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PSEUDO-RIEMANNIAN SASAKI SOLVMANIFOLDS

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Abstract. We study a class of left-invariant pseudo-Riemannian Sasaki metrics on solvable Lie groups, which can be characterized by the property that the zero level set of the moment map relative to the action of some one-parameter subgroup $\{\exp tX\}$ is a normal nilpotent subgroup commuting with $\{\exp tX\}$, and X is not lightlike. We characterize this geometry in terms of the Sasaki reduction and its pseudo-Kähler quotient under the action generated by the Reeb vector field.

We classify pseudo-Riemannian Sasaki solvmanifolds of this type in dimension 5 and those of dimension 7 whose Kähler reduction in the above sense is abelian.

Introduction

Sasaki manifolds were introduced in [16] as an odd-dimensional counterpart to Kähler geometry; they are characterized by an almost contact metric structure (ϕ, ξ, η, g) which is both normal and contact. Beside the analogy, they bear a strong relation to Kähler geometry in that both the cone over a Sasaki manifold and the space of leaves of the Reeb foliation carry a Kähler structure. For pseudo-Riemannian metrics, a completely analogous definition of Sasaki structure can be given, which was first considered in [17]; the relation to pseudo-Kähler geometry is the same as in the definite setting.

Arguably, the most interesting Sasaki metrics are those satisfying the Einstein condition $\mathrm{ric} = 2ng$, where the Einstein constant is fixed by the dimension. Both in the Riemannian and indefinite case, Einstein-Sasaki metrics are characterized by the existence of a Killing spinor (see [2]), which makes them relevant for general relativity and supersymmetry (see [9, 18]).

In this paper we focus on the homogeneous case, and particularly on invariant pseudo-Riemannian Sasaki metrics on solvmanifolds. Although we do not insist on the Einstein condition here, the prospect of applying the machinery to produce Einstein-Sasaki metrics leads us to consider *standard* solvmanifolds,

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corresponding to semidirect products $\mathfrak{g} \rtimes \mathfrak{a}$, where $\mathfrak{g}$ is nilpotent, $\mathfrak{a}$ abelian and their sum orthogonal. Indeed, all Riemannian Einstein solvmanifolds are of this type (see [12, 13]), and even in the indefinite case the standard condition has proved quite effective to produce examples (see [6, 7]). In fact, the most studied standard Lie algebras are those of Iwasawa type (or pseudo-Iwasawa, for indefinite signature), namely those for which $\operatorname{ad} X$ is symmetric for all X in $\mathfrak{a}$.

Restricting to left-invariant pseudo-Riemannian Sasaki metrics on solvable Lie groups allows us to work at the Lie algebra level; we shall therefore refer to the structures under consideration as Sasaki structures on a Lie algebra. Our first result (Proposition 2.6) is that Sasaki Lie algebras cannot be of pseudo-Iwasawa type. This motivates us to study the more general class of standard Lie algebras, though restricting for simplicity to one-dimensional abelian factors, i.e., $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$. In Proposition 3.3, we characterize the Sasaki condition on $\tilde{\mathfrak{g}}$ in terms of the induced structure on $\mathfrak{g}$. The resulting conditions on $\mathfrak{g}$ are somewhat unwieldy.

However, the situation simplifies if we impose that $\mathfrak{g}$ is the zero-level set of a moment map relative to the action of a one-parameter subgroup. In practice, this means that $\phi(e_0)$ lies in the center $\mathfrak{z}(\mathfrak{g})$. We dub this particular class of Sasaki structures $\mathfrak{z}$-*standard*. One can then take the Sasaki reduction in the sense of contact geometry, obtaining a new Sasaki nilmanifold with additional structure, namely a derivation D commuting with ϕ and satisfying a quadratic equation of the form

$$[D^s, D^a] = hD^s - 2(D^s)^2,$$

where h is a real constant, and D^s, D^a denote the symmetric and antisymmetric part of D (Corollary 4.3). In this setting, the Reeb field ξ is central, so one can take a further quotient and obtain a pseudo-Kähler nilmanifold in three dimensions less (Corollary 4.4); equivalently, one can interpret this quotient as a Kähler reduction of the pseudo-Kähler Lie algebra $\tilde{\mathfrak{g}}/\operatorname{Span}\{\xi\}$.

