), skewness (skew), and excess kurtosis (kurt). We also report Sharpe ratio (SR), conditional value-at-risk (CVaR), mean-to-CVaR (MtC) for each index. The correlation (corr) between returns of home and RoW, EME_1, and EME_2 is displayed in the last column. The CVaR and MtC are computed at the 5% confidence level. The monthly returns span from January 1, 1999, to December 31, 2019. The mean, std, and Sharpe ratios are annualized

Table 2 Means and covariance ambiguity intervals

	(A) Mean			(B) Covariance					
				Min		Max		Size	
	RoW	Home		RoW	Home	RoW	Home	RoW	Home
Min	− 0.09	− 0.08	RoW	0.02	0.01	0.07	0.05	0.05	0.04
Max	0.27	0.19	Home	0.01	0.01	0.05	0.04	0.04	0.03
Size	0.36	0.27							

This table reports the ambiguity intervals of returns mean (Panel A) and covariance (Panel B) for home (US) and rest of the world (RoW) indices where RoW is constructed as an equally-weighted portfolio of 40 countries excluding the US. The maximum ambiguity intervals are estimated using block bootstrapping with a block size of 6 and a simulation size of 5000. They are specified by the minimum (min), maximum (max), and width of the interval (size). The data used for simulation are the monthly returns and span from January 1, 1999, to December 31, 2019. The reported statistics are annualized

The home ambiguity intervals of the US means (Panel A) and variances (Panel B, diagonal elements) are generally smaller than RoW. Moreover, the data provided for covariance ambiguity intervals (Panel B) implies an interval of [0.71, 0.90] for the correlation. The mean ambiguity intervals for EME_1 and EME_2 are [− 0.05, 0.35] and [− 0.07, 0.32], respectively, and are larger (0.40 and 0.39) than the US interval (0.27).

4.2 The US equity home bias puzzle

We test whether ambiguity in means or covariance can explain the home bias for the US with respect to the RoW. Following Sect. 3, we consider interval ambiguity sets controlled by a shrinkage factor.[13] We use the ambiguity interval data from Table 2

[13] With zero shrinkage for both home and RoW, we run the MtC model using actual home and RoW returns time series (252 observations).

and solve the robust model for varying values of the shrinkage factors of the US and RoW countries. We test whether the optimal asset allocations for ambiguity in means or covariance match the observed US home bias.

4.2.1 Ambiguity in means

We test for ambiguity in the means and solve model (7), with $\Sigma_{-} = \Sigma_{+}$ and equal to the unambiguous covariance calculated using the data from Table 1.

We show the results in Fig. 3 displaying as the red-shaded area the combinations of shrinkage factors for which the optimal home allocation matches or exceeds the actual home allocations for EHB = 0.74. Asset allocations biased towards the home for US shrinkage factors greater or equal to foreign factors signal that the observed ambiguity determines the bias. (If we need to assume a lower shrinkage factor for the home than foreign to explain the puzzle, then our model results are not driven by the data but by the assumption.) Further, if the allocation matches the actual allocation (as is the case at some points of the figure), then the relative ambiguity can fully explain the home bias puzzle. Finally, an optimal allocation equal to the actual at the corner point of no shrinkage implies that the ambiguity derived from market data can explain the puzzle.

From the figure, we observe that the optimal allocations are tilted toward the US when the shrinkage factor of the home is less than the RoW (above the 45-degree line), i.e., where the home is relatively less ambiguous compared to RoW. These results support that market ambiguity drives home bias. Importantly, the model generates allocations to the home consistent with the observed allocation at the corner point. The model with the market-implied ambiguity estimates in Table 2 may explain the puzzle for US investors in the aggregate case of two assets.

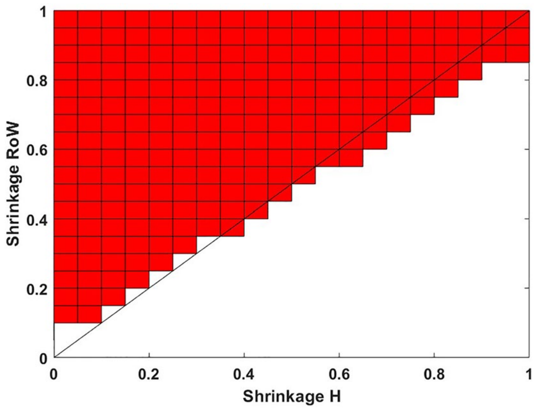

Fig. 3 Ambiguity in mean returns and equity home bias puzzle. This figure illustrates with the red-shaded area the optimal allocation to home (US) equity that is greater than or equal to the observed US allocation when mean returns are ambiguous, and shrinkage factors of both home and RoW indices vary between 0 and 1. The model optimizes the robust MtC model over the home and the rest of world (RoW) indices, where the RoW index is constructed as an equally-weighted portfolio of 40 countries excluding home. The maximum ambiguity intervals for means are estimated using block bootstrap methodology and are as described in Table 2. The un-ambiguous covariance is from Table 1

