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ORIGINAL PAPER

On heat equations associated with fractional harmonic oscillators

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Abstract

We establish some fixed-time decay estimates in Lebesgue spaces for the fractional heat propagator $e^{-tH^{\beta}}$, $t, \beta > 0$, associated with the harmonic oscillator $H = -\Delta + |x|^2$. We then prove some local and global wellposedness results for nonlinear fractional heat equations.

Keywords Fractional harmonic oscillator (primary) · dissipative estimates · heat equations · wellposedness

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1 Introduction

Consider the heat equation associated with the fractional harmonic oscillator, namely

$$\begin{cases} \partial_t u(t,x) + H^\beta u(t,x) = 0 \\ u(0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.1}$$

where $H^\beta = (-\Delta + |x|^2)^\beta$, $\beta > 0$, and $u(t,x) \in \mathbb{C}$.

Strictly speaking, the corresponding fractional heat semigroup e^{-tH^β} is defined in terms of the spectral decomposition of the standard Hermite operator $H = H^1 = -\Delta + |x|^2$. To be precise, recall that

$$H = \sum_{k=0}^{\infty} (2k+d) P_k,$$

where P_k stands for the orthogonal projection of $L^2(\mathbb{R}^d)$ onto the eigenspace corresponding to the eigenvalue $(2k+d)$ – see Section 2.1 below for further details. As a consequence of the spectral theorem, we can consider the family of fractional powers of H defined by

$$H^\beta = \sum_{k=0}^{\infty} (2k+d)^\beta P_k, \quad \beta > 0.$$

The heat semigroup e^{-tH^β} is then defined accordingly by

$$e^{-tH^\beta} f = \sum_{k=0}^{\infty} e^{-t(2k+d)^\beta} P_k f, \quad f \in L^2(\mathbb{R}^d).$$

While there is a wealth of literature on the semigroup $e^{-t(-\Delta)^\beta}$ (see e.g., [21], [34]), stimulated by the very wide range of physics-inspired models involving the fractional Laplacian [16], [11], the current research of the semigroup e^{-tH^β} is rather limited, even in fundamental settings such as the Lebesgue spaces. This is particularly striking in view of the role played by the Hermite operator H and its fractional powers H^β in several aspects of quantum physics and mathematical analysis [18, 30].

The purpose of this note is to advance the knowledge of the fractional heat semigroup, in the wake of a research program initiated by the authors in [3]. In particular, our main result is a set of fixed-time decay estimates for e^{-tH^β} in the Lebesgue space setting.

Theorem 1 *For* $1 \le p, q \le \infty$ *and* $\beta > 0$, *set*

$$\sigma_\beta := \frac{d}{2\beta}\left|\frac{1}{p} - \frac{1}{q}\right|.$$

1. *If $p, q \in (1, \infty)$, or $(p, q) = (1, \infty)$, or $p = 1$ and $q \in [2, \infty)$, or $p \in (1, \infty)$ and $q = 1$, then there exists a constant $C > 0$ such that*

$$\|e^{-tH^\beta} f\|_{L^q} \leq \begin{cases} Ce^{-td^\beta}\|f\|_{L^p} & \text{if } \ t \geq 1 \\ Ct^{-\sigma_\beta}\|f\|_{L^p} & \text{if } \ 0 < t \leq 1. \end{cases} \tag{1.2}$$

2. *If $0 < \beta \leq 1$, then the above estimate holds for $p, q \in [1, \infty]$.*

To the best of our knowledge, the dissipative estimate in Theorem 1 is new even for the Hermite operator ($\beta = 1$). We also stress that the time decay at infinity in (1.2) is sharp for any choice of Lebesgue exponents. Moreover, since the power of t is never positive for small time, we infer that there is a singularity near the origin for $p \neq q$.

It is worth emphasizing that the fractional Hermite propagator e^{-tH^β} is not a Fourier multiplier, hence we cannot rely on the arguments typically used to establish $L^p - L^q$ space-time estimates for the fractional heat propagator $e^{-t(-\Delta)^\beta}$ – see for instance [21, Lemma 3.1]. In fact, we will resort to techniques of pseudodifferential calculus to deal with the operators e^{-tH^β} and e^{-tH} (cf. [22, Section 4.5]), and also to Bochner's subordination formula in order to express the heat semigroup e^{-tH^β}, $0 < \beta \leq 1$, in terms of solutions of the heat equation e^{-tH} (see (3.4)).

As an application of Theorem 1, we investigate the wellposedness of

$$\begin{cases} \partial_t u(t, x) + H^\beta u(t, x) = |u(t, x)|^{\gamma - 1} u(t, x) \\ u(0, x) = u_0(x), \end{cases} \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.3}$$

with $u(t, x) \in \mathbb{C}$, $\beta > 0$ and $\gamma > 1$.

First, let us highlight that, due to the occurrence of the quadratic potential $|x|^2$, the problem (1.3) has no scaling symmetry. Nevertheless, the companion fractional heat equation

$$\begin{cases} \partial_t u(t, x) + (-\Delta)^\beta u(t, x) = |u(t, x)|^{\gamma - 1} u(t, x) \\ u(0, x) = u_0(x), \end{cases} \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.4}$$

is invariant under the following scaling transformation. For $\lambda > 0$, set

$$u_\lambda(t, x) = \lambda^{\frac{2\beta}{\gamma - 1}} u(\lambda^{2\beta} t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda^{\frac{2\beta}{\gamma - 1}} u_0(\lambda x).$$

If $u(t, x)$ is a solution of (1.4) with initial datum $u_0(x)$, then $u_\lambda(t, x)$ is also a solution of (1.4) with initial datum $u_{0,\lambda}(x)$. The L^p space is invariant under the above scaling only when $p = p_c^\beta := \frac{d(\gamma - 1)}{2\beta}$. Motivated by this remark, we shall say that (1.3) is

$$L^p - \begin{cases} \text{sub-critical} & \text{if } \ 1 \leq p < p_c^\beta \\ \text{critical} & \text{if } \ p = p_c^\beta \\ \text{super-critical} & \text{if } \ p > p_c^\beta. \end{cases}$$

Concerning the wellposedness of (1.3), our result can be stated as follows.

Theorem 2 *Assume that $u_0 \in L^p(\mathbb{R}^d)$, $1 < p < \infty$ and $\beta > 0$.*

1. *(Local well-posedness) If $p > p_c^\beta$, then there exists a $T > 0$ such that (1.3) has a solution $u \in C([0,T], L^p(\mathbb{R}^d))$. Moreover, u extends to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_{L^p} = \infty$.*
2. *(Lower blow-up rate) Consider $p > p_c^\beta$ and suppose that $T_{\max} < \infty$, where $T_{\max}$ is the existence time of the resulting maximal solution of (1.3). Then*

$$\|u(t)\|_{L^p} \geq C\,(T_{\max} - t)^{\frac{d}{2p\beta} - \frac{1}{\gamma-1}}, \quad \text{for all } t \in [0, T_{\max}).$$

3. *(Global existence) If $p = p_c^\beta$ and $\|u_0\|_{L^{p_c^\beta}}$ is sufficiently small, then $T_{max} = \infty$.*

Let us briefly recall the literature to better frame our results. Weissler [34] proved local wellposeness for (1.4) in L^p for super-critical indices $p > p_c^1 \geq 1$. Concerning the sub-critical regime $p < p_c^1$, there is no general theory of existence, see [34], [6]. Actually, Haraux-Weissler [15] proved that if $1 < p_c^1 < \gamma + 1$ then there is a global solution of (1.4) (with zero initial data) in $L^p(\mathbb{R}^d)$ for $1 \leq p < p_c^1$, but no such solution exists when $\gamma + 1 < p_c$. In the critical case where $p = p_c^1$ it is proved that the solution exists globally in time for small initial data. Some results in the same vein have been proved for the fractional heat equation (1.4) by Miao, Yuan and Zhang in [21, Theorem 4.1].