This construction can be inverted: starting from a pseudo-Kähler nilmanifold with a derivation as above, one obtains a pseudo-Kähler solvmanifold in two dimensions higher, then giving a $\mathfrak{z}$-standard Sasaki solvmanifold by taking a circle bundle (Proposition 5.1). This procedure differs from the double extension procedure considered in [3], in that the two "extra" dimensions span a definite two-plane, rather than neutral.

We show that up to isometry, when D^s is both a derivation and diagonalizable over $\mathbb{C}$ it can be assumed to be a projection, giving a simple explicit form to the resulting Sasaki structure (Corollary 5.6). Making use of this fact, we classify $\mathfrak{z}$-standard Sasaki solvmanifolds in dimension 5 (Theorem 5.7), and all those in dimension 7 whose Kähler reduction is abelian (Theorem 5.8).

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1. Pseudo-Riemannian Sasaki structures

In this section we recall some basic definitions and facts on pseudo-Riemannian Sasaki structures. For further details we refer to [5, 17].

Definition. An *almost contact structure* on a $(2n+1)$-dimensional manifold M is a triple (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form, such that

$$\eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi^2 = -\operatorname{Id} + \eta \otimes \xi.$$

Given a pseudo-Riemannian metric g on M, the quadruple (ϕ, ξ, η, g) is called *an almost contact metric structure* if (ϕ, ξ, η) is an almost contact structure and

$$g(\xi, \xi) = \epsilon \in \{\pm 1\}, \quad \eta = \epsilon \xi^\flat, \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y)$$

for any vector fields X, Y.

We will assume $\epsilon = 1$ in the sequel.

Note that if (ϕ, ξ, η, g) is an almost contact metric structure with $g(\xi, \xi) = \epsilon = -1$, then defining $\bar{g} = -g$ we have that $(\phi, \xi, \eta, \bar{g})$ is another almost contact metric structure such that $\bar{g}(\xi, \xi) = \bar{\epsilon} = 1$, so our assumption does not entail a loss of generality.

Remark 1.1. The generalized eigenspace of 0 for ϕ is generated by ξ. Therefore 0 is an eigenvalue and ξ is an eigenvector, i.e., $\phi(\xi) = 0$.

Remark 1.2. The endomorphism ϕ is always skew-symmetric: indeed,

$$\begin{aligned} g(\phi(X), Y) &= -g(\phi X, \phi^2 Y - \eta(Y)\xi) \\ &= -g(X, \phi(Y)) + \eta(X)\eta(\phi(Y)) = -g(X, \phi(Y)). \end{aligned}$$

In fact, if ϕ is assumed to be skew-symmetric, $g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y)$ is equivalent to $\phi^2 = -\operatorname{Id} + \eta \otimes \xi$.

We define the *fundamental* 2*-form* associated to the almost contact metric structure (ϕ, ξ, η, g) as

$$\Phi = g(\cdot, \phi \cdot).$$

In addition, in analogy with the Nijenhuis tensor field for complex manifolds, we define

$$N_\phi = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

Definition. An almost contact metric structure (ϕ, ξ, η, g) is said to be *Sasaki* if (ϕ, ξ, η, g) satisfies $N_\phi + d\eta \otimes \xi = 0$ and $d\eta = 2\Phi$.

Sasaki structures can be characterized in terms of the covariant derivative $\nabla\phi$; as usual, we indicate by ∇ the Levi-Civita connection, by R its curvature tensor, by ric its Ricci tensor.

Lemma 1.3 ([17, Proposition 1])**.** *Given an almost contact metric structure (ϕ, ξ, η, g) on a manifold of dimension $2n+1$ such that*

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

the following hold:

1. $\nabla_X \xi = -\phi(X)$;
2. ξ *is a Killing vector field*;
3. $d\eta(X,Y) = 2\Phi(X,Y)$;
4. $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$;
5. $\mathrm{ric}(\xi, X) = 2n\eta(X)$.

