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Thermal convection with a Cattaneo heat flux model

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The problem of thermal convection in a layer of viscous incompressible fluid is analysed. The heat flux law is taken to be one of Cattaneo type. The time derivative of the heat flux is allowed to be a material derivative, or a general objective derivative. It is shown that only one objective derivative leads to results consistent with what one expects in real life. This objective derivative leads to a Cattaneo–Christov theory, and the results for linear instability theory are in agreement with those for a material derivative. It is further shown that none of the theories allow a standard nonlinear, energy stability analysis. A further heat flux due to P.M. Mariano is added and then an analysis is performed for stationary convection, oscillatory convection, and fully nonlinear theory. For the material derivative case, the analysis proceeds and global nonlinear stability is achieved. For Cattaneo–Christov theory, it appears necessary to add a regularization term in the equation for the heat flux, and even then the analysis only works in two space dimensions, and is conditional upon the size of the initial data. For the three-dimensional situation, it is shown how a nonlinear stability analysis may be achieved with a Navier–Stokes–Voigt fluid rather than a Navier–Stokes one.

1. Introduction

The problem of thermal convection where a fluid layer is heated from below and under appropriate conditions the fluid rises and forms into a pattern of convection cells is well known. The phenomenon was observed by Thompson [1] and by Bénard [2], see e.g. the historical article by Wesfreid [3]. A complete mathematical analysis for the linearized instability theory is lucidly described in the brilliant book by Chandrasekhar [4].

The classical thermal convection problem, or the Bénard problem as it is often known, is based on the heat flux being described by Fourier's Law, namely, the heat flux is a linear function of the temperature gradient. In a highly cited article, Cattaneo [5] introduced a generalization of Fourier's Law which could avoid the problem whereby a temperature disturbance travels with an infinite speed. Cattaneo's heat flux law involves a relaxation time for the heat flux and is now well appreciated in the field.

Straughan & Franchi [6] introduced the Cattaneo law into the instability problem for thermal convection. Various works followed this and Christov [7] replaced the material derivative in the Cattaneo relation by a Lie derivative on the basis that this derivative is objective. The resulting theory for thermal motion in a fluid employing the Cattaneo heat flux law is known as Cattaneo–Christov theory, see e.g. Ciarletta & Straughan [8], Straughan [9], Papanicolaou [10], Tibullo & Zampoli [11] and Straughan [12]. This generalized theory of thermal convection has become very popular in the research literature and continues to attract much attention, see e.g. Bissell [13,14], Eltayeb [15,16], Eltayeb *et al.* [17], Hughes *et al.* [18,19], Capone & Gianfrani [20], Shivakumara *et al.* [21], Hema *et al.* [22], Mamatha *et al.* [23], Dávalos Orozco & Diaz [24], Riaz Khan & Mao [25] and Straughan [26].

Much of the attention in the Cattaneo–Christov theory of thermal convection is driven by application to fundamental areas of real life. For example, this theory is employed in convection stars, see e.g. Falcón [27], Falcón & Labrador [28], Herrera & Falcón [29–31], Herrera & Santos [32,33], Herrera & Martinez [34–36], Herrera & Pavón [37] and Herrera [38]. It has been used in studying collapse in stellar interiors, Govender & Govinder [39], Govender & Thirukkanesh [40] and Govender *et al.* [41]. A particularly interesting application is to volcanic action in planets, Bargmann *et al.* [42]. A further recent application is to thermal convection in micro channels, see e.g. Khadrawi [43]. Yet further application is in the important field of potential for possible hydrogen energy production, see Riza Khan & Mao [25].

There are differing attitudes towards analysing non-isothermal fluid motion with a Cattaneo heat flux law. One avenue treats the Cattaneo equation as a balance law and uses the material time derivative for the relaxation term. The other school of thought argues that in the limit of relaxation one recovers Fourier's Law and as this is a constitutive equation then Cattaneo's equation should likewise represent a constitutive equation. In this case, the material time derivative should be replaced by an objective time derivative. The goal of this work is to give a complete analysis of linear instability theory *and* investigate fully nonlinear energy stability theory for thermal convection when Cattaneo's law is used. We employ both a material derivative for the heat flux and a general form of objective derivative for the same quantity. In this way, we obtain a complete mathematical analysis which is then compared with what is expected to be physically realistic behaviour.

2. Cattaneo models and thermal convection

The equations for fluid motion are comprised of the balance of linear momentum, balance of mass, balance of energy and an equation governing the evolutionary behaviour of the heat flux. Let $v_i(\mathbf{x},t)$, $p(\mathbf{x},t)$, $T(\mathbf{x},t)$ and $Q_i(\mathbf{x},t)$ denote the velocity field at position $\mathbf{x}$ and time t, the pressure, the temperature and the heat flux vector, respectively. The balance of linear momentum with a Boussinesq approximation may be written as, cf. Chandrasekhar [4], Barletta [44], Breugem & Rees [45],

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu\Delta v_i + \alpha g k_i T, \tag{2.1}$$

where ρ_0 is a constant reference density, ν is the kinematic viscosity, α is the coefficient of expansion of the fluid, g is gravity, $\mathbf{k}=(0,0,1)$ and Δ is the three-dimensional Laplacian.