Remark 1 Let us discuss some aspects of the previous results. In particular, we highlight some intriguing related problems that we plan to explore in future work.

- The sign in power type non-linearity (focusing or defocusing) will not play any role in our analysis. Therefore, we have chosen to consider the defocusing case for the sake of concreteness.
- Using properties of Hermite functions and interpolation, in [35, Theorem 1.6] Wong proved that $\|e^{-tH} f\|_{L^2(\mathbb{R})} \lesssim (\sinh t)^{-1} \|f\|_{L^p(\mathbb{R})}$ for $t > 0$ and $1 \leq p \leq 2$. We note that Theorem 1 recaptures and improves Wong's result.
- It is known that (1.4) is ill-posed on Lebesgue spaces in the sub-critical regime, see [15]. There is reason to believe that the same conclusion holds for (1.3). However, a thorough analysis of this problem is beyond the scope of this note.
- It is expected that Theorem 1 could be useful in dealing with other types of non-linearities in (1.3), such as exponential and inhomogeneous type non-linearity (which are also extensively studied in the literature).
- In Section 5 we discuss another application of Theorem 1, namely Strichartz estimates for the fractional heat semigroup. Our approach here relies on a standard technique (i.e., $TT^\star$ method and real interpolation), whereas a refined phase-space analysis of H^β is expected to reflect into better estimates.

2 Preliminaries

Notation. The symbol $X \lesssim Y$ means that the underlying inequality holds with a suitable positive constant factor:

$$X \lesssim Y \implies \exists C > 0 : X \leq CY.$$

2.1 On the fractional harmonic oscillator H^β

Let us briefly review some facts concerning the spectral decomposition of the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^d$.

Let $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^d$, be the normalized d-dimensional Hermite functions, that is

$$\Phi_\alpha(x) = \Pi_{j=1}^d h_{\alpha_j}(x_j), \quad h_k(x) = (\sqrt{\pi}2^k k!)^{-1/2}(-1)^k e^{\frac{1}{2}x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

The Hermite functions Φ_α are eigenfunctions of H with eigenvalues $(2|\alpha| + d)$, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Moreover, they form an orthonormal basis of $L^2(\mathbb{R}^d)$. The spectral decomposition of H is thus given by

$$H = \sum_{k=0}^{\infty} (2k + d) P_k, \qquad P_k f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha \rangle \Phi_\alpha,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^d)$.

In general, given a bounded function $m : \mathbb{N} \to \mathbb{C}$, the spectral theorem allows us to define the operator $m(H)$ such that

$$m(H)f = \sum_{\alpha \in \mathbb{N}^d} m(2|\alpha| + d)\langle f, \Phi_\alpha \rangle \Phi_\alpha = \sum_{k=0}^{\infty} m(2k + d) P_k f, \quad f \in L^2(\mathbb{R}^d).$$

In view of the Plancherel theorem for the Hermite expansions, $m(H)$ is bounded on $L^2(\mathbb{R}^d)$. We refer to [30] for further details, in particular for Hörmander multiplier-type results for $m(H)$ on $L^p(\mathbb{R}^d)$.

2.2 Some relevant function spaces

For the benefit of the reader we review some basic facts of time-frequency analysis – see for instance [7, 14], [1] for comprehensive treatments.

Recall that the short-time Fourier transform of a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to a window function $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space) is defined by

$$V_g f(x, \xi) = \langle f, g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t - x)} e^{-2\pi i \xi \cdot t} dt, \ (x, \xi) \in \mathbb{R}^{2d},$$

where the brackets $\langle \cdot, \cdot \rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the L^2 inner product.

Modulation spaces, introduced by Feichtinger [9], have proved to be extremely useful in a wide variety of contexts, ranging from analysis of PDEs to mathematical physics – among the most recent contributions, see e.g., [8], [20], [2], [23], [10]. Modulation spaces are defined as follows. For $1 \leq p, q \leq \infty$ we have

$$M^{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^{p,q}} := \left\| \|V_g f(x,\xi)\|_{L^p_x} \right\|_{L^q_\xi} < \infty \right\}.$$

The Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{R}^d)$ are defined by

$$\mathcal{F}L^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{F}L^p} := \|\hat{f}\|_{L^p} < \infty \right\}.$$

We recall from [3, Theorem 1.1] some bounds for the fractional heat semigroup on modulation spaces.

Theorem 3 *Let $\beta > 0$, $0 < p_1, p_2, q_1, q_2 \leq \infty$ and set*

$$\frac{1}{\tilde{p}} := \max\left\{ \frac{1}{p_2} - \frac{1}{p_1}, 0 \right\}, \quad \frac{1}{\tilde{q}} := \max\left\{ \frac{1}{q_2} - \frac{1}{q_1}, 0 \right\}, \quad \sigma_\beta := \frac{d}{2\beta}\left(\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \right).$$

Then

$$\|e^{-tH^\beta} f\|_{M^{p_2,q_2}} \leq \begin{cases} Ce^{-td^\beta} \|f\|_{M^{p_1,q_1}} & \text{if} \quad t \geq 1, \\ Ct^{-\sigma_\beta} \|f\|_{M^{p_1,q_1}} & \text{if} \quad 0 < t \leq 1, \end{cases}$$

where $C > 0$ is a universal constant.

We briefly recall some properties of the Shubin classes Γ^s, which play a central role as symbol classes in the theory of pseudodifferential operators – we refer to [19, 22] for additional details. For $s \in \mathbb{R}$ we define Γ^s as the space of functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying the following condition: for every $\tilde{\alpha} \in \mathbb{N}^{2d}$ there exists $C_{\tilde{\alpha}} > 0$ such that

$$|\partial^{\tilde{\alpha}} a(x,\xi)| \leq C_{\tilde{\alpha}} (1 + |(x,\xi)|)^{s - |\tilde{\alpha}|}, \qquad (x,\xi) \in \mathbb{R}^{2d},$$

This space becomes a Fréchet space endowed with the obvious seminorms.

It is important for our purposes to recall that the fractional Hermite propagator is a pseudodifferential operator with symbol in a suitable Shubin class, as proved in [3, Proposition 2.3].