Arguing as in [4, Theorem 7.3.16], one obtains:

Proposition 1.4. *Let (ϕ, ξ, η, g) be an almost contact pseudo-Riemannian metric structure on M. The following are equivalent*:

1. (ϕ, ξ, η, g) *is Sasaki*;
2. *the cone* $(\mathbb{R}^+ \times M, J, \omega)$ *is pseudo-Kähler*;
3. $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$;
4. $\nabla_X \Phi = \eta \wedge X^\flat$.

Pseudo-Sasaki manifolds are related to pseudo-Kähler geometry in the following way. Recall that a pseudo-Kähler structure on a manifold M is an almost-pseudo-Hermitian structure (J, g, ω), with the convention that $\omega = g(\cdot, J\cdot)$, such that J is integrable and ω is closed; equivalently, ω is parallel with respect to the Levi-Civita connection.

Like in the Riemannian case, we have the following:

Proposition 1.5 ([14])**.** *Let M have a pseudo-Riemannian Sasaki structure (ϕ, ξ, η, g). Then the space of leaves of the Reeb foliation has an induced pseudo-Kähler structure.*

Finally, we recall that given a Sasaki structure (ϕ, ξ, η, g) and a positive constant a, we can define another Sasaki structure by

$$\hat{\phi} = \phi, \qquad \hat{\xi} = a^{-1}\xi, \qquad \hat{\eta} = a\eta, \qquad \hat{g} = ag + (a^2 - a)\eta \otimes \eta.$$

Such a transformation is called a *$\mathcal{D}$-homothety*. This defines an equivalence relation between Sasaki structures on a given manifold.

2. Sasaki Lie algebras

Throughout the paper, we consider left-invariant structures on Lie groups, which can be characterized at the Lie algebra level. Accordingly, we shall refer to pseudo-Riemannian metrics on a Lie algebra, Sasaki structures etc. to mean objects defined at the Lie algebra level and silently extended to the Lie group by left translation.

Recall from [6] that a *standard decomposition* on a Lie algebra $\tilde{\mathfrak{g}}$ endowed with a pseudo-Riemannian metric is an orthogonal decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{g}$ nilpotent and $\mathfrak{a}$ abelian. A standard decomposition is *pseudo-Iwasawa* if $\operatorname{ad} X$ is symmetric for all $X \in \mathfrak{a}$. These definitions mimick and generalize analogous definitions for Riemannian metrics (see [12]), and they have proved useful in the study of Einstein metrics ([6]).

It is well known that nonisomorphic Lie algebras can be isometric, meaning that the corresponding pseudo-Riemannian manifolds are isometric. The method to obtain such isometries is recalled below in Proposition 2.2. A natural question is whether one can choose a representative in an isometry class of Sasaki Lie algebras which admits a pseudo-Iwasawa decomposition. We show that this is never the case: indeed, no Sasaki Lie algebras admits a pseudo-Iwasawa decomposition. This will motivate the study of the more general standard case in the following sections.

We begin this section with an example of a standard Sasaki Lie algebra.

Example 2.1. Consider the 5-dimensional Lie algebra

$$\mathfrak{g} = (0, -2e^{12} - 2e^{34}, -3e^{45} - e^{13} + 3e^{24}, 3e^{35} - 3e^{23} - e^{14}, 2e^{12} + 2e^{34});$$

with notation as in [15]; explicitly, we have a fixed basis $\{e_i\}$ of $\mathfrak{g}$ such that the dual basis $\{e^i\}$ of $\mathfrak{g}^*$ satisfies $de^1 = 0$, $de^2 = -2e^1 \wedge e^2 - 2e^3 \wedge e^4$ and so on, with $d\colon \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ denoting the Chevalley-Eilenberg operator. As observed in [8, Example 5.6], the Lie algebra $\mathfrak{g}$ carries an Einstein-Sasaki structure given by

$$g = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5,$$
$$\xi = e_5, \qquad \Phi = e^{12} + e^{34}.$$

This has a standard decomposition $\operatorname{Span}\{e_1\} \ltimes \operatorname{Span}\{e_2, e_3, e_4, e_5\}$. Notice that this metric can be obtained from the Riemannian η-Einstein-Sasaki metric on the Lie algebra $\mathfrak{g}_0$ of [1] by reversing the sign of the metric along the Reeb vector field.