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Throughout, we employ standard indicial notation in conjunction with the Einstein summation convention. Hence, for example

$$\frac{\partial v_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

where $\mathbf{v} = (u, v, w)$ and $\mathbf{x} = (x, y, z)$. For another example,

$$\begin{aligned} v_i T_{,i} &\equiv \sum_{i=1}^{3} v_i \frac{\partial T}{\partial x_i} \\ &= u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}. \end{aligned}$$

Since the fluid is incompressible

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{2.2}$$

The equation for the balance of energy may be written as

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = -\frac{\partial Q_i}{\partial x_i}. \tag{2.3}$$

The equation for the heat flux may, in principle, assume various forms. In the first case, one may follow, for example, Christov & Jordan [46], Jou *et al.* [47], Sellitto *et al.* [48], and write

$$\tau\left(\frac{\partial Q_i}{\partial t} + v_j \frac{\partial Q_i}{\partial x_j}\right) + Q_i = -\kappa \frac{\partial T}{\partial x_i}, \tag{2.4}$$

where (2.4) represents a form of Cattaneo's equation in a moving fluid, τ is a relaxation coefficient and κ may be taken to be the thermal diffusivity. Equation (2.4) regards the heat flux as a fundamental variable and this equation may be thought of as a conservation law for Q_i.

There is an alternative school of thought which argues that as $\tau \to 0$ the equation for Q_i becomes a constitutive equation, namely Fourier's Law, $\mathbf{Q} = -\kappa \nabla T$. Then, the derivative in the balance law for $\mathbf{Q}$ should not be the material derivative as in (2.4), but should employ an objective time derivative for $\mathbf{Q}$, as discussed in Morro [49,50], see also Capriz & Mariano [51]. A general form of objective derivative is given by Morro [49,50] and then one may replace equation (2.4) by

$$\tau\left(\frac{\partial Q_i}{\partial t} + v_j \frac{\partial Q_i}{\partial x_j} - W_{ij}Q_j + \gamma D_{ij}Q_j\right) + Q_i = -\kappa \frac{\partial T}{\partial x_i}, \tag{2.5}$$

where γ we take as a constant, and W_{ij}, D_{ij} are the skew and symmetric parts of the velocity gradient $v_{i,j}$, defined by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \quad \text{and} \quad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

We refer to the model given by (2.1), (2.2), (2.3) and (2.4) as describing material derivative theory. The model given by (2.1), (2.2), (2.3) and (2.5), we describe as Cattaneo–Morro theory.

Special cases of equation (2.5) have been analysed in the literature, for example, when $\gamma = 0$ the objective derivative is a Jaumann derivative and the resulting theory is known as Cattaneo–Fox theory, cf. Straughan & Franchi [6], Straughan [9, pp. 228–232]. When $\gamma = 1$ the objective derivative is a Cotter–Rivlin one, see Morro [49,50]. A particular case which has attracted a lot of attention is when $\gamma = -1$ for which the derivative is associated with the names of Truesdell [52], Christov [7], and the resulting theory is known as Cattaneo–Christov theory, cf. Ciarletta & Straughan [8], Straughan [53], Tibullo & Zampoli [11], Straughan [12] and Straughan [9, pp. 233–237].

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We suppose the fluid occupies the horizontal layer $\{(x,y)\in\mathbb{R}^2\}\times\{0<z<d\}$ with the boundary conditions

$$v_i=0,\ z=0,d;\quad T=T_L,\ z=0;\quad T=T_U,\ z=d; \tag{2.6}$$

where T_L, T_U are constants with $T_L>T_U$.

For each of the models introduced above, i.e. the material derivative and the Cattaneo–Morro for any γ, the steady solution subject to the boundary conditions (2.6) is

$$\bar{v}_i\equiv 0,\quad \bar{T}=-\beta z+T_L,\ \ \bar{\mathbf{Q}}=(0,0,\kappa\beta), \tag{2.7}$$

where β is the temperature gradient

$$\beta=\frac{T_L-T_U}{d}>0.$$

The steady pressure $\bar{p}(z)$ is found from the momentum equation.

In the next section, we investigate the instability of this stationary solution.

3. Linear instability theory

Perturbations u_i,θ,π,q_i are introduced to $\bar{v}_i,\bar{T},\bar{p},\bar{Q}_i$ and these are non-dimensionalized with the scales

$$\begin{gathered}x_i=dx_i^*,\quad t=\mathcal{T}t^*,\quad u_i=Uu_i^*,\quad \theta=T^\sharp\theta^*,\\ q_i=Q^\sharp q_i^*,\quad \pi=P\pi^*,\quad T^\sharp=U\sqrt{\frac{\beta\nu}{\kappa\alpha g}},\quad Sg=\frac{\tau\nu}{d^2}\\ \mathcal{T}=\frac{d^2}{\nu},\quad P=\frac{\rho_0\nu U}{d},\quad U=\frac{\nu}{d},\quad Q^\sharp=\frac{\kappa T^\sharp}{d},\quad Pr=\frac{\nu}{\kappa},\end{gathered} \tag{3.1}$$

and the Rayeigh number is defined as

$$Ra=R^2=\frac{\alpha\beta g d^4}{\kappa\nu}.$$

The non-dimensional fully nonlinear perturbation equations are now for the material derivative system

