Upper bounds for the resultant and Diophantine applications

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for $l \geq 5$, where

$$\Phi(r) = \frac{D}{2r}\log(2D\,\Delta_F) + \frac{rd}{2}\left(2m\log l + \log(2D\,\Delta_G)\right):$$

We choose

$$r = 1 + \left[\sqrt{\frac{D\log(2D\,\Delta_F)}{d(2m\log l + \log(2D\,\Delta_G))}}\right]:$$

Therefore

$$\frac{D}{r} \leq \sqrt{\frac{dD(2m\log l + \log(2D\,\Delta_G))}{\log(2D\,\Delta_F)}} \tag{3:9}$$

and

$$rd \leq d + \sqrt{\frac{dD\log(2D\,\Delta_F)}{2m\log l + \log(2D\,\Delta_G)}}: \tag{3:10}$$

Moreover (3.8) implies

$$r \leq \frac{D}{3d} + \sqrt{\frac{D}{3d}} \leq \frac{D}{d}: \tag{3:11}$$

By (3.9), (3.10) and (3.11) we have

$$\Phi(r) \leq \frac{d}{2}\log(2D\,\Delta_G) + \sqrt{dD\log(2D\,\Delta_F)(2m\log l + \log(2D\,\Delta_G))}:$$

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4. Lower bounds for $\left|\prod(\alpha_h - \beta_k)\right|$.

In this section we assume that F and G are integral co-prime polynomials. Then

$$|\mathrm{Res}(F;G)| \geq 1; \quad \Delta(F) \leq M(F)^2; \quad \Delta(G) \leq M(G)^2:$$

Hence theorem 1.1 and theorem 3.1 provides lower bounds for $\left|\prod(\alpha_h - \beta_k)\right|$. As a simple example, choose $l = m = 1$ in theorem 1.1. We find the following improved version of (1.6):

Corollary 4.2. Let $\alpha, \beta \neq 0$ be non-conjugate algebraic numbers of degrees D and d. Then

$$|\alpha - \beta|^{-1} \leq e(2D)^{1+d/2}M(\alpha)^{2d}M(\beta)^{D}\exp\left(\sqrt{dD\log(2DM(\alpha)^2)\log\left(2DM(\beta)^2\right)}\right)$$

provided that

$$3d\max\left(\frac{\log(2DM(\beta)^2)}{\log(2DM(\alpha)^2)};1\right) \leq D:$$

G_p *is irreducible (see [R] p. 139). Hence, for any real* x*, the polynomials* G *and* $F = \;\; {}_{p \;\; x} G_p$ *are square-free and coprime. We choose* $x = (\log d)^2 = (\log\log d)$*. By the Prime Number Theorem,*

$$D := \deg F = \;\; {}_{p \;\; x}\, d \quad \frac{d(\log d)^2}{2(\log\log d)^2} \quad \frac{d^2}{2}; \tag{6:1}$$

$$\log M(F) = \;\; {}_{p \;\; x}\, p \log M(\;\;) \quad \frac{(\log d)^4 \log M(\;\;)}{4(\log\log d)^3} \quad \log d; \tag{6:2}$$

and, since ${}_{p \;\; x}\, p^d \; | \; Res(F; G)$ *by lemma 2 of [D],*

$$\log |Res(F; G)| \quad {}_{p \;\; x}\, d \log p \quad {}_1 \frac{d(\log d)^2}{\log\log d}: \tag{6:3}$$

We also have $D \;\; 9d$ *and, by (6.2),*

$$\max \quad \frac{\log(2DM(F)^2)}{\log(2DM(\;\;)^2)}; 1 \quad 1 + \frac{2\log M(F)}{\log d} \quad 3 \quad \frac{D}{3d}:$$

Therefore we can apply theorem 1.1 (with $l = m = 0$*), which gives*

$$\log |Res(F; G)| \quad \frac{d}{2}\log(2D) + d\log M(F) + (D + d) \\ + \quad \overline{dD \log(2DM(F)^2) \log(2DM(\;\;)^2)}: \tag{6:4}$$

By (6.1) and (6.2) we have

$$\frac{d}{2}\log(2D) + d\log M(F) + (D + d)\log M(\;\;) \quad 2d\log d \tag{6:5}$$

and

$$dD \log(2DM(F)^2) \log(2DM(\;\;)^2) \\ {}_2 \frac{d^2(\log d)^4}{4(\log\log d)^2} \quad 2 + \frac{(\log d)^3 \log M(\;\;)}{(\log\log d)^3} \quad : \tag{6:6}$$

Substituting (6.3), (6.5) and (6.6) into (6.4) we obtain

$$ {}_1 \frac{d(\log d)^2}{\log\log d} \quad 2d\log d + \frac{d(\log d)^2}{2\log\log d} \quad \overline{2 + \frac{(\log d)^3 \log M(\;\;)}{(\log\log d)^3}} \\ {}_2 \frac{d(\log d)^2}{2\log\log d} \quad \overline{2 + \frac{(\log d)^3 \log M(\;\;)}{(\log\log d)^3}} \; ;$$

whence

$$\log M(\;\;) \quad 4 \;\; {}^6 \;\; 2 \quad \frac{\log\log d}{\log d}^{\,3} \quad (2 \quad ") \quad \frac{\log\log d}{\log d}^{\,3}:$$

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