Proposition 1 *Let $\beta > 0$. The fractional Hermite operator $H^\beta = (-\Delta + |x|^2)^\beta$ is a pseudodifferential operator with Weyl symbol $a_\beta \in \Gamma^{2\beta}$. More precisely, we have*

$$a_\beta(x,\xi) = (|x|^2 + |\xi|^2)^\beta + r(x,\xi), \quad |x| + |\xi| \geq 1, \tag{2.1}$$

where $r \in \Gamma^{2\beta - 2}$.

We also recall some facts concerning the so-called Shubin-Sobolev (also known as Hermite-Sobolev) spaces Q^s, $s \in \mathbb{R}$ – see [25], [12, Theorem 2.1] for further details. In particular, Q^s is the space of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{Q^s}^2 := \|H^{s/2} f\|_{L^2}^2 = \sum_{k=0}^{\infty} ||P_k f||_{L^2}^2 (2k+d)^s < \infty.$$

Given the polynomial weight $v_s(x,\xi) := (1+|x|+|\xi|)^s$ with $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ and $s \in \mathbb{R}$, consider the weighted modulation space $M^{2,2}_{v_s}(\mathbb{R}^d)$ endowed with the norm $\|f\|_{M^{2,2}_{v_s}} := \left\| v_s V_g f \right\|_{L^2(\mathbb{R}^{2d})}$. In view of the characterization $Q^s = M^{2,2}_{v_s}$ (see for instance [7, Lemma 4.4.19]), Hölder's inequality and the inclusion relations of Shubin-Sobolev spaces (see e.g., [7, Theorem 2.4.17]), it is well known that

$$Q^s \hookrightarrow M^{p,q} \hookrightarrow M^{\infty} \hookrightarrow Q^{-s}$$

for all $p, q \in [1,\infty]$ and $s > d$ – see also [14] and references therein.

3 Proof of Theorem 1

3.1 Proof of Part (1)

It is well known that

$$L^p \hookrightarrow M^{p,\infty} \text{ and } M^{q,1} \hookrightarrow L^q \text{ for } 1 \le p, q \le \infty,$$

see e.g., [7, 13, 28]. In light of this embedding and Theorem 3, for $t > 1$ we obtain the desired estimate

$$\|e^{-tH^{\beta}} f\|_{L^q} \lesssim e^{-td^{\beta}} \|f\|_{L^p}, \quad \forall p, q \in [1,\infty].$$

Let us consider now the case where $0 < t \le 1$. In view of Proposition 1 we think of H^{β} as a pseudodifferential operator with Weyl symbol $a_{\beta} \in \Gamma^{2\beta}$, where

$$a_{\beta}(x,\xi) = (|x|^2 + |\xi|^2)^{\beta} + r(x,\xi), \quad |x| + |\xi| \ge 1,$$

for a suitable $r \in \Gamma^{2\beta-2}$. We may further rewrite

$$a_{\beta}(x,\xi) = a(x,\xi) + r'(x,\xi), \quad x, \xi \in \mathbb{R}^d,$$

for some $r' \in \Gamma^{2\beta-2}$, where $a \in \Gamma^{2\beta}$ satisfies

$$a(x,\xi) \ge (1+|x|+|\xi|)^{2\beta}, \quad x, \xi \in \mathbb{R}^d. \tag{3.1}$$

Note that the same conclusion holds for the Kohn-Nirenberg symbol of H^β (see [22, Proposition 1.2.9]). Therefore, we assume in the sequel that the above functions $a(x,\xi)$ and $r'(x,\xi)$ denote the Kohn-Nirenberg symbols of the corresponding operators.

It follows from [22, Theorem 4.5.1] that the heat semigroup e^{-tH^β} has a Kohn-Nirenberg symbol with the following structure[1]:

$$b_t(x,\xi) = e^{-ta(x,\xi)} + e^{-ta(x,\xi)} \sum_{j=1}^{J-1} \sum_{l=1}^{2j} t^l u_{l,j}(x,\xi) + r_t''(x,\xi),$$

where $J \geq 1$ is arbitrarily chosen, $u_{l,j} \in \Gamma^{2\beta l - 2j}$ and r_t'' satisfy

$$\left|\partial_x^\alpha \partial_\xi^\gamma r_t''(x,\xi)\right| \leq C_{\alpha,\gamma}(1+|x|+|\xi|)^{-2J-|\alpha|-|\gamma|}$$

for a constant $C_{\alpha,\gamma}$ independent of $t \in (0,1)$, for every $\alpha, \gamma \in \mathbb{N}^d$.

Since $r_t''(x,D)\colon Q^{-J} \to Q^J$, for J large enough, we have

$$\|r_t''(x,D)f\|_{L^q} \leq C\|f\|_{L^p}.$$

Let us focus now on the symbol

$$C_t(x,\xi) := e^{-ta(x,\xi)} \sum_{j=1}^{J-1} \sum_{l=1}^{2j} t^l u_{l,j}(x,\xi).$$

By virtue of the Leibniz rule, the chain rule and (3.1), one can verify the estimates

$$\left|\partial_x^\alpha \partial_\xi^\gamma [e^{\frac{t}{4}\langle x\rangle^{2\beta}} C_t(x,\xi)]\right| \leq C_{\alpha,\gamma}(1+|\xi|)^{-|\gamma|}, \tag{3.2}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. In fact, it suffices to observe that $\partial_x^\alpha e^{\frac{t}{4}\langle x\rangle^{2\beta}}$ is a finite linear combination of terms of the type

$$e^{\frac{t}{4}\langle x\rangle^{2\beta}} \quad \partial^{\alpha_1}[t\langle x\rangle^{2\beta}] \cdots \partial^{\alpha_k}[t\langle x\rangle^{2\beta}],$$

with $\alpha_1 + \cdots + \alpha_k = |\alpha|$, so that

$$\left|\partial_x^\alpha e^{\frac{t}{4}\langle x\rangle^{2\beta}}\right| \leq e^{\frac{t}{4}2\langle x\rangle^{2\beta}} \;\langle x\rangle^{-|\alpha|}.$$

Similarly, since $a \in \Gamma^{2\beta}$ satisfies (3.1), we have

$$\left|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\right| \leq a(x,\xi)\,(1+|x|+|\xi|)^{-|\alpha|-|\gamma|},$$

[1] Note that the mentioned result is stated for the Weyl quantization, but again a straightforward change of variables shows that the same conclusion holds for the Kohn-Nirenberg quantization.

so that, arguing as above,

$$\left|\partial_x^\alpha \partial_\xi^\gamma e^{-ta(x,\xi)}\right| \le e^{-\frac{t}{2}a(x,\xi)} (1+|x|+|\xi|)^{-|\alpha|-|\gamma|},$$

hence we infer

$$\left|\partial_x^\alpha \partial_\xi^\gamma [t^l u_{l,j}(x,\xi)]\right| \le t^l a(x,\xi)^l (1+|x|+|\xi|)^{-2j-|\alpha|-|\gamma|}.$$

The claimed bound thus follows by the Leibniz rule.