$$\begin{aligned}&u_{i,t}+u_ju_{i,j}=-\pi_{,i}+\Delta u_i+Rk_i\theta,\\ &u_{i,i}=0,\\ &Pr(\theta_{,t}+u_i\theta_{,i})=Rw-q_{i,i},\\ &Sg(q_{i,t}+u_jq_{i,j})=-\theta_{,i}-q_i,\end{aligned} \tag{3.1}$$

whereas, for the Cattaneo–Morro system, one has

$$\begin{aligned}&u_{i,t}+u_ju_{i,j}=-\pi_{,i}+\Delta u_i+Rk_i\theta,\\ &u_{i,i}=0,\\ &Pr(\theta_{,t}+u_i\theta_{,i})=Rw-q_{i,i},\\ &Sg\left(q_{i,t}+u_jq_{i,j}-\left[\frac{1-\gamma}{2}\right]u_{i,j}q_j+\left[\frac{1+\gamma}{2}\right]u_{j,i}q_j\right)=\\ &\qquad -\theta_{,i}-q_i+\frac{SgR}{2Pr}(1-\gamma)u_{i,3}-\frac{SgR}{2Pr}(1+\gamma)w_{,i}.\end{aligned} \tag{3.2}$$

These equations hold on $\mathbb{R}^2\times\{z\in(0,1)\}\times\{t>0\}$, with the boundary conditions

$$u_i=0,\quad \theta=0,\ \ z=0,1, \tag{3.3}$$

together with the fact that u_i,θ,π,q_i satisfy plane tiling periodic boundary conditions in x,y. The forms of the convection cell resulting from the periodicity are triangles, rectangles and hexagons, and full details may be found in Chandrasekhar [4, pp. 43–53]. These equations are linearized and

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then one puts $u_i = e^{\sigma t}u_i(\mathbf{x})$, $\theta = e^{\sigma t}\theta(\mathbf{x})$, $q_i = e^{\sigma t}q_i(\mathbf{x})$, $\pi = e^{\sigma t}\pi(\mathbf{x})$. This yields the linearized system of equations Cattaneo–Morro theory as

$$\begin{aligned}
&\sigma u_i = -\pi_{,i} + \Delta u_i + Rk_i\theta,\\
&u_{i,i} = 0,\\
&Pr\sigma\theta = Rw - q_{i,i}\,,\\
&Sg\sigma q_i = -\theta_{,i} - q_i + \frac{SgR}{2Pr}(1-\gamma)u_{i,3} - \frac{SgR}{2Pr}(1+\gamma)w_{,i},
\end{aligned} \tag{3.4}$$

where the analogous material derivative equations do *not* contain the $SgR/2Pr$ terms.

Equations (3.4) are handled by a normal mode analysis, cf. Chandrasekhar [4, pp. 22–43], Barletta [54,55]. We take curlcurl of $(3.4)_1$ and retain the third term, and we take the derivative of $(3.4)_4$. This results in the system of equations

$$\begin{aligned}
&-\sigma\Delta w = -\Delta^2 w - R\Delta^*\theta,\\
&Pr\sigma\theta = Rw - h,\\
&Sg\sigma h = -\Delta\theta - h - \frac{SgR}{2Pr}(1+\gamma)\Delta w,
\end{aligned} \tag{3.5}$$

where $h = q_{i,i}$ and $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$. One now puts $w = W(z)\phi(x,y)$, $\theta = \Theta(z)\phi(x,y)$, and $h = H(z)\phi(x,y)$, where ϕ satisfies $\Delta^*\phi = -a^2\phi$, a being a wavenumber, Chandrasekhar [4]. As in Hughes [18], we analyse the situation for two stress-free surfaces, and then W, Θ, H may be written as a sin series, e.g. $W = \sum_{i=1}^{\infty} W_n \sin n\pi z$, for which system (3.5) yields a determinant which results in a cubic equation for σ. The real and imaginary parts of this equation are found and these lead to the following equations:

$$R^2_{\text{stat}} = \frac{\Lambda^3}{a^2}\,\frac{1}{[1-(1+\gamma)SgPr\Lambda/2]} \tag{3.6}$$

and

$$R^2_{\text{osc}} = \frac{\Lambda^2}{a^2}\,\frac{Pr}{Sg[1+(1+\gamma)/2Pr]} + \frac{\Lambda}{a^2}\,\frac{(Pr+1)}{Sg^2[1+(1+\gamma)/2Pr]}, \tag{3.7}$$

where R^2_{stat} and R^2_{osc} are the Rayleigh numbers for stationary and oscillatory convection, respectively, and $\Lambda = \pi^2 + a^2$. The critical Rayleigh numbers are found by minimizing (3.6) and (3.7) in a^2.