We may have an optimal home-biased allocation even if the home ambiguity is slightly higher than the RoW (see the small shaded area below the 45-degree line). This is because the US has a better ambiguity interval relative to RoW regarding size and location. In particular, the ambiguity interval of the US is [− 0.08, 0.19] (size of 0.27), while the ambiguity interval of RoW is [− 0.09, 0.27] (size of 0.36). This is consistent with the findings of the second controlled experiment, where we show that mean return ambiguity may shift optimal allocations toward more (or less) ambiguous assets due to the size or location of the ambiguity intervals. This observation highlights the nuanced ambiguity effects in explaining the puzzle.

4.2.2 Ambiguity in covariance

We repeat the above test with ambiguous standard deviations and unambiguous means at the values from Table 1. The ambiguity interval is obtained from Table 2 (Panel B). The US standard deviation lies in [0.10, 0.20] (size 0.10), and the standard deviation of RoW lies in [0.14, 0.26] (size 0.12). Hence, the home is slightly less ambiguous than RoW. Also, comparing the ambiguity interval of standard deviations with that of the mean (Panel A), we observe the mean ambiguity interval is about three times larger than the standard deviation interval, and the covariance ambiguity is relatively low. The correlations are assumed to be unambiguous at the fixed values given in the table; recall that standard deviation ambiguity implies covariance ambiguity even if the correlation is fixed.

Solving the robust MtC model for ambiguity in the standard deviations for different shrinkage factors, we obtain the optimal allocation shown in Fig. 4. The red-shaded area is in line with the results of the controlled experiments (Figs. 1

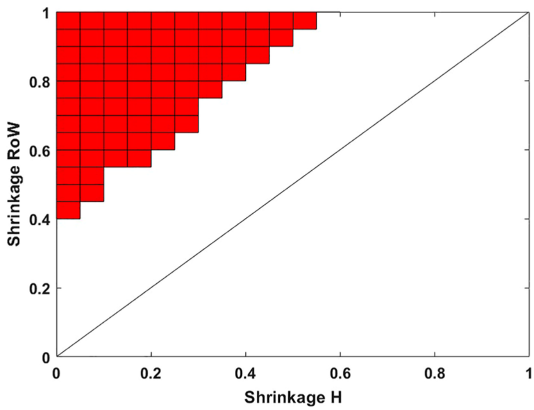

Fig. 4 Covariance ambiguity and the equity home bias puzzle. This figure illustrates with the red-shaded area the optimal allocation to home equity index when standard deviations are ambiguous for shrinkage factors of home and RoW between 0 and 1. The model optimizes the robust MtC model over the home and the rest of world (RoW) indices, where the RoW index is constructed as an equally-weighted portfolio of 40 countries excluding home. The maximum ambiguity intervals for standard deviations are estimated using block bootstrapping as described in Table 2. The un-ambiguous mean returns are from Table 1

and 2, Panels B) where we observed that allocations increase for less ambiguous assets. However, the optimal home allocations reach or exceed the actual US allocations for home shrinkage factors below 0.50. These results show that the relative ambiguity of the standard deviation between the US and the RoW induces home biases. Albeit, the observed relative ambiguity cannot explain the puzzle without significantly shrinking the home ambiguity. The standard deviations ambiguity has a second-order effect on the bias and although it induces bias it can not explain the puzzle. This is a new result in the literature.

4.2.3 US investors and emerging markets

The results of Sect. 4.2.1 advance the results from the two-country model of Epstein and Miao (2003). When they applied their model to the US data under the assumption of a greater ambiguity in the foreign market, they found that ambiguity moves allocations in the right direction towards home bias. However, their model did not match the observed allocations, and the authors point out the need for a multi-country extension.