To summarize, for every $p \in (1,\infty)$ we have

$$\|e^{\frac{t}{4}\langle x\rangle^{2\beta}} C_t(x,D) f\|_{L^p} \le \|f\|_{L^p}, \quad 0 < t < 1,$$

by the L^p boundedness of pseudodifferential operators with symbol in Hörmander's class $S^0_{1,0}$ – see for instance [27, Proposition 4, p. 250]. For $1 \le q \le p \le \infty$ we have, by Hölder inequality,

$$\|e^{-\frac{t}{4}\langle x\rangle^{2\beta}} f\|_{L^q} \le C t^{\frac{d}{2\beta}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^p}.$$

Hence we obtain, for $1 \le q \le \infty,\ 1 < p < \infty,\ q \le p$,

$$\|C_t(x,D) f\|_{L^q} \le C t^{\frac{d}{2\beta}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^p}, \quad 0 < t < 1.$$

On the other hand, we also have

$$\left|C_t(x,\xi)\right| \le C e^{-\frac{t}{2}|\xi|^{2\beta}}, \quad 0 < t < 1,$$

and the integral kernel of the operator $C_t(x,D)$ given by

$$K(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} C_t(x,\xi)\, d\xi$$

is readily seen to satisfy

$$\left|K(x,y)\right| \le C t^{-\frac{d}{2\beta}}.$$

This gives the desired continuity result $L^1 \to L^\infty$, while the remaining bounds follow by interpolation with the above $L^p \to L^q$ estimates.

Remark 2 Note that some endpoint cases can be obtained in a straightforward way. For instance, from $L^1 \to L^\infty$ continuity we also obtain $L^1 \to L^2$ bounds as follows: if $f \in L^2(\mathbb{R}^d)$ then

$$\left|\langle e^{-tH^\beta} f, e^{-tH^\beta} f\rangle\right| = \left|\langle e^{-2tH^\beta} f, f\rangle\right| \le C t^{-\frac{d}{2\beta}} \|f\|^2_{L^1}$$

so that

$$\|e^{-tH^{\beta}} f\|_{L^2} \leq C t^{-\frac{d}{4\beta}} \|f\|_{L^1}, \quad 0 < t < 1.$$

By interpolation with $L^1 \to L^\infty$ one also gets the desired estimate $L^1 \to L^q$ for $2 \leq q \leq \infty$.

Remark 3 Some endpoint cases (e.g., if $p, q \in \{1, \infty\}$) are not covered in the results above. A deeper investigation of the kernel $K(x, y)$ of $C_t(x, D)$ could likely give some result in this connection (for example $L^1 \to L^1$, $L^\infty \to L^\infty$), but it will not be essential for the applications to the nonlinear problem in Theorem 2. Nevertheless, the dispersive estimate $L^1 \to L^\infty$ is covered.

3.2 Proof of Part (2)

In order to prove the second claim in Theorem 1, some preparatory work is needed. First, we recast e^{-tH} as the Weyl transform of a function on $\mathbb{C}^d$, which allows us to think of e^{-tH} as a pseudodifferential operator.

Recall that the Weyl transform $W(F)$ of a function $F\colon \mathbb{C}^d \to \mathbb{C}$ is defined by

$$W(F)\phi(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\xi-\eta)\cdot y} b\left(\frac{\xi+\eta}{2}, y\right) \phi(\eta)\, dy d\eta,$$

for $\phi \in L^2(\mathbb{R}^d)$, where the symbol $b(\xi, \eta)$ is the full inverse Fourier transform of F in both variables. In particular, the Weyl transform $W(F)$ is a pseudodifferential operator in the Weyl calculus with symbol b.

Let us highlight that the Weyl symbol of the Hermite semigroup e^{-tH} is given by the function $a_t(x, \xi) = C_d(\cosh t)^{-d} e^{-(\tanh t)(|x|^2+|\xi|^2)}$, see [31]. Thus,

$$\begin{aligned} &e^{-tH} f(x) \\ &= C_d(\cosh t)^{-d} (2\pi)^{-d} \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-\eta)\cdot y} e^{-(\tanh t)|y|^2} e^{-(\tanh t)(|\frac{x+\eta}{2}|^2)} f(\eta)\, dy d\eta}_{=I}. \end{aligned}$$

In order to bound the above integral I, we first recast the latter expression in terms of convolution. Recall that the Fourier transform of the Gaussian function $f(y) = e^{-\pi a|y|^2}$ with $a > 0$ is given by $\hat{f}(x) = a^{-d/2} e^{-\pi|x|^2/a}$, and note that

$$\frac{|x-\eta|^2}{4} - \frac{|x|^2}{2} - \frac{|\eta|^2}{2} = -\frac{|x+\eta|^2}{4}.$$

As a result, we have

$$\begin{aligned} (\tanh t)^{d/2} I &= \int_{\mathbb{R}^d} e^{-(\frac{1}{4\tanh t} - \frac{\tanh t}{4})|x-\eta|^2} e^{-\frac{\tanh t}{2}(|x|^2+|\eta|^2)} f(\eta)\, d\eta \\ &= e^{-\frac{\tanh t}{2}|x|^2} \left(e^{-\frac{1}{2\sinh 2t}|\cdot|^2} * g\right)(x), \end{aligned}$$

where we set $g(\cdot)=e^{-\frac{\tanh t}{2}|\cdot|^2}f(\cdot)$. Note that

$$(\cosh t)^{-d}\,(\tanh t)^{-d/2}=(\sinh(2t))^{-d/2}\,.$$

Hence

$$e^{-tH}f(x)=\tilde{C}_d\,(\sinh(2t))^{-d/2}\,e^{-\frac{\tanh t}{2}|x|^2}\left(e^{-\frac{1}{2\sinh 2t}|\cdot|^2}*g\right)(x). \tag{3.3}$$

Lemma 1 *Let* $1\le p,q\le\infty$ *and* $t>0$. *Then*

$$\|e^{-tH}f\|_{L^q}\le C(\tanh t)^{-\frac{d}{2}\left|\frac{1}{q}-\frac{1}{p}\right|}\|f\|_{L^p},$$

for some constant $C>0$ *that depends only on* d.

Proof Using Mehler's formula for the Hermite functions (see e.g., [30]), the kernel $K_t(x,y)$ of the semigroup e^{-tH} is explicitly given by

$$K_t(x,y)=c_d(\sinh 2t)^{-d/2}e^{-\frac{1}{4}(\coth t)|x-y|^2}e^{-\frac{1}{4}(\tanh t)|x+y|^2}.$$

For $1<p<q<\infty$, set $\alpha=d(1/p-1/q)$. Then we have

$$\begin{aligned}K_t(x,y)&=c_d(\sinh 2t)^{-d/2}(\tanh t)^{(d-\alpha)/2}|x-y|^{\alpha-d}\\&\quad\times((\coth t)|x-y|^2)^{(d-\alpha)/2}e^{-\frac{1}{4}(\coth t)|x-y|^2}e^{-\frac{1}{4}(\tanh t)|x+y|^2},\end{aligned}$$

from which we obtain the estimate

$$K_t(x,y)\le C(\cosh t)^{-d}(\tanh t)^{-\alpha/2}|x-y|^{\alpha-d}.$$

Since the Riesz potential

$$R_\alpha f(x)=c_\alpha\int_{\mathbb{R}^d}f(y)|x-y|^{\alpha-d}dy$$

is bounded from L^p to L^q for $1<p<q<\infty$, we get

$$\|e^{-tH}f\|_{L^q}\le C(\cosh t)^{-d}(\tanh t)^{-\alpha/2}\|f\|_{L^p}$$

for $1<p<q<\infty$.