One notes that (3.6) allows the possibility of convective fluid motion (instability) when heating from above if the denominator is negative. Thus, for physically correct behaviour, we require $\gamma \leq -1$. However, for γ in this range (3.7) may then allow R^2_{osc} to be negative. Thus, one may argue in favour of neglecting the Cattaneo–Fox and Cotter–Rivlin theories since $\gamma = 0$ and 1, respectively. The Cattaneo–Christov theory has $\gamma = -1$ and this agrees exactly (in the linear case) with the material derivative theory. For this case

$$R^2_{\text{stat}} = \frac{\Lambda^3}{a^2} \quad \text{and} \quad R^2_{\text{osc}} = \frac{Pr}{Sg}\,\frac{\Lambda^2}{a^2} + \frac{(Pr+1)}{Sg^2}\,\frac{\Lambda}{a^2},$$

cf. Straughan [53], Straughan [9, pp. 233–237]. Hence for two stress-free surfaces $Ra_{\text{stat}} = 27\pi^4/4$, $a^2_{\text{stat}} = \pi^2/2$ and a^2_{osc}, Ra_{osc} may be found in equations (8.39) and (8.40) of Straughan [9], and oscillatory convection may only occur for Sg large enough. Oscillatory convection is important in that when this occurs the ensuing convective motion oscillates periodically in time which is a phenomenon which affects the physics substantially.

The critical Rayleigh number as found here is essential in that if the Rayleigh number exceeds this critical value then instability via a convective motion will arise.

4. Nonlinear energy stability

It is worth observing that existence results for a suitable weak solution to the equations when Cattaneo's Law holds are provided by Boukrouche *et al.* [56]. As the system of equations begins with the Navier–Stokes equations, the overall global existence situation is unknown, just as it is for a solution to the Navier–Stokes equations.

If we follow the classical procedure to construct a nonlinear stability analysis for either equations (3.1) or (3.2) then we define a convection cell V to be the periodic cell in the (x, y) plane (which we take as a hexagon) extended to a three-dimensional cell over $z \in (0, 1)$. The procedure begins with multiplying $(3.1)_1$ or $(3.2)_1$ by u_i and integrating over V. Next, multiply $(3.1)_3$ or $(3.2)_3$ by θ and integrate over V. After integration by parts and use of the boundary conditions we obtain for either system the 'energy' identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}||\mathbf{u}||^2 = -||\nabla\mathbf{u}||^2 + R(\theta, w) \tag{4.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{Pr}{2}||\theta||^2 = R(\theta, w) - (q_{i,i}, \theta). \tag{4.2}$$

We next multiply $(3.1)_4$ by q_i and $(3.2)_4$ by q_i and integrate each over V. In this way, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{Sg}{2}||\mathbf{q}||^2 = -||\mathbf{q}||^2 - (\theta_{,i}, q_i), \tag{4.3}$$

for the material derivative system, or

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Sg}{2}||\mathbf{q}||^2 &= \left(\frac{1-\gamma}{2}\right)\int_V u_{i,j}q_iq_j\,\mathrm{d}x - \left(\frac{1+\gamma}{2}\right)\int_V u_{j,i}q_iq_j\,\mathrm{d}x - ||\mathbf{q}||^2 - (\theta_{,i}, q_i)\\ &\quad + \frac{SgR}{2Pr}(1-\gamma)(u_{i,3}, q_i) - \frac{SgR}{2Pr}(1+\gamma)(w_{,i}, q_i),\end{aligned} \tag{4.4}$$

for the Cattaneo–Morro system.

To obtain a general energy equation for each system, one now adds (4.1) to (4.2) and then adds (4.3) or (4.4) depending on whether material derivative or Cattaneo–Morro is considered.

Define the energy function E by

$$E(t) = \frac{1}{2}||\mathbf{u}||^2 + \frac{Pr}{2}||\theta||^2 + \frac{Sg}{2}||\mathbf{q}||^2, \tag{4.5}$$

and the dissipation function D by

$$D(t) = ||\nabla\mathbf{u}||^2 + ||\mathbf{q}||^2, \tag{4.6}$$

and define the production terms I and I_2 by

$$I = 2(w, \theta) \tag{4.7}$$

and

$$I_2 = 2(w, \theta) + \frac{Sg}{2Pr}[(1-\gamma)(u_{i,3}, q_i) - (1+\gamma)(w_{,i}, q_i)]. \tag{4.8}$$

The energy equation for material derivative theory is then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = RI - D \leq -D\left(1 - \frac{R}{R_E}\right),$$

where

$$\frac{1}{R_E} = \max_H \frac{I}{D}, \tag{4.9}$$

and where H is the space of admissible solutions.

Consider the maximum problem for R_E, namely

$$\frac{1}{R_E} = \max_H \frac{2(\theta, w)}{\|\nabla \mathbf{u}\|^2 + \|\mathbf{q}\|^2}. \tag{4.10}$$

The space H consists of $\{\mathbf{u} \in (H^1(V))^3\}$ where $u_{i,i} = 0$ in V and u_i satisfies the boundary conditions, $\{\theta \in H^1(V)\}$ with the boundary conditions $\theta = 0$ on $z = 0, 1$, and is spatially periodic in (x, y), and $\{\mathbf{q} \in L^2(V)\}$ together with periodicity in (x, y). The definition of R_E immediately introduces a fundamental problem, in that since θ is not in the denominator the maximum does not exist.

A similar outcome arises for Cattaneo–Morro theory but now another parameter R_E arises, namely R_{EM} where

$$\frac{1}{R_{EM}} = \max_H \frac{I_2}{D}.$$

In addition, for Cattaneo–Morro theory, the energy equation is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = RI_2 - D - \gamma \int_V d_{ij} q_i q_j \,\mathrm{d}x, \tag{4.11}$$

where $d_{ij} = (u_{i,j} + u_{j,i})/2$ and the cubic term in $d_{ij} q_i q_j$ introduces a further complication.