Our model naturally extends to multiple assets, so we now test the ambiguity effects from potential investment in the two emerging market indices. The results in Fig. 5 display the home allocation for combinations of shrinkage factors of the emerging markets. With higher expected returns and lower correlations than with RoW, the optimal allocations to home are zero when the home is fully ambiguous for all combinations of foreign markets shrinkage factor (Panel A of Fig. 5). When we assume a shrinkage factor of 0.70 for the US the model can generate allocations that match the observed ones (Panel B of Fig. 5). Whereas the interval ambiguity model explains the US home bias in the case of two assets, it does not

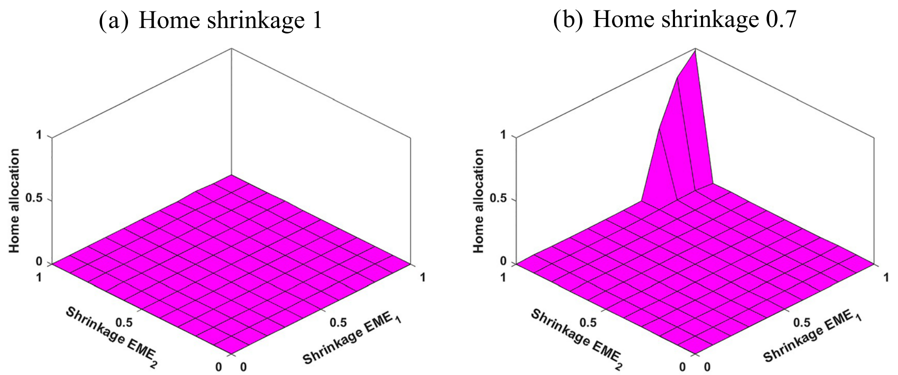

Fig. 5 Does mean return interval ambiguity fully explain equity home bias? This figure illustrates the optimal allocation to home (US) equity index when mean returns of home, EME_1, and EME_2 are ambiguous with shrinkage factors in [0, 1]. In Panel A, the shrinkage factor of the home is equal to 1, and in Panel B it is equal to 0.7. The model optimizes the robust MtC model over the home, EME_1, and EME_2 indices. EME_1 and EME_2 are constructed as equally-weighted portfolios of Asian or European and Latin American emerging market indices, respectively

do so with more assets that can provide diversification benefits to the US investor. The model is not a good candidate to put to the full data for a comprehensive empirical study to explain the equity home bias puzzle. We consider this negative result an important advance. It suggests that we must consider some aspects of the problem missing from the interval ambiguity model, namely the ambiguity of correlated returns.

4.3 Alternative model specifications

We conclude by showing that our empirical findings are not driven by the risk measure we have chosen or by outliers picked by the extreme (0.95) tail risk.

4.3.1 Deviation risk measures

We consider an alternative model based on a deviation risk measure. Portfolios obtained using variance risk satisfy two fund separation theorem (Tobin 1958), and fund separation was also established for generalized deviations, including CVaR deviation (Rockafellar et al. 2006). However, our interest in the puzzle is in choosing between two risky and ambiguous assets and not the risk-free rate. Hence, we opted for the SSD consistent MtC model for our main test and show that our findings hold with a deviation measure.

We use the classical mean-variance model and, from its robust counterpart (Goldfarb and Iyengar 2003) obtain the robust maximum Sharpe ratio and mean-variance efficient portfolios. The models do not follow directly from Goldfarb and Iyengar (2003), who consider ambiguity in the parameters of a linear factor model of market returns corresponding to the confidence regions of the statistical estimation procedures. Instead, we work with direct interval estimates of the covariance following Lotfi and Zenios (2018).

Robust Sharpe ratio maximization under the ambiguity of Definition 2.3 is as follows:

$$\max_{x \in \mathbb{X}} \min_{(\bar{r}, \Sigma) \in U_I} \frac{\mathbb{E}\left(\tilde{r}_p - r_f\right)}{\sigma\left(\tilde{r}_p - r_f\right)}. \tag{8}$$

We give the SDCP formulation of robust Sharpe, akin to Theorem 2.2 for robust MtC.

Theorem 4.1 *The robust Sharpe portfolio optimization model* (8) *can be cast as follows*:

$$
\begin{aligned}
\max_{v'_+, v'_- \in \mathbb{R}^n, \Lambda_+, \Lambda_- \in \mathbb{R}^{n\times n}} \quad & (\bar{r}_- - r_f e)^\top v'_- - (\bar{r}_+ - r_f e)^\top v'_+ \\
s.t. \quad & \\
& tr(\Lambda_+ \Sigma_+) - tr(\Lambda_- \Sigma_-) \leq 1 \\
& \begin{bmatrix} \Lambda_+ - \Lambda_- & v'_- - v'_+ \\ (v'_- - v'_+)^\top & 1 \end{bmatrix} \succeq 0 \\
& v'_- - v'_+ \geq 0 \\
& e^\top (v'_- - v'_+) > 0 \\
& v'_+, v'_- \geq 0, \; \Lambda_+, \Lambda_- \geq 0,
\end{aligned} \tag{9}
$$

where tr denotes the trace operator. Given the optimal solutions of (9) v'^*_+ *and* v'^*_-, *we obtain the optimal solution of* (8) *as* $x^* = \frac{1}{e^\top v'^*} v'^*$ *where* $v'^* = v'^*_- - v'^*_+$.