To prove the remaining cases, we use the identity (3.3). We consider the case $1\le q\le p\le\infty$ first. Set $\frac{1}{q}=\frac{1}{p}+\frac{1}{\tilde{q}}$ and note that

$$\|e^{-\frac{\tanh t}{2}|\cdot|^2}\|_{L^{\tilde{q}}}\sim(\tanh t)^{-d/2\tilde{q}}=(\tanh t)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

By (3.3) and invoking Hölder and Young's inequalities, we obtain

$$\begin{aligned}
\|e^{-tH} f\|_{L^q} &\lesssim (\sinh 2t)^{-d/2} \, \|e^{-\frac{\tanh t}{2}|\cdot|^2}\|_{L^{\tilde{q}}} \, \|e^{-\frac{1}{2\sinh 2t}|\cdot|^2} * g\|_{L^p} \\
&\lesssim (\sinh 2t)^{-d/2} \, (\tanh t)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \, \|e^{-\frac{1}{2\sinh 2t}|\cdot|^2}\|_{L^1} \|g\|_{L^p} \\
&\lesssim (\tanh t)^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^p}.
\end{aligned}$$

Let $1 \le q \le \infty$ and note that

$$\|e^{-\frac{1}{2\sinh 2t}|\cdot|^2}\|_{L^q} \approx (\sinh(2t))^{d/2q}.$$

By (3.3) and Young inequality, we have

$$\begin{aligned}
\|e^{-tH} f\|_{L^q} &\lesssim (\sinh 2t)^{-d/2} \|e^{-\frac{1}{2\sinh 2t}|\cdot|^2} * g\|_{L^q} \\
&\lesssim (\sinh 2t)^{-d/2} \|e^{-\frac{1}{2\sinh 2t}|\cdot|^2}\|_{L^q} \|g\|_{L^1} \\
&\lesssim (\sinh 2t)^{-\frac{d}{2}\left(1-\frac{1}{q}\right)} \|f\|_{L^1} \\
&\lesssim (\cosh t)^{-d\left(1-\frac{1}{q}\right)} \, (\tanh t)^{-\frac{d}{2}\left(1-\frac{1}{q}\right)} \|f\|_{L^1}.
\end{aligned}$$

This completes the proof. □

Note that Lemma 1 essentially gives the desired fixed-time estimate of Theorem 1 (2) for $\beta = 1$ – see also Remark 4 below. In order to deal with the case $0 < \beta < 1$, Bochner's subordination formula and the property of probability density function (see (3.6)) will play a crucial role. To be precise, Bochner's subordination formula allows us to express the heat semigroup $e^{-t\sqrt{H}}$ in terms of solutions of the heat equation:

$$e^{-t\sqrt{H}} f(x) = \pi^{-1/2} \int_0^\infty e^{-y} \, e^{-\frac{t^2}{4y}H} f(x) \, y^{-1/2} \, dy, \tag{3.4}$$

which ultimately follows from the identity

$$e^{-a} = \pi^{-1/2} \int_0^\infty e^{-y} \, e^{-\frac{a^2}{4y}} \, y^{-1/2} \, dy \quad (a > 0).$$

The Macdonald function $K_\nu(z)$ is defined, for $z > 0$, by

$$K_\nu(z) = 2^{-\nu-1} \, z^\nu \int_0^\infty e^{-y-\frac{z^2}{4y}} \, y^{-\nu-1} \, dy.$$

A straightforward change of variables shows that

$$z^\nu K_\nu(z) = 2^{\nu-1} \int_0^\infty e^{-y-\frac{z^2}{4y}} \, y^{\nu-1} \, dy = z^\nu K_{-\nu}(z).$$

Then $z^\nu K_\nu(z)$ converges to $2^{\nu-1}\Gamma(\nu)$ as $z \to 0$. Moreover, it is known that $K_\nu(z)$ has exponential decay at infinity (see [17]). Consider now the Gaussian kernel of the form

$$g_t(x) = (4\pi t)^{-d/2}\, e^{-\frac{|x|^2}{4t}},\ t > 0,\ x \in \mathbb{R}^d.$$

We set $p_t(x, y) = p_t(x - y)$, where

$$p_t(x) = \int_0^\infty g_s(x)\, \eta_t(s)\, ds,$$

g_s is the Gaussian kernel defined above and $\eta_t \geq 0$ is the density function of the distribution of the β-stable subordinator at time t, see e.g., [4], [5]. Therefore, $\eta_t(s) = 0$ for $s \leq 0$ and, for $0 < \beta < 1$, we have

$$\int_0^\infty e^{-us}\, \eta_t(s)\, ds = e^{-tu^\beta}, \quad u \geq 0. \tag{3.5}$$

The fractional heat semi group e^{-tH^β} is thus given in terms of solutions of the heat equation:

$$e^{-tH^\beta} f(x) = \int_0^\infty e^{-sH} f(x)\, \eta_t(s)\, ds. \tag{3.6}$$

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1 – Part (2) The case $t > 1$ follows from the proof of Part (1) of Theorem 1, as it holds for all $p, q \in [1, \infty]$. We then assume $0 < t \leq 1$ from now on. In view of the identity (3.6) and Lemma 1 for the case $\beta = 1$, we obtain

$$\|e^{-tH^\beta} f\|_{L^q} \leq C \left[\int_0^\infty (\tanh s)^{-\alpha/2} \eta_t(s) ds\right] \|f\|_{L^p},$$

where we set $\alpha = d|1/p - 1/q|$. Splitting the integral above into two parts, the integral taken over $[1, \infty)$ is bounded by

$$\int_0^\infty \eta_t(s) ds = 1.$$

The remaining integral is bounded by

$$\int_0^\infty s^{-\alpha/2} \eta_t(s) ds = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \left(\int_0^\infty e^{-us} u^{\alpha/2-1} du\right) \eta_t(s) ds.$$

Changing the order of integration, and using (3.5), for a suitable constant $C > 0$ we obtain

$$\int_0^\infty s^{-\alpha/2}\eta_t(s)ds \leq C\int_0^\infty u^{\alpha/2-1}e^{-tu^\beta}du.$$

Finally, the change of variables $v = u^\beta$ gives the estimate

$$\int_0^\infty u^{\alpha/2-1}e^{-tu^\beta}du \leq C\int_0^\infty v^{(\alpha/2\beta)-1}e^{-tv}dv = C_{\alpha,\beta}t^{-(\alpha/2\beta)}.$$

This completes the proof for the case $0 < t \leq 1$. □

Remark 4 We would like to have also a representation in the vein of (3.3) for the fractional heat propagator e^{-tH^β} with $\beta > 1$ in terms of the Weyl transform. On the other hand, we have a convolution formula for the classical fractional heat propagator $e^{-t(-\Delta)^\beta}$. Regretfully, we do not know how to get fixed-time estimates for $\beta > 1$ via the Weyl transform at the time.