The point is that for all theories, either that with the material derivative or *any* of Cattaneo–Morro type (which includes Cattaneo–Christov, Cattaneo–Fox or Cotter–Rivlin theory) the maximum of the production divided by the dissipation does not exist. Therefore, an entirely different strategy is required.

5. Balance of energy equation with a Mariano flux

To overcome the problem of §4 and derive a meaningful nonlinear energy stability analysis, we employ an idea of Mariano [57] and modify the balance of energy equation (2.3). The idea is to split the heat flux into two parts. Thus, one replaces equation (2.3) by

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = -\frac{\partial Q_i}{\partial x_i} - \frac{\partial F_i}{\partial x_i}, \tag{5.1}$$

where the total heat flux is now $\mathbf{Q} + \mathbf{F}$. The F_i term is given by a Fourier's Law so

$$F_i = -\hat{\zeta} \frac{\partial T}{\partial x_i}, \tag{5.2}$$

where $\hat{\zeta} > 0$ is a constant. The flux Q_i is given by equation (2.4) for a material derivative theory and by (2.5) for the Cattaneo–Morro theory.

Thus, the governing equations for a material derivative theory now become

$$\left.\begin{aligned}
&\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \Delta v_i + \alpha g k_i T, \\
&\frac{\partial v_i}{\partial x_i} = 0, \\
&\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = -\frac{\partial Q_i}{\partial x_i} + \hat{\zeta} \Delta T, \\
&\tau \left(\frac{\partial Q_i}{\partial t} + v_j \frac{\partial Q_i}{\partial x_j} \right) + Q_i = -\kappa \frac{\partial T}{\partial x_i},
\end{aligned}\right\} \tag{5.3}$$

whereas those for Cattaneo–Morro theory take the form

$$\begin{aligned}
&\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x_i}+\nu\Delta v_i+\alpha g k_i T,\\
&\frac{\partial v_i}{\partial x_i}=0,\\
&\frac{\partial T}{\partial t}+v_i\frac{\partial T}{\partial x_i}=-\frac{\partial Q_i}{\partial x_i}+\hat{\zeta}\Delta T,\\
&\tau\left(\frac{\partial Q_i}{\partial t}+v_j\frac{\partial Q_i}{\partial x_j}-W_{ij}Q_j+\gamma D_{ij}Q_j\right)+Q_i=-\kappa\frac{\partial T}{\partial x_i}.
\end{aligned}\tag{5.4}$$

The basic state in the case of (5.3) or (5.4) remains the same as (2.7). We now proceed to analyse the stability and instability of this conduction solution.

6. Unconditional nonlinear energy stability for material derivative theory with a Mariano flux

The non-dimensional perturbation equations in the case of the material derivative theory are

$$\begin{aligned}
&u_{i,t}+u_j u_{i,j}=-\pi_{,i}+\Delta u_i+Rk_i\theta,\\
&u_{i,i}=0,\\
&Pr(\theta_{,t}+u_i\theta_{,i})=Rw-q_{i,i}+\zeta\Delta\theta,\\
&Sg(q_{i,t}+u_j q_{i,j})=-\theta_{,i}-q_i,
\end{aligned}\tag{6.1}$$

where ζ is a non-dimensional form of $\hat{\zeta}$.

One may proceed to develop a linear instability analysis as in §3 employing a normal mode technique, cf. Barletta [54,55]. The details are similar to those of §3 and one may show that the Rayleigh number for stationary convection is given by

$$R^2_{\text{stat}}=(\zeta+1)\frac{\Lambda^3}{a^2},\tag{6.2}$$

whereas the oscillatory convection boundary follows from

$$R^2_{\text{osc}}=\zeta\frac{\Lambda^3}{a^2}+\left[\frac{\zeta+Pr^2+\zeta^2+2\zeta Pr}{Sg(\zeta+Pr)}\right]\frac{\Lambda^2}{a^2}+\left[\frac{Pr^2+(1+\zeta)Pr}{Sg^2(\zeta+Pr)}\right]\frac{\Lambda}{a^2}.\tag{6.3}$$

In either case, one has to minimize R^2_{stat} or R^2_{osc} in a^2 to find the critical Rayleigh and wavenumbers. Upon performing this minimization, one sees that for stationary convection with two stress-free surfaces

$$R^2_{\text{stat}}=\frac{27\pi^4}{4}(\zeta+1)\quad\text{and}\quad a^2_{\text{stat}}=\frac{\pi^2}{2}.\tag{6.4}$$

The minimum of the oscillatory convection boundary is found by minimizing (6.3) in a^2 by numerical means. Numerical results are given after the following energy stability analysis.