Given a Lie algebra $\mathfrak{g}$ with a metric g, for any endomorphism $f\colon \mathfrak{g} \to \mathfrak{g}$ we write $f = f^s + f^a$, where f^s is symmetric and f^a is skew-symmetric relative to the metric, i.e.,

$$f^s = \frac{1}{2}(f + f^*), \qquad f^a = \frac{1}{2}(f - f^*).$$

Consider a semidirect product $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{a}$ abelian, and fix any metric. In [10, Section 1.8] and [6, Proposition 1.19] it was shown that under certain conditions one can obtain an isometric Lie algebra by projecting on the symmetric part. These results assume that the decomposition is standard; however, the proof holds more generally, without assuming that the metric is standard and taking more general projections:

Proposition 2.2. *Let $\tilde{\mathfrak{g}}$ be a pseudo-Riemannian Lie algebra (not necessarily standard) of the form $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$; let $\chi\colon \mathfrak{a} \to \mathrm{Der}(\mathfrak{g})$ be a Lie algebra homomorphism such that, extending $\chi(X)$ to $\tilde{\mathfrak{g}}$ by declaring it to be zero on $\mathfrak{a}$,*

$$\chi(X)^s = (\operatorname{ad} X)^s, \qquad [\chi(X), \operatorname{ad} Y] = 0,\ \ X, Y \in \mathfrak{a}. \tag{2.1}$$

Let $\tilde{\mathfrak{g}}^*$ be the Lie algebra $\mathfrak{g} \rtimes_\chi \mathfrak{a}$. Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$, with the corresponding left-invariant metrics, whose differential at e is the identity of $\mathfrak{g} \oplus \mathfrak{a}$ as a vector space.

Proof. Observe that for every X in $\mathfrak{a}$, $\chi(X)$ is a derivation of $\mathfrak{g}$ that commutes with $\operatorname{ad} \mathfrak{a}$ by (2.1), and therefore a derivation of $\tilde{\mathfrak{g}}$. For X in $\mathfrak{a}$, write $\operatorname{ad} X = A(X)+\chi(X)$, where $A(X)$ is an antisymmetric derivation of $\tilde{\mathfrak{g}}$. By construction, $A(X)$ is zero on $\mathfrak{a}$.

The rest of the proof is identical to [6, Proposition 1.19], except that one replaces $(\operatorname{ad} X)^a$ with $A(X)$, and one cannot assume that $\exp \mathfrak{g} \exp \mathfrak{a}$ equals the whole connected, simply-connected group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$; however, it is clear that $\exp A(X)$ fixes the connected subgroup with Lie algebra $\mathfrak{a}$, which is what is needed. □

As a consequence we have a result analogous to [6, Proposition 1.19] for nonstandard metrics:

Corollary 2.3. *Let $\tilde{\mathfrak{g}}$ be a pseudo-Riemannian Lie algebra of the form $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$; suppose that, for every X in $\mathfrak{a}$, $(\operatorname{ad} X)^*$ is a derivation of $\tilde{\mathfrak{g}}$ vanishing on $\mathfrak{a}$, and furthermore*

$$[(\operatorname{ad} X)^*, \operatorname{ad} Y] = 0, \quad X, Y \in \mathfrak{a}.$$

Define $\chi\colon \mathfrak{a} \to \mathrm{Der}(\mathfrak{g})$ as $\chi(X) = (\operatorname{ad} X)^s$. Let $\tilde{\mathfrak{g}}^*$ be the solvable Lie algebra $\mathfrak{g} \rtimes_\chi \mathfrak{a}$. Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$, with the corresponding left-invariant metrics, whose differential at e is the identity of $\mathfrak{g} \oplus \mathfrak{a}$ as a vector space.