For the proof see Appendix 1.

We test the deviation-based model on the problem of Sect. 4.2.2 when home and RoW are fully ambiguous with shrinkage factors 1. At the maximum expected return portfolio there is no difference between a CVaR-based and a deviation-based model. Differences become more pronounced when we minimize the risk measure. Hence, we compare the two models on the efficient frontier, including the tangency portfolios. To test the sensitivity of our results, we run in addition to the robust MtC and robust Sharpe, the robust mean-variance (Appendix 1) and robust mean-CVaR (Lotfi and Zenios 2018, Theorem 5) with interval ambiguities.

We obtain ten equally spaced points on the efficient frontier. At the minimum-risk portfolio, the home allocation is 0.89 with mean-CVaR and 0.90 with mean-variance. As we move to higher expected returns and lower risks, the home shifts to 0.79, 0.69, and 0.60 with mean-CVaR, and 0.80, 0.70, and 0.60 with mean-variance. For the remaining points towards the maximum expected return portfolios, including the tangency portfolios, the allocations are identical to two decimal points. Hence, our main findings relating to the home bias remain valid when using a deviation risk measure.

4.3.2 Tails

We test that outliers do not drive our results, given the 0.95 tail risk measure we use. CVaR is not a robust statistic, and although outliers are, usually not of practical concern for large datasets, as in our case, we solve the model for different confidence levels α in the CVaR estimation. For the problem of subection 4.2.2 and the standard value $\alpha = 0.95$, the home allocation for the robust minimum CVaR point, which is the most sensitive to the parameter, is 0.89. For $\alpha = 0.95$ it increases to 0.90, and for $\alpha = 0.80$ is reduced to 0.88. Hence, our results do not seem to be driven by outliers picked by the choice of α.

5 Conclusion

This paper develops the robust MtC model under ambiguity in means and covariance and empirically studies its performance. The model is SSD consistent and is formulated as a semi-definite positive cone program that is efficiently solvable.

We show, using controlled experiments, that the effects of ambiguity can be nuanced. The ambiguity effect dominates the diversification benefits when means are ambiguous, but covariance ambiguity is less important, and diversification benefits dominate for negatively correlated assets. Furthermore, the ambiguity interval location (higher returns or lower risk) and size (smaller ambiguity interval) can affect the optimal allocations such that the more ambiguous asset may become preferable in optimal allocations.

We apply the model to market data for a US investor. Market estimates of ambiguity can explain the US equity home bias puzzle in a two-country setup. We show that the model generates optimal home allocations that match the actual home allocations under ambiguity in mean returns. However, this is not the case when considering ambiguity only in the covariance. Covariance ambiguity also induces home bias but the intervals obtained from market data cannot generate the actual home allocations. In a three-country example with high expected returns and lower correlation of emerging markets, we find that ambiguity in the means induces bias but can not match the observed allocation.

Our work uncovers two important facts relating to the literature exploring ambiguity as a potential explanation of the home equity bias. First, that covariance ambiguity induces bias but can not completely explain the puzzle. Second, interval ambiguity which ignores that ambiguity in the means refers to correlated returns also does not fully explain the puzzle. Ambiguity in the means of correlated returns is investigated in Lotfi and Zenios (2023) who arrive at a complete explanation of the home equity bias puzzle through the ambiguity channels.

Appendix 1: Robust MtC model

We provide the proof of Theorem 2.2.