To develop a nonlinear energy stability analysis, we multiply $(6.1)_1$ by u_i and integrate over V. Then multiply $(6.1)_3$ by θ and integrate over V. Finally, we multiply $(6.1)_4$ by q_i and integrate over V. The resulting equations are added and after integration by parts and use of the boundary conditions one may arrive at an energy equation

$$\frac{\mathrm{d}E}{\mathrm{d}t}=RI-D,$$

where

$$E=\frac{1}{2}||\mathbf{u}||^2+\frac{Pr}{2}||\theta||^2+\frac{Sg}{2}||\mathbf{q}||^2,$$

the production term I is

$$I = 2(w, \theta),$$

and the dissipation D is

$$D = ||\nabla \mathbf{u}||^2 + \zeta ||\nabla \theta||^2 + ||\mathbf{q}||^2.$$

Define now R_E by

$$R_E = \max_H \frac{I}{D}, \tag{6.5}$$

where H is the space of admissible solutions. Then

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -D\left(1 - \frac{R}{R_E}\right),$$

and so if $R < R_E$ we find with $b = 1 - R/R_E > 0$, and use of Poincaré's inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -bc_1 E,$$

for a constant $c_1 > 0$. Hence, E decays rapidly and one obtains global stability.

Due to the absence of q_i in the numerator, the Euler–Lagrange equations corresponding to (6.5) may be taken to be

$$\begin{aligned} \Delta u_i + R_E \theta k_i &= \varphi_{,i}, \\ u_{i,i} &= 0, \\ \zeta \Delta \theta + R_E w &= 0, \end{aligned} \tag{6.6}$$

where φ is a Lagrange multiplier. For two free surfaces, one solves (6.6) to find

$$R_E^2 = \zeta \frac{\Lambda^3}{a^2}, \tag{6.7}$$

and so the critical nonlinear energy Rayleigh number is given by

$$R_E^2 = \frac{27\pi^4 \zeta}{4}. \tag{6.8}$$

It is important to note that

$$R^2_{\text{stat}} \ge R_E^2 \quad \text{and} \quad R^2_{\text{osc}} \ge R_E^2.$$

The thresholds R^2_{stat} and R^2_{osc} represent linear instability thresholds. The values of R_E^2 represent a threshold for global nonlinear stability.

Tables 1 and 2 display values for R^2_{osc} after the minimization has been performed. The critical value of a^2_{osc} is also given. For comparison, the values of R^2_{stat} and R_E^2 are also given in these tables. In the tables $Pr = 20$, $\zeta = 1$ in table 1 whereas $\zeta = 20$ in table 2. It is seen in table 1 that when $Sg = 2, \ldots, 10$ oscillatory convection will occur. In table 2, oscillatory convection occurs when $Sg = 3, \ldots, 10$. The oscillatory wavenumber a^2_{osc} is decreasing in both tables as Sg increases. Thus, increasing the coefficient τ leads to the convection cells becoming more narrow.

7. Nonlinear energy stability for Cattaneo–Christov–Mariano theory with a Navier–Stokes fluid

In the fully nonlinear case of Cattaneo–Christov–Mariano theory, we have not been able to develop an energy stability analysis employing equations (3.2) together with a Mariano extra flux term. Instead, it appears that one requires extra dissipation in the evolution equation for q_i. This is thus a very important difference between the material derivative theory and the Cattaneo–Christov–Mariano theory. We thus appeal to work of Ván & Fülöp [58, eq. (5)], see also Mariano [57, eq. (1.71)], where one includes a Laplacian of q_i in addition to the Mariano flux. Further

Table 1. Critical values for the Rayleigh number for stationary convection, Ra_{stat}, for oscillatory convection, Ra_{osc}, and for global nonlinear stability, Ra_{en}, for various values of Sg. The a^2_{osc} values are the values of a^2 for oscillatory convection. The equivalent value for a^2 for stationary convection is $a^2 = \pi^2/2$. Here, $Pr = 20$ and $\zeta = 1$.

Ra_{stat}	Ra_{en}	Ra_{osc}	a^2_{osc}	Sg
1315.02	657.511	1591.99	7.050	1
1315.02	657.511	1121.20	6.226	2
1315.02	657.511	966.605	5.872	3
1315.02	657.511	889.503	5.672	4
1315.02	657.511	843.248	5.543	5
1315.02	657.511	812.396	5.453	6
1315.02	657.511	790.345	5.386	7
1315.02	657.511	773.796	5.334	8
1315.02	657.511	760.917	5.293	9
1315.02	657.511	750.609	5.260	10

Table 2. Critical values for the Rayleigh number for stationary convection, Ra_{stat}, for oscillatory convection, Ra_{osc}, and for global nonlinear stability, Ra_{en}, for various values of Sg. The a^2_{osc} values are the values of a^2 for oscillatory convection. The equivalent value for a^2 for stationary convection is $a^2 = \pi^2/2$. Here, $Pr = 20$ and $\zeta = 20$.

Ra_{stat}	Ra_{en}	Ra_{osc}	a^2_{osc}	Sg
13807.7	13150.2	14989.7	5.265	1
13807.7	13150.2	14059.8	5.102	2
13807.7	13150.2	13754.3	5.046	3
13807.7	13150.2	13602.5	5.019	4
13807.7	13150.2	13511.6	5.002	5
13807.7	13150.2	13451.2	4.991	6
13807.7	13150.2	13408.0	4.983	7
13807.7	13150.2	13375.7	4.977	8
13807.7	13150.2	13350.6	4.972	9
13807.7	13150.2	13330.5	4.968	10

relevant articles on this aspect are those of Rogolino & Cimmelli [59] and Carlomagno *et al.* [60]. The relevant nonlinear perturbation equations are then

$$\begin{aligned} &u_{i,t} + u_j u_{i,j} = -\pi_{,i} + \Delta u_i + R\theta k_i, \\ &u_{i,i} = 0, \\ &Pr(\theta_{,t} + u_i\theta_{,i}) = Rw - q_{i,i} + \zeta\Delta\theta, \\ &Sg(q_{i,t} + u_j q_{i,j} - u_{i,j}q_j) = \frac{RSg}{Pr}\, u_{i,3} - q_i - \theta_{,i} + \epsilon\Delta q_i. \end{aligned} \tag{7.1}$$