Example 2.4. We can apply Proposition 2.2 to Example 2.1 with $\mathfrak{a} = \operatorname{Span}\{e_5\}$, $\mathfrak{g} = \operatorname{Span}\{e_1, e_2 - e_5, e_3, e_4\}$ to obtain an isometric Lie algebra

$$\tilde{\mathfrak{g}} = (0, -2e^{12} - 2e^{34}, -e^{13}, -e^{14}, 2e^{12} + 2e^{34}),$$
$$g = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5,$$
$$\xi = e_5, \qquad \Phi = e^{12} + e^{34}.$$

This can be written as $\operatorname{Span}\{e_2, e_3, e_4, e_5\} \rtimes \operatorname{Span}\{e_1\}$, with

$$\operatorname{Span}\{e_2, e_3, e_4, e_5\} \cong (-2E^{23}, 0, 0, 2E^{23})$$

and

$$\operatorname{ad} e_1 = 2e^2 \otimes (e_2 - e_5) + e^3 \otimes e_3 + e^4 \otimes e_4.$$

This is standard but not pseudo-Iwasawa, consistently with Proposition 2.6 below.

In the following, we will need the explicit formula for the Levi-Civita connection of a metric on a Lie algebra, namely

$$\nabla_w v = -\operatorname{ad}(v)^s w - \frac{1}{2}(\operatorname{ad} w)^* v. \tag{2.2}$$

The formula follows immediately from the Koszul formula. In order to specialize to the standard case, we will need to fix an orthogonal basis $\{e_s\}$ on the abelian factor $\mathfrak{a}$ such that $\tilde{g}(e_s, e_s) = \epsilon_s$.

Lemma 2.5. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$. Then*

$$\widetilde{\nabla}_H X = \widetilde{\operatorname{ad}}(H)^a(X), \qquad \widetilde{\nabla}_X H = -\widetilde{\operatorname{ad}}(H)^s(X),$$

for all $H \in \mathfrak{a}$, $X \in \tilde{\mathfrak{g}}$. In addition, if $\{e_i\}$ is an orthogonal basis of $\mathfrak{a}$ and $v, w \in \mathfrak{g}$, we have

$$\widetilde{\nabla}_w v = -\operatorname{ad}(v)^s w - \frac{1}{2}(\operatorname{ad} w)^* v + \sum_s \epsilon_s \tilde{g}(\widetilde{\operatorname{ad}}(e_s)^s v, w) e_s, \quad v, w \in \mathfrak{g}.$$

Proof. If we apply (2.2) to $\widetilde{\nabla}$, we get

$$\begin{aligned}
\widetilde{\nabla}_H X &= -\widetilde{\operatorname{ad}}(X)^s H - \frac{1}{2}(\widetilde{\operatorname{ad}} H)^* X \\
&= -\frac{1}{2}\widetilde{\operatorname{ad}}(X) H - \frac{1}{2}\widetilde{\operatorname{ad}}(X)^* H - \frac{1}{2}\widetilde{\operatorname{ad}}(H)^* X = \widetilde{\operatorname{ad}}(H)^a(X), \\
\widetilde{\nabla}_X H &= -\widetilde{\operatorname{ad}}(H)^s X - \frac{1}{2}(\widetilde{\operatorname{ad}} X)^* H = -\widetilde{\operatorname{ad}}(H)^s X.
\end{aligned}$$

Now observe that $\widetilde{\operatorname{ad}}(v)^* w = \operatorname{ad}(v)^* w + \sum_s \epsilon_s \tilde{g}([v, e_s], w) e_s$. Therefore,

$$\begin{aligned}
\widetilde{\nabla}_w v &= -\frac{1}{2}\widetilde{\operatorname{ad}}(v) w - \frac{1}{2}\widetilde{\operatorname{ad}}(v)^* w - \frac{1}{2}\widetilde{\operatorname{ad}}(w)^* v \\
&= -\frac{1}{2}\operatorname{ad}(v) w - \frac{1}{2}\operatorname{ad}(v)^* w - \frac{1}{2}\operatorname{ad}(w)^* v \\
&\quad - \frac{1}{2}\sum_s \epsilon_s \tilde{g}([v, e_s], w) e_s - \frac{1}{2}\sum_s \epsilon_s \tilde{g}([w, e_s], v) e_s \\
&= -\operatorname{ad}(v)^s w - \frac{1}{2}\operatorname{ad}(w)^* v \\
&\quad + \frac{1}{2}\sum_s \epsilon_s \big(\tilde{g}(\operatorname{ad}(e_s) v, w) + \tilde{g}(\operatorname{ad}(e_s)^* v, w)\big) e_s.
\end{aligned}$$ □