Remark 5 Using the fact that e^{-tH} commutes with the Fourier transform, i.e., $\widehat{e^{-tH}f} = e^{-tH}\hat{f}$, one obtains

$$\|e^{-tH}f\|_{\mathcal{F}L^q} \leq C(\tanh t)^{-\frac{d}{2}\left|\frac{1}{q}-\frac{1}{p}\right|}\|f\|_{\mathcal{F}L^p}.$$

4 Proof of Theorem 2

4.1 Part (1) – local wellposedness

Fix $M_1 \geq \|u_0\|_{L^p}$.

The proof strategy is quite standard. Let $T > 0$ and set

$$Y_T = L^\infty\left((0,T), L^p(\mathbb{R}^d)\right) \cap L^\infty\left((0,T), L^{p\gamma}(\mathbb{R}^d)\right),$$

endowed with a norm

$$\|u\|_{Y_T} = \max\left\{\sup_{0<t<T}\|u(t)\|_{L^p},\ \sup_{0<t<T} t^{\frac{d(\gamma-1)}{2p\gamma\beta}}\|u(t)\|_{L^{p\gamma}}\right\}.$$

Moreover, consider

$$B_{M+1} = \{u \in Y_T : \|u\|_{Y_T} \leq M+1\}$$

where $M > 0$ is chosen in such a way that $\|e^{-tH^\beta}u_0\|_{Y_T} \leq CM_1 \leq M$. Note that M depends only on $\|u_0\|_{Y_T}$ – in particular, it is independent of t.

Consider the mapping $\Phi \colon B_{M+1} \to Y_T$ defined by

$$\Phi[u](t) = e^{-tH^\beta} u_0 + \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau. \tag{4.1}$$

We shall show that in fact Φ is a mapping from B_{M+1} into B_{M+1}. Indeed, consider $u \in B_{M+1}$. By Theorem 1, for $q \in \{p, p\gamma\}$, we have

$$\begin{aligned}
&\left\| \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau \right\|_{L^q} \\
&\leq C \int_0^t (t-\tau)^{-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]} \|u(\tau)\|_{L^{p\gamma}}^{\gamma} d\tau \\
&\leq C(M+1)^\gamma \int_0^t (t-\tau)^{-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]} \tau^{-\frac{d(\gamma-1)}{2p\beta}} d\tau \\
&= C(M+1)^\gamma t^{1-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]-\frac{d(\gamma-1)}{2p\beta}} \times \int_0^1 (1-\tau)^{-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]} \tau^{-\frac{d(\gamma-1)}{2p\beta}} d\tau.
\end{aligned}$$

Since $q = p$ or $q = p\gamma$, $\gamma > 1$ and $p > p_c^\beta$, we have

$$\int_0^1 (1-\tau)^{-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]} \tau^{-\frac{d(\gamma-1)}{2p\beta}} d\tau < \infty.$$

Therefore, we infer

$$t^{\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}]} \left\| \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau \right\|_{L^q} \leq C(M+1)^\gamma T^{1-\frac{d(\gamma-1)}{2p\beta}}. \tag{4.2}$$

If we take $q = p$ or $q = p\gamma$ in (4.2), then

$$\left\| \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau \right\|_{L^p} \leq C_1(M+1)^\gamma T^{1-\frac{d(\gamma-1)}{2p\beta}} \tag{4.3}$$

or

$$t^{\frac{d(\gamma-1)}{2p\gamma\beta}} \left\| \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau \right\|_{L^{p\gamma}} \leq C_2(M+1)^\gamma T^{1-\frac{d(\gamma-1)}{2p\beta}}.$$

As a result, we conclude that

$$\|\Phi[u]\|_{Y_T} \leq M + \max\{C_1, C_2\} (M+1)^\gamma T^{1-\frac{d(\gamma-1)}{2p\beta}}.$$

Moreover, for a sufficiently small $T > 0$, we have

$$\max\{C_1, C_2\} (M+1)^\gamma T^{1-\frac{d(\gamma-1)}{2p\beta}} \leq 1.$$

This shows that Φ is a mapping from B_{M+1} into B_{M+1}, as claimed.

We now prove that $\Phi\colon B_{M+1} \to Y_T$ is a contraction mapping. Recall that

$$\left| |u|^{\gamma-1}u - |v|^{\gamma-1}v \right| \lesssim_\gamma \left(|u|^{\gamma-1} + |v|^{\gamma-1} \right) |u-v|. \tag{4.4}$$

By (4.4) and Hölder inequality, we have

$$\| |u|^{\gamma-1}u - |v|^{\gamma-1}v \|_{L^p} \le \gamma \left(\|u\|_{L^{p\gamma}}^{\gamma-1} + \|v\|_{L^{p\gamma}}^{\gamma-1} \right) \|u-v\|_{L^{p\gamma}}.$$

In light of the previous computation, for $u, v \in B_{M+1}$ and $q \in \{p, p\gamma\}$ we have

$$\begin{aligned} &\|\Phi[u](t) - \Phi[v](t)\|_{L^q} \\ &\le \gamma \int_0^t (t-\tau)^{-\frac{d}{2\beta}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u(\tau)\|_{L^{p\gamma}}^{\gamma-1} + \|v(\tau)\|_{L^{p\gamma}}^{\gamma-1} \right) \|u(\tau) - v(\tau)\|_{L^{p\gamma}} d\tau \\ &\le C_3 (M+1)^{\gamma-1}\, t^{1-\frac{d}{2\beta}[\frac{1}{p}-\frac{1}{q}] - \frac{d(\gamma-1)}{2p\beta}} \|u-v\|_{Y_T} \end{aligned} \tag{4.5}$$

for a constant $C_3 > 0$. By taking $q = p$ or $q = p\gamma$ in (4.5), we similarly obtain

$$\|\Phi[u](t) - \Phi[v](t)\|_{Y_T} \le C_4 (M+1)^{\gamma-1}\, T^{1-\frac{d(\gamma-1)}{2p\beta}} \|u-v\|_{Y_T}$$

for a constant $C_4 > 0$. Since $1 - \frac{d(\gamma-1)}{2p\beta} > 0$, for a sufficiently small $T > 0$ we have

$$C_4 (M+1)^{\gamma-1}\, T^{1-\frac{d(\gamma-1)}{2p\beta}} \le \frac{1}{2}.$$

We have thus proved that the mapping Φ is the contraction mapping for a sufficiently small T. By Banach fixed point theorem, there exists a unique fixed point u of the mapping Φ in B_{M+1} and, in light of Duhamel's principle, the latter is a solution of (1.3).