The ϵ term in (7.1) is described by Ván & Fülöp [58] and by Mariano [57], and as observed by Capriz *et al.* [61, after eq. (102)], is of a class of regularization terms frequently added. It is important to observe that an energy balance law with extra regularizing terms like $(7.1)_4$ is believed to be relevant also when one is dealing with everyday temperatures, as is explained in the excellent article of Ván *et al.* [62].

It transpires that the difficulty with an energy stability analysis for (7.1) arises from the term $-Sg\, u_{i,j}q_j$. The subsequent manipulation of this depends on the choice of energy functional and the Sobolev inequality. In fact, with a Navier–Stokes theory like (7.1), we only proceed in two space dimensions. To address the three-dimensional problem appears to need extra regularity via a Navier–Stokes–Voigt theory, and this is developed in the next section.

Hence, we now proceed on the understanding that the period cell is a two-dimensional one. We multiply $(7.1)_1$ by u_i and integrate over V. We also multiply $(7.1)_3$ by θ, equation $(7.1)_4$ by q_i and integrate each over V. In this way, we obtain the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}||\mathbf{u}||^2 = -||\nabla\mathbf{u}||^2 + R(\theta, w), \tag{7.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{Pr}{2}||\theta||^2 = -\zeta||\nabla\theta||^2 + R(\theta, w) - (q_{i,i}, \theta), \tag{7.3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{Sg}{2}||\mathbf{q}||^2 = Sg\int_V d_{ij}q_iq_j\,\mathrm{d}x + \frac{RSg}{Pr}(u_{i,3}, q_i) - \epsilon||\nabla\mathbf{q}||^2 - (q_i, \theta_{,i}) - ||\mathbf{q}||^2. \tag{7.4}$$

Now form the sum of these three equations to obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = RI + Sg\int_V d_{ij}q_iq_j\,\mathrm{d}x + \frac{RSg}{Pr}(u_{i,3}, q_i) - ||\nabla\mathbf{u}||^2 - \epsilon||\nabla\mathbf{q}||^2 - ||\mathbf{q}||^2 - \zeta||\nabla\theta||^2, \tag{7.5}$$

where $d_{ij} = (u_{i,j} + u_{j,i})/2$ and $I = 2(\theta, w)$ and

$$E = \frac{1}{2}||\mathbf{u}||^2 + \frac{Pr}{2}||\theta||^2 + \frac{Sg}{2}||\mathbf{q}||^2.$$

We outline the remainder of the analysis since the stability obtained is conditional upon the size of the initial data.

Employ the arithmetic-geometric mean inequality in the form

$$(u_{i,3}, q_i) \le \frac{1}{2\xi}\,||\nabla\mathbf{u}||^2 + \frac{\xi}{2}\,||\mathbf{q}||^2,$$

for $\xi > 0$ at our disposal. Select $\xi = SgR/Pr$. Then, require Sg, Pr and R to be such that

$$2Pr^2 \ge Sg^2R^2. \tag{7.6}$$

From (7.5), one may now deduce

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le RI - D_1 + Sg\int_V d_{ij}q_iq_j\,\mathrm{d}x - \epsilon||\nabla\mathbf{q}||^2, \tag{7.7}$$

where

$$D_1 = \frac{1}{2}||\nabla\mathbf{u}||^2 + \zeta||\nabla\theta||^2.$$

Define

$$\frac{1}{R_{E1}} = \max_J \frac{I}{D_1}, \tag{7.8}$$

where J is the space of solutions for u_i and θ, i.e. $\mathbf{u} \in (H^1(V))^3$, $\theta \in H^1(V)$, $u_{i,i} = 0$, and u_i and θ satisfy the appropriate boundary condtions.

One may solve the Euler–Lagrange equations for (7.8) and one derives

$$R_{E1}^2 = \frac{\zeta}{2}\,\frac{\Lambda^3}{a^2}.$$

Thus minimizing in a^2 yields $R_{E1}^2 = 27\pi^4\zeta/8$.

Now put $D = D_1 + \epsilon||\nabla \mathbf{q}||^2$. Then from (7.7), one may show

$$\frac{dE}{dt} \le -D\left(1 - \frac{R}{R_{E1}}\right) + Sg\int_V d_{ij}q_iq_j\, dx. \tag{7.9}$$

Consider $R < R_{E1}$ and put $b = 1 - R/R_{E1} > 0$. Then from (7.9), we use the Cauchy–Schwarz inequality to find

$$\frac{dE}{dt} \le -bD + ||\mathbf{q}||_4^2\, ||\nabla \mathbf{u}||,$$

where $||\cdot||_4$ denotes the norm on $L^4(V)$. Using a sharp form of the Sobolev inequality in two dimensions, namely

$$||\mathbf{q}||_4^4 \le c_1^2||\mathbf{q}||^2\, ||\nabla \mathbf{q}||^2, \tag{7.10}$$

cf. Payne [63, pp. 132, 133], one then finds

$$\begin{aligned}\frac{dE}{dt} &\le -bD + c_1||\mathbf{q}||\, ||\nabla \mathbf{q}||\, ||\nabla \mathbf{u}||\\ &\le -bD + kE^{1/2}D,\end{aligned} \tag{7.11}$$

where $k = 2c_1/\sqrt{\epsilon Sg}$.