Proof We define $\xi = \text{CVaR}((\tilde{r} - r_f e)^\top x) > 0$, and break the objective function (5) in two components to obtain

$$
\begin{aligned}
&\max_{x \in \mathbb{R}^n,\, \xi \in \mathbb{R}} \quad \bar{r}^\top \frac{x}{\xi} - r_f \frac{1}{\xi} \\
&\text{s.t.} \quad \text{CVaR}\left(\left(\tilde{r} - r_f e \right)^\top x \right) \leq \xi \\
&e^\top x = 1 \\
&x \geq 0,\ \xi > 0.
\end{aligned}
\tag{10}
$$

We define $x' = \frac{x}{\xi}$ and $\nu = \frac{1}{\xi}$, and rewrite the above as:

$$\begin{aligned} \max_{x' \in \mathbb{R}^n,\, \nu \in \mathbb{R}} \quad & \bar{r}^\top x' - r_f \nu \\ \text{s.t.} \quad & \text{CVaR}\left(\left(\tilde{r} - r_f e \right)^\top x' \right) \leq 1 \\ & e^\top x' = \nu \\ & x' \geq 0,\ \nu > 0. \end{aligned} \tag{11}$$

Replacing $e^\top x'$ for ν in the objective and $e^\top x' > 0$ for $\nu > 0$ in the constraint, we get

$$\begin{aligned} \max_{x' \in \mathbb{R}^n} \quad & \left(\bar{r} - r_f e \right)^\top x' \\ & s.t. \\ & \text{CVaR}\left(\left(\tilde{r} - r_f e \right)^\top x' \right) \leq 1 \\ & e^\top x' > 0 \\ & x' \geq 0. \end{aligned} \tag{12}$$

Therefore the robust counterpart of maximum MtC is as follows:

$$\begin{aligned} \max_{x' \in \mathbb{R}^n} \quad & \min_{(\bar{r}, \Sigma) \in U_I,\, \pi \in \mathbb{D}} \left(\bar{r} - r_f e \right)^\top x' \\ & s.t. \\ & \max_{(\bar{r}, \Sigma) \in U_I,\, \pi \in \mathbb{D}} \text{CVaR}\left(\left(\tilde{r} - r_f e \right)^\top x' \right) \leq 1 \\ & e^\top x' > 0 \\ & x' \geq 0. \end{aligned} \tag{13}$$

Using the representation of CVaR in Theorem 2.1, one can write the above as:

$$\begin{aligned} \max_{x' \in \mathbb{R}^n, \gamma \in \mathbb{R}} \quad & \min_{(\bar{r}, \Sigma) \in U_I,\, \pi \in \mathbb{D}} \left(\bar{r} - r_f e \right)^\top x' \\ & s.t. \\ & \max_{(\bar{r}, \Sigma) \in U_I,\, \pi \in \mathbb{D}} F\left(\left(\tilde{r} - r_f e \right)^\top x', \gamma \right) \leq 1 \\ & e^\top x' > 0 \\ & x' \geq 0. \end{aligned} \tag{14}$$

This is equivalent to:

$$
\begin{aligned}
&\max_{x' \in \mathbb{R}^n} \quad \min_{(\bar{r},\Sigma)\in U_I} \left(\bar{r} - r_f e\right)^\top x' \\
&s.t. \\
&\max_{(\bar{r},\Sigma)\in U_I} -\left(\bar{r} - r_f e\right)^\top x' + \frac{\sqrt{\alpha}}{\sqrt{1-\alpha}} \sqrt{x'^\top \Sigma x'} \le 1 \\
&e^\top x' > 0 \\
&x' \ge 0,
\end{aligned}
\tag{15}
$$

by Lotfi and Zenios (2018, Proposition 1). From Appendix A4 of the same paper we reformulate the inner minimization in the objective function and the maximization that appeared in the first constraint to get the following formulation.

$$
\begin{aligned}
&\max_{v'_+, v'_- \in \mathbb{R}^n,\, \upsilon \in \mathbb{R},\, \Lambda, \Lambda_+, \Lambda_- \in \mathbb{R}^{n\times n}} \quad \left(\bar{r}_- - r_f e\right)^\top v'_- - \left(\bar{r}_+ - r_f e\right)^\top v'_+ \\
&s.t. \\
&\left(\bar{r}_+ - r_f e\right)^\top v'_+ - \left(\bar{r}_- - r_f e\right)^\top v'_- + \\
&\frac{\alpha}{1-\alpha}\upsilon + tr(\Lambda_+ \Sigma_+) - tr(\Lambda_- \Sigma_-) \le 1 \\
&\begin{bmatrix} \Lambda & \frac{v'_- - v'_+}{2} \\ \frac{(v'_- - v'_+)^\top}{2} & \upsilon \end{bmatrix} \succeq 0 \\
&\Lambda \preceq \Lambda_+ - \Lambda_- \\
&v'_- - v'_+ \ge 0 \\
&e^\top (v'_- - v'_+) > 0 \\
&v'_+, v'_- \ge 0,\ \Lambda_+,\ \Lambda_- \ge 0.
\end{aligned}
\tag{16}
$$

This completes the proof.