Let us finally prove that $u \in C([0,T], L^p(\mathbb{R}^d))$. For $u_0 \in L^p(\mathbb{R}^d)$, let the solution map $\Phi_{u_0}\colon [0,T] \to L^p(\mathbb{R}^d)$ given by

$$\Phi_{u_0} u(t) = e^{-tH^\beta} u_0 + \int_0^t e^{-(t-\tau)H^\beta} \left(|u(\tau)|^{\gamma-1} u(\tau) \right) d\tau.$$

In view of (4.3), we obtain

$$\|u(t) - u(0)\|_{L^p} \le \|e^{-tH^\beta} u_0 - u_0\|_{L^p} + C_1 (M+1)^\gamma\, t^{1-\frac{d(\gamma-1)}{2p\beta}}.$$

The solution map $u(t)$ is then continuous at $t = 0$ if $\|e^{-tH^\beta} u_0 - u_0\|_{L^p} \to 0$ as $t \to 0$. In fact, one can similarly show that it is continuous on $[0,T]$, hence we have $u \in C([0,T], L^p(\mathbb{R}^d))$. It only remains to show that $e^{-tH^\beta} \to I$ as $t \to 0$ in the strong operator topology on $L^p(\mathbb{R}^d)$. To this aim, let us note first that Theorem 1 implies that the semigroup e^{-tH^β} is bounded on $L^p(\mathbb{R}^d)$, uniformly with respect to $t \in [0, 1]$, hence it is enough to prove the claim on a dense subspace of $L^p(\mathbb{R}^d)$. It is well known that finite linear combinations of Hermite functions are dense in $\mathcal{S}(\mathbb{R}^d)$ (see e.g., [26, Theorem 6.4.4]), and in this case the proof of $\|e^{-tH^\beta} f - f\|_{L^p} \to 0$ as $t \to 0$ is an immediate consequence of the dominated convergence theorem.

Remark 6 We shall also mention that the result of Part (1) can be alternatively derived from the abstract theorem of Weissler [33, Theorem 1]. To this aim, we define $K_t(u) = e^{-tH^\beta}(|u|^{\gamma-1}u)$. Then for $t > 0$, $K_t : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is locally Lipschitz and

$$\begin{aligned}
\|K_t(u) - K_t(v)\|_{L^p} &\lesssim t^{-\frac{d}{2\beta}\left(\frac{\gamma}{p}-\frac{1}{p}\right)} \||u|^{\gamma-1}u - |v|^{\gamma-1}v\|_{L^{\frac{p}{\gamma}}} \\
&\lesssim t^{-\frac{d}{2\beta}\left(\frac{\gamma}{p}-\frac{1}{p}\right)} \left(\|u\|_{L^p}^{\gamma-1} + \|v\|_{L^p}^{\gamma-1}\right) \|u - v\|_{L^p} \\
&\lesssim t^{-\frac{d}{2\beta}\left(\frac{\gamma}{p}-\frac{1}{p}\right)} M^{\gamma-1} \|u - v\|_{L^p},
\end{aligned}$$

for $\|u\|_{L^p} \le M$ and $\|v\|_{L^p} \le M$. Since $p > \frac{d(\gamma-1)}{2\beta}$, we have $t^{-\frac{d}{2\beta}\left(\frac{\gamma}{p}-\frac{1}{p}\right)} \in L^1_{\text{loc}}(0, \infty)$. Note that $t \mapsto \|K_t(0)\|_{L^p} = 0 \in L^1_{\text{loc}}(0, \infty)$ and $e^{-sH} K_t = K_{t+s}$ for $t, s > 0$. Then (1) follows by [33, Theorem 1].

4.2 Part (2) – lower blow-up rate

Let $u_0 \in L^p(\mathbb{R}^d)$ be such that $T_{\max} < \infty$, and let $u \in C\left([0, T_{\max}), L^p(\mathbb{R}^d)\right)$ be the maximal solution of (1.3). Fix $s \in [0, T_{\max})$ and set

$$w(t) = u(t + s), \quad t \in [0, T_{\max} - s), \text{ and } w(0) = u(s).$$

Then, as in the proof of Part (1), we claim that

$$\|u(s)\|_{L^p} + K M^\gamma (T_{\max} - s)^{1-\frac{d(\gamma-1)}{2p\beta}} > M, \quad \forall M > 0, \tag{4.6}$$

for some constant $K > 0$. Assuming the contrary, then for some $M > 0$ we would have

$$\|u(s)\|_{L^p} + K M^\gamma (T_{\max} - s)^{1-\frac{d(\gamma-1)}{2p\beta}} \le M,$$

and w would be defined on $[0, T_{\max} - s]$ – in particular, $u(T_{\max})$ would be well defined, a contradiction. Hence, (4.6) is verified, for any $t \in [0, T_{\max})$ fixed and for all $M > 0$.

Set then $M = 2\|u(t)\|_{L^p}$. By (4.6), we infer

$$\|u(t)\|_{L^p} + K2^\gamma \|u(t)\|_{L^p}^\gamma (T_{\max} - t)^{1-\frac{d(\gamma-1)}{2p\beta}} > 2\|u(t)\|_{L^p}, \quad \forall t \in [0, T_{\max}).$$

Hence, we have

$$\|u(t)\|_{L^p} \geq C\,(T_{\max} - t)^{\frac{d}{2p\beta} - \frac{1}{\gamma-1}} \quad \text{for all } t \in [0, T_{\max}).$$

4.3 Part (3) – global existence

Given $\gamma > 1$, one can choose r in such a way that

$$\frac{2\beta}{d\gamma(\gamma-1)} < \frac{1}{r} < \frac{2\beta}{d(\gamma-1)}.$$

Let r be fixed once for all and set

$$\delta = \frac{1}{\gamma-1} - \frac{d}{2r\beta}.$$

We observe that

$$\delta + 1 - \frac{d(\gamma-1)}{2r\beta} - \delta\gamma = 0.$$

Suppose that $\rho > 0$ and $M > 0$ satisfy the inequality

$$\rho + KM^{\gamma} \leq M,$$

where $K = K(\gamma, d, r) > 0$ is a constant and can explicitly be computed. We claim that if

$$\sup_{t>0} t^{\delta} \|e^{-tH^{\beta}} u_0\|_{L^r} \leq \rho \tag{4.7}$$

then there is a unique global solution u of (1.3) such that

$$\sup_{t>0} t^{\delta} \|u(t)\|_{L^r} \leq M. \tag{4.8}$$

In order to prove our claim, consider

$$X = \left\{ u\colon (0,\infty) \to L^r(\mathbb{R}^d) : \sup_{t>0} t^{\delta} \|u(t)\|_{L^r} < \infty \right\},$$

$$X_M = \left\{ u \in X : \sup_{t>0} t^{\delta} \|u(t)\|_{L^r} \leq M \right\}, \quad d(u,v) = \sup_{t>0} t^{\delta} \|u(t) - v(t)\|_{L^r}.$$

It is easy to realize that (X_M, d) is a complete metric space.