If now $E^{1/2}(0) < b/k$ then one may employ a continuity argument to show that $E(t) \to 0$ in an exponential manner, cf. Straughan [64, pp. 14–16]. Thus, nonlinear stability is established in two dimensions provided $E^{1/2}(0) < b/k$ and (7.6) holds.

Since the stability so found is conditional, we do not include details of numerical values for the critical Rayleigh numbers of linear and nonlinear stability.

8. Nonlinear energy stability for Cattaneo–Christov–Mariano theory with a Navier–Stokes–Voigt fluid

We have not seen how to obtain a nonlinear energy stability analysis for equations (7.1) in three dimensions. Instead, it would appear necessary to add a Kelvin–Voigt regularization term to the momentum equation. Thus, instead of (7.1), we employ an analogous system but with a Navier–Stokes–Voigt fluid, cf. Damazio *et al.* [65] and Straughan [66]. The equations are then

$$\begin{aligned}&u_{i,t} + u_ju_{i,j} - \lambda\Delta u_{i,t} = -\pi_{,i} + \Delta u_i + R\theta k_i,\\ &u_{i,i} = 0,\\ &Pr(\theta_{,t} + u_i\theta_{,i}) = Rw - q_{i,i} + \zeta\Delta\theta,\\ &Sg(q_{i,t} + u_jq_{i,j} - u_{i,j}q_j) = \frac{RSg}{Pr}u_{i,3} - q_i - \theta_{,i} + \epsilon\Delta q_i,\end{aligned} \tag{8.1}$$

where $\lambda > 0$ is a constant.

We do not describe in detail the analysis of nonlinear stability for these equations. The procedure is exactly as in §7, but now one obtains as an energy functional

$$E(t) = \frac{1}{2}||\mathbf{u}||^2 + \frac{\lambda}{2}||\nabla \mathbf{u}||^2 + \frac{Pr}{2}||\theta||^2 + \frac{Sg}{2}||\mathbf{q}||^2.$$

In three space dimensions, the Sobolev inequality (7.10) does not hold and must be replaced by an inequality of form

$$||\mathbf{q}||_4^4 \le c_2^2||\mathbf{q}||\, ||\nabla \mathbf{q}||^3, \tag{8.2}$$

cf. Payne [63, pp. 133–135]. The nonlinear analysis then finds when estimating $\int_V d_{ij}q_iq_j\, dx$,

$$\begin{aligned}\int_V d_{ij}q_iq_j\, dx &\le ||\mathbf{q}||_4^2\, ||\nabla \mathbf{u}||\\ &\le c_2||\mathbf{q}||^{1/2}\, ||\nabla \mathbf{q}||^{3/2}\, ||\nabla \mathbf{u}||\\ &\le c_2k_1E^{1/2}D,\end{aligned}$$

for some computable constant $k_1 > 0$.

The nonlinear Rayleigh number threshold is as in §7 and the stability analysis is again conditional. The stationary convection linear values are as in §7. However, the oscillatory convection values will *not* be the same as in §7. The presence of the Kelvin–Voigt term will influence these values.

9. Conclusion

We have investigated the thermal convection Bénard problem when a Cattaneo heat flux law is employed instead of Fourier's Law.

For the Cattaneo law, we considered several possibilities. One of these employs a material derivative for the rate of change of the heat flux **Q**. Additionally, we allowed the material derivative to be replaced by a general objective derivative for **Q**. This objective derivative involves a constant γ. We showed that unless $\gamma = -1$, then the objective derivative will in general lead to unphysical behaviour. When $\gamma = -1$ this is known as Cattaneo–Christov theory. We showed that in the *linearized* case Cattaneo–Christov and the material derivative theories lead to the same critical Rayleigh numbers.

For the nonlinear stability theory, the situation is very different. It would appear that none of the Cattaneo theories are amenable to an energy stability analysis. Instead, we appealed to an extra flux theory of Mariano [57]. If we employ the extra flux, then the resulting material derivative theory yields unconditional nonlinear stability and the critical Rayleigh numbers are physically acceptable for both linear instability and nonlinear stability. In the case of Cattaneo–Christov theory, we were unable to make progress as in the material derivative case unless we include an additional regularity term in the heat flux equation, cf. [57,58]. Even then, the energy stability analysis appears to work only in two space dimensions, and is conditional upon the size of the inital data. In the three-dimensional case, we were unable to use Navier–Stokes theory and had to replace this with a Navier–Stokes–Voigt theory.

In conclusion, it appears that one has to be very careful when analysing thermal convection with a Cattaneo heat flux law. When a Mariano [57] extra flux theory is employed a well-defined mathematical analysis is possible.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. M.G.: conceptualization, data curation, formal analysis, methodology, writing—original draft, writing—review and editing; B.S.: conceptualization, data curation, formal analysis, methodology, software, supervision, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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