Appendix 2: Robust deviation risk measure models

We give the proof of the SDCP formulation of robust Sharpe ratio maximization and the SDCP formulation for robust mean-variance efficient frontiers.

Appendix 2.1: Sharpe ratio maximization

We provide the proof of Theorem 4.1

Proof The Sharpe ratio maximization model is as follows:

$$\begin{aligned} &\max_{x \in \mathbb{R}^n} \quad \frac{\mathbb{E}(\tilde{r}_p - r_f)}{\sigma\left(\tilde{r}_p - r_f\right)} \\ &s.t. \\ &e^\top x = 1 \\ &x \geq 0. \end{aligned} \tag{17}$$

By setting $\xi = \sigma(\tilde{r}_p - r_f) > 0$, we can write the objective function in two components:

$$\begin{aligned} &\max_{x \in \mathbb{R}^n, \xi \in \mathbb{R}} \quad \bar{r}^\top \frac{x}{\xi} - r_f \frac{1}{\xi} \\ &s.t. \\ &\sigma\left(\left(\tilde{r} - r_f e\right)^\top x\right) \leq \xi \\ &e^\top x = 1 \\ &x \geq 0, \ \xi > 0. \end{aligned} \tag{18}$$

We define $x' = \frac{x}{\xi}$ and $\nu = \frac{1}{\xi}$, then the above can be written as follows:

$$\begin{aligned} &\max_{x' \in \mathbb{R}^n, \nu \in \mathbb{R}} \quad \bar{r}^\top x' - r_f \nu \\ &s.t. \\ &\sigma\left(\left(\tilde{r} - r_f e\right)^\top x'\right) \leq 1 \\ &e^\top x' = \nu \\ &x' \geq 0, \ \nu > 0. \end{aligned} \tag{19}$$

Replacing $e^\top x'$ for ν in the objective and $e^\top x' > 0$ for $\nu > 0$ in the constraint, we get

$$\begin{aligned} &\max_{x' \in \mathbb{R}^n} \quad \left(\bar{r} - r_f e\right)^\top x' \\ &s.t. \\ &\sigma\left(\left(\tilde{r} - r_f e\right)^\top x'\right) \leq 1 \\ &e^\top x' > 0 \\ &x' \geq 0. \end{aligned} \tag{20}$$

Therefore, the robust counterpart of maximum Sharpe is as follows:

$$\begin{aligned} &\max_{x' \in \mathbb{R}^n} \quad \min_{(\bar{r},\Sigma) \in U_I} \left(\bar{r} - r_f e\right)^\top x' \\ &s.t. \\ &\max_{(\bar{r},\Sigma) \in U_I} \sigma\left(\left(\tilde{r} - r_f e\right)^\top x'\right) \leq 1 \\ &e^\top x' > 0 \\ &x' \geq 0. \end{aligned} \tag{21}$$

Lotfi and Zenios (2018, Appendix A4) allows us to reformulate the inner minimization in the objective function to obtain:

$$\begin{aligned} &\max_{v'_+, v'_- \in \mathbb{R}^n,} \quad (\bar{r}_- - r_f e)^\top v'_- - \left(\bar{r}_+ - r_f e\right)^\top v'_+ \\ &s.t. \\ &\max_{(\bar{r},\Sigma) \in U_I} \sigma\left(\left(\tilde{r} - r_f e\right)^\top (v'_- - v'_+)\right) \leq 1 \\ &v'_- - v'_+ \geq 0 \\ &e^\top (v'_- - v'_+) > 0 \\ &v'_+, v'_- \geq 0. \end{aligned} \tag{22}$$

The dual formulation of the first constraint maximization problem is as follows.

$$\begin{aligned} &\min_{\Lambda_-, \Lambda_+ \in \mathbb{R}^{n\times n}} \quad tr(\Lambda_+ \Sigma_+) - tr(\Lambda_- \Sigma_-) \\ &s.t. \\ &\Lambda_+ - \Lambda_- \succeq (v'_- - v'_+)(v'_- - v'_+)^\top \\ &\Lambda_+, \Lambda_- \geq 0, \end{aligned} \tag{23}$$

which can be rewritten using Schur complement as the following:

$$\begin{aligned} &\min_{\Lambda_-, \Lambda_+ \in \mathbb{R}^{n\times n}} \quad tr(\Lambda_+ \Sigma_+) - tr(\Lambda_- \Sigma_-) \\ &s.t. \\ &\begin{bmatrix} \Lambda_+ - \Lambda_- & (v'_- - v'_+) \\ (v'_- - v'_+)^\top & 1 \end{bmatrix} \succeq 0 \\ &\Lambda_+, \Lambda_- \geq 0. \end{aligned} \tag{24}$$

One can replace the above with the first constraint maximization problem in (22) to get the required model. This completes the proof. □

.