Consider now the mapping

$$\mathcal{J}_{u_0}(u)(t) = e^{-tH^\beta}u_0 + \int_0^t e^{-(t-s)H^\beta}(|u(s)|^{\gamma-1}u(s))ds. \tag{4.9}$$

Let u_0 and v_0 satisfy (4.7) and choose $u, v \in X_M$. Clearly, we have

$$\begin{aligned} t^\delta \|\mathcal{J}_{u_0}u(t) - \mathcal{J}_{v_0}v(t)\|_{L^r} &\le t^\delta \|e^{-tH^\beta}(u_0 - v_0)\|_{L^r} \\ &+ t^\delta \int_0^t \|e^{-(t-s)H^\beta}(|u(s)|^{\gamma-1}u(s) - |v(s)|^{\gamma-1}v(s))\|_{L^r} ds. \end{aligned}$$

Using Theorem 1 with exponents $(p, q) = (r/\gamma, r)$, (4.4) and Hölder's inequality, we obtain

$$\begin{aligned} &\|e^{-(t-s)H^\beta}(|u(s)|^{\gamma-1}u(s) - |v(s)|^{\gamma-1}v(s))\|_{L^r} \\ &\qquad\lesssim (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} \||u(s)|^{\gamma-1}u(s) - |v(s)|^{\gamma-1}v(s)\|_{L^{\frac{r}{\gamma}}} \\ &\qquad\lesssim (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} \gamma \left(\|u(s)\|_{L^r}^{\gamma-1} + \|v(s)\|_{L^r}^{\gamma-1} \right) \|u(s) - v(s)\|_{L^r} \\ &\qquad\lesssim (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} \gamma s^{-\delta\gamma} M^{\gamma-1} d(u, v). \end{aligned}$$

Using this inequality, we get

$$\begin{aligned} &t^\delta \|\mathcal{J}_{u_0}u(t) - \mathcal{J}_{v_0}v(t)\|_{L^r} \\ &\le t^\delta \|e^{-tH^\beta}(u_0 - v_0)\|_{L^r} + t^\delta \gamma M^{\gamma-1} d(u, v) \int_0^t (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} s^{-\delta\gamma} ds \\ &\le t^\delta \|e^{-tH^\beta}(u_0 - v_0)\|_{L^r} + K\, M^{\gamma-1} d(u, v), \end{aligned} \tag{4.10}$$

where $K = t^\delta \gamma \int_0^t (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} s^{-\delta\gamma} ds$ is a finite positive constant. Indeed, since

$$\delta\gamma < 1, \quad \frac{d(\gamma - 1)}{2r\beta} < 1,$$

we see that

$$\int_0^t (t-s)^{-\frac{d(\gamma-1)}{2r\beta}} s^{-\delta\gamma} ds = t^{1-\frac{d(\gamma-1)}{2r\beta} - \delta\gamma} \int_0^1 (1-s)^{-\frac{d(\gamma-1)}{2r\beta}} s^{-\delta\gamma} ds < \infty.$$

Setting $v_0 = 0$ and $v = 0$ in (4.10) we have

$$t^\delta \|\mathcal{J}_{u_0}u(t)\|_{L^r} \le \rho + K M^\gamma \le M.$$

That is, $\mathcal{J}_{u_0}$ maps X_M into itself. Letting $u_0 = v_0$ in (4.10), we note that

$$d(\mathcal{J}_{u_0}u(t), \mathcal{J}_{u_0}v(t)) \le K\, M^{\gamma-1} d(u, v).$$

Since $KM^{\gamma-1} < 1$, then $\mathcal{J}_{u_0}$ is a strict contraction on X_M. Therefore, $\mathcal{J}_{u_0}$ has a unique fixed point u in X_M, which is a solution of (4.9).

Finally, using Theorem 1 with exponents $(p,q) = \left(\frac{d(\gamma-1)}{2\beta}, r\right)$, we see that if $\|u_0\|_{L^{p_c^\beta}}$ is sufficiently small then (4.7) is satisfied.

5 Concluding remarks

In this concluding section we illustrate another application of Theorem 1, that is a set of Strichartz estimates for the fractional heat propagator. We emphasize that Strichartz estimates are indispensable tools for a thorough study of the wellposedness theory for nonlinear equations – see e.g., [29], [32]. Since the proof is based on a standard machinery, via $TT^\star$ method and real interpolation (see for instance [21, Lemma 3.2] and [36, Theorem 1.4] and the references therein), we omit the details.

We say that (q,p,r) is an β*-admissible triple* of indices if

$$\frac{1}{q} = \frac{d}{2\beta}\left(\frac{1}{r} - \frac{1}{p}\right),$$

where

$$1 < r \le p < \begin{cases} \frac{dr}{d-2\beta}, & for\ \ d > 2r\beta \\ \infty & for\ \ d \le 2r\beta. \end{cases}$$

Theorem 4 *Consider $I = [0,T)$ for some $T \in (0,\infty]$, and $\beta > 0$.*

1. *Let (q,p,r) be any β-admissible triple and consider $f \in L^r(\mathbb{R}^d)$. Then $e^{-tH^\beta} f \in L^q(I, L^p(\mathbb{R}^d)) \cap C_b(I, L^r(\mathbb{R}^d))$ and there exists a constant $C > 0$ such that*

$$\|e^{-tH^\beta} f\|_{L^q(I,L^p)} \le C\|f\|_{L^r}.$$

2. *Denote by $p_1' = \frac{p_1}{p_1-1}$ the Hölder conjugate index of $p_1 \in [1,\infty]$. Let $p_1', p \in (1,\infty)$, or $(p_1', p) = (1,\infty)$, or $p_1' = 1$ and $p \in [2,\infty)$, or $p_1' \in (1,\infty)$ and $p = 1$. Assume that (q,p) and (q_1,p_1) satisfy $p_1' \neq p$, $1 < q_1' < q < \infty$ and*

$$\frac{1}{q_1'} + \frac{d}{2\beta}\left|\frac{1}{p_1'} - \frac{1}{p}\right| = 1 + \frac{1}{q}. \tag{5.1}$$

Then, there exists a constant $C > 0$ such that

$$\left\|\int_0^t e^{-(t-s)H^\beta} F(s)ds\right\|_{L^q(I,L^p(\mathbb{R}^d))} \le C\|F\|_{L^{q_1'}(I,L^{p_1'}(\mathbb{R}^d))}. \tag{5.2}$$

We note that Pierfelice [24] studied Strichartz estimates for (1.3) with $H = -\Delta$, while Miao-Yuan-Zhang [21] and Zhai [36] obtained Strichartz estimates for the fractional Laplacian $(-\Delta)^\beta$.

Remark 7 We have several comments for Theorem 4.

1. Taking Theorem 1 into account, part (1) of Theorem 4, can be proved in analogy with [21, Lemma 3.2], while part (2) of Theorem 4 can be proved similarly to [36, Theorem 1.4].
2. The property (5.1) is weaker than the admissibility of triples $(q, p, 2)$ and $(q_1, p_1, 2)$.
3. The hypothesis (5.1) and the constraint $p_1' \neq p,\ 1 < q_1' < q < \infty$ appear as a consequence of the Hardy-Littlewood-Sobolev inequality.
4. In order to prove (5.2) we use Theorem 1, hence the assumptions on (p_1', p) : $p_1', p \in (1, \infty)$, or $(p_1', p) = (1, \infty)$, or $p_1' = 1$ and $p \in [2, \infty)$, or $p_1' \in (1, \infty)$ and $p = 1$. See [36, Section 3.2] for details.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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