We can now prove the following:

Proposition 2.6. *Let $\tilde{\mathfrak{g}}$ be a solvable Lie algebra with a Sasaki pseudo-Riemannian metric g. Then there is no pseudo-Iwasawa decomposition.*

Proof. Assume for a contradiction that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ is a pseudo-Iwasawa decomposition. Then by Lemma 2.5 and Lemma 1.3 we have

$$0 = \widetilde{\nabla}_H \xi = -\phi(H), \quad H \in \mathfrak{a}.$$

This implies that $\mathfrak{a}$ is one-dimensional and spanned by ξ. We have

$$-\phi X = \widetilde{\nabla}_X \xi = -\widetilde{\mathrm{ad}}(\xi) X.$$

However ϕ is skew-symmetric, while $\widetilde{\mathrm{ad}}(\xi)$ is symmetric, giving a contradiction. □

3. Sasaki structures on rank-one standard Lie algebras

In this section we consider standard decompositions of rank one, meaning that the abelian factor $\mathfrak{a}$ is one-dimensional. Accordingly, $\tilde{\mathfrak{g}}$ will be a solvable Lie algebra endowed with a standard decomposition $\mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$, with D a derivation of $\mathfrak{g}$ and $\operatorname{ad} e_0 = D$; we will denote by $[\,,\,]$ and d the Lie bracket and exterior derivative on $\mathfrak{g}$.

Lemma 3.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation, and let $\tau = \pm 1$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that*

$$\tilde{g} = g + \tau e^0 \otimes e^0, \qquad \widetilde{\nabla}\xi = -\phi$$

if and only if $\xi \in \mathfrak{g}$ and, writing $b = D^a(\xi)$, for all $u, w \in \mathfrak{g}$

$$\phi(w) = \frac{1}{2}(\operatorname{ad} w)^*(\xi) + \tau g(b, w) e_0, \qquad \phi(e_0) = -b, \tag{3.1}$$

$$D(\xi) = 0, \qquad (\operatorname{ad}\xi)^s = 0, \qquad (\operatorname{ad} b)^*(\xi) = 0, \tag{3.2}$$

$$g(w, u) = g(\xi, w) g(\xi, u) + \tau g(b, w) g(b, u) + \frac{1}{4} g((\operatorname{ad} w)^*\xi, (\operatorname{ad} u)^*\xi). \tag{3.3}$$

Proof. Given $\tilde{g} = g + \tau e^0 \otimes e^0$ and $\xi \in \tilde{\mathfrak{g}}$, define $\eta = \xi^\flat$ and $\phi = -\widetilde{\nabla}\xi$.

Write

$$\xi = v + a e_0, \quad v \in \mathfrak{g}, a \in \mathbb{R}.$$

By Lemma 2.5, we have

$$\widetilde{\nabla}_w \xi = \widetilde{\nabla}_w v + a \widetilde{\nabla}_w e_0 = -\operatorname{ad}(v)^s w - \frac{1}{2}(\operatorname{ad} w)^* v + \tau \tilde{g}(D^s(w), v) e_0 - a D^s(w),$$
$$\widetilde{\nabla}_{e_0}\xi = D^a(v).$$

Since $\widetilde{\phi}(X) = -\widetilde{\nabla}_X \xi$, we can write

$$\phi(w) = \operatorname{ad}(v)^s w + \frac{1}{2}(\operatorname{ad} w)^* v - \tau \tilde{g}(D^s(w), v) e_0 + a D^s(w),$$
$$\phi(e_0) = -D^a(v).$$

This determines an almost-contact metric structure if and only if ϕ is skew-symmetric and