Appendix 2.2: Mean-variance efficient frontiers

Mean-variance optimization under ambiguity in the means and covariance is given by:

$$\begin{aligned} &\max_{x \in \mathbb{X}} \quad \min_{(\bar{r},\Sigma)\in U_I} \ \mathbb{E}(\tilde{r}_p) \\ &s.t. \\ &\max_{(\bar{r},\Sigma)\in U_I} \sigma(\tilde{r}_p) \leq \sigma_0. \end{aligned} \tag{25}$$

Varying σ_0 we can trace the mean-variance efficient frontier.

Theorem 1 *The robust mean-variance portfolio optimization model* (25) *can be cast as follows*:

$$\begin{aligned} &\max_{v'_+, v'_- \in \mathbb{R}^n,\, \Lambda_+,\, \Lambda_- \in \mathbb{R}^{n\times n}} \quad (\bar{r}_- - r_f e)^\top v'_- - (\bar{r}_+ - r_f e)^\top v'_+ \\ &s.t. \\ &tr(\Lambda_+ \Sigma_+) - tr(\Lambda_- \Sigma_-) \leq \sigma_0 \\ &\begin{bmatrix} \Lambda_+ - \Lambda_- & (v'_- - v'_+) \\ (v'_- - v'_+)^\top & 1 \end{bmatrix} \succeq 0 \\ &v'_- - v'_+ \geq 0 \\ &e^\top (v'_- - v'_+) = 1 \\ &v'_+, v'_- \geq 0,\ \Lambda_+,\ \Lambda_- \geq 0, \end{aligned} \tag{26}$$

where tr denotes the trace operator. Given the optimal solutions of (26) v'^*_+ *and* v'^*_-, *we obtain the optimal solution of* (25) *as* $x^* = v'^*_- - v'^*_+$.

The proof follows from the proof of Theorem 4.1 above.

Acknowledgements Financial support was provided, in part, by the Cyprus Academy of Sciences, Letters, and Arts.

Funding Open access funding provided by the Cyprus Libraries Consortium (CLC).

Data availability statement The data used in this study are available from Datastream and Thomson Reuters Eikon but restrictions apply; these data were used under University of Cyprus license for the current study and so are not publicly available. Data are, however, available from the authors upon reasonable request and with proper permission.

References

Artzner P, Delbaen F, Eber JM, Heath D (1999) Coherent measures of risk. Math Finance 9:203–228
Ben-Tal A, El Ghaoui L, Nemirovski A (2009) Robust optimization. Princeton University Press, Princeton
Best MJ, Grauer RR (1991) Sensitivity analysis for mean-variance portfolio problems. Manag Sci 37:980–989
Bossaerts P, Ghirardato P, Guarnaschelli S, Zame WR (2010) Ambiguity in asset markets: theory and experiment. Rev Financ Stud 23:1325–1359
Boyle P, Garlappi L, Uppal R, Wang T (2012) Keynes meets Markowitz: the trade-off between familiarity and diversification. Manag Sci 58:253–272
Broadie M (1993) Computing efficient frontiers using estimated parameters. Ann Oper Res 45:21–58
Cao HH, Wang T, Zhang HH (2005) Model uncertainty, limited market participation, and asset prices. Rev Financ Stud 18:1219–1251
Ceria S, Stubbs RA (2006) Incorporating estimation errors into portfolio selection: robust portfolio construction. J Asset Manag 7:109–127
Chan LK, Karceski J, Lakonishok J (1999) On portfolio optimization: forecasting covariances and choosing the risk model. Rev Financ Stud 12:937–974
Chopra VK, Ziemba WT (1993) The effect of errors in means, variances, and covariances on optimal portfolio choice. J Portf Manag 19:6–11
Cooper I, Sercu P, Vanpée R (2012) The equity home bias puzzle: a survey. Found Trends Finance 7:289–416
Dow BYJ, Da SR, Werlang C (1992) Uncertainty aversion, risk aversion, and the optimal choice of portfolio. Econometrica 60:197–204
Easley D, O'Hara M (2009) Ambiguity and nonparticipation: the role of regulation. Rev Financ Stud 22:1817–1843
El Ghaoui L, Oks M, Oustry F (2003) Worst-case value-at-risk and robust portfolio optimization: a conic programming approach. Oper Res 51(4):543–556
Epstein LG, Miao J (2003) A two-person dynamic equilibrium under ambiguity. J Econ Dyn Control 27:1253–1288
Epstein LG, Wang T (1994) Intertemporal asset pricing under Knightian uncertainty. Econometrica 62:283–322
Farinelli S, Ferreira M, Rossello D, Thoeny M, Tibiletti L (2008) Beyond sharpe ratio: optimal asset allocation using different performance ratios. J Bank Finance 32:2057–2063
French KR, Poterba JM (1991) Investor diversification and international equity markets. Am Econ Rev 81:222–226
Gala VD, Pagliardi G, Zenios SA (2023) Global political risk and international stock returns. J Empirical Finance 72:78–102. https://dio.org/10.1016/j.jempfin.2023.03.004
Gao J, Zhou K, Li D, Cao X (2017) Dynamic mean-LPM and mean-CVaR portfolio optimization in continuous-time. SIAM J Control Optim 55:1377–1397
Gilboa I, Schmeidler D (1989) Maxmin expected utility with non-unique prior. J Math Econ 18:141–153
Goldfarb D, Iyengar G (2003) Robust portfolio selection problems. Math Oper Res 28:1–38
Gotoh J, Takano Y (2007) Newsvendor solutions via conditional value-at-risk minimization. Eur J Oper Res 179:80–96
Grant M, Boyd S (2014) CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx
Hara C, Honda T (2022) Implied ambiguity: mean-variance inefficiency and pricing errors. Manag Sci 68:4246–4260
Hong LJ, Liu G (2009) Simulating sensitivities of conditional value at risk. Manag Sci 55:281–293
Jagannathan R, Ma T (2003) Risk reduction in large portfolios: why imposing the wrong constraints helps. J Finance 58:1651–1683
Kaut M, Vladimirou H, Wallace SW, Zenios SA (2007) Stability analysis of portfolio management with conditional value-at-risk. Quant Finance 7(4):397–409
Klibanoff P, Marinacci M, Mukerji S (2005) A smooth model of decision making under ambiguity. Econometrica 73:1849–1892

Lotfi S, Zenios S (2018) Robust VaR and CVaR optimization under joint ambiguity in distributions, means, and covariances. Eur J Oper Res 269(2):556–576

Lotfi S, Zenios SA (2023) An explanation of under-diversification puzzles through correlated returns ambiguity. Available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4638276

Mulvey J, Vanderbei R, Zenios S (1995) Robust optimization of large-scale systems. Oper Res 43:264–281

Paç AB, Pınar MÇ (2014) Robust portfolio choice with CVaR and VaR under distribution and mean return ambiguity. TOP 22:875–891

Pagliardi G, Lotfi S, Paparoditis E, Zenios SA (2021) Hedging political risk in international equity portfolios. Available at https://ssrn.com/abstract=3891070

Pang T, Karan C (2018) A closed-form solution of the Black–Litterman model with conditional value at risk. Oper Res Lett 46:103–108

Paparoditis E, Politis DN (2003) Residual-based block bootstrap for unit root testing. Econometrica 71(3):813–855

Peijnenburg K (2018) Life-cycle asset allocation with ambiguity aversion and learning. J Financ Quant Anal 53:1963–1994

Rachev ST, Stoyanov SV, Fabozzi FJ (2008) Advanced stochastic models, risk assessment, and portfolio optimization. Wiley, Hoboken

Rockafellar R, Uryasev S (2002) Conditional value-at-risk for general loss distributions. J Bank Finance 26:1443–1471

Rockafellar RT, Uryasev SP, Zabarankin M (2006) Master funds in portfolio analysis with general deviation measures. J Bank Finance 30:743–778

Sharpe W (1994) The sharpe ratio. J Portf Manag 21:49–58

Stoyanov SV, Rachev ST, Fabozzi FJ (2013) Sensitivity of portfolio VaR and CVaR to portfolio return characteristics. Ann Oper Res 205:169–187

Tobin J (1958) Liquidity preference as behavior towards risk. Rev Econ Stud 25(2):65–86

Tütüncü RH, Koenig M (2004) Robust asset allocation. Ann Oper Res 132:157–187

Uppal R, Wang T (2003) Model misspecification and underdiversification. J Finance 58:2465–2486

Ye K, Parpas P, Rustem B (2012) Robust portfolio optimization: a conic programming approach. Comput Optim Appl 52:463–481

Zhu S, Fukushima M (2009) Worst-case conditional value-at-risk with application to robust portfolio management. Oper Res 57:1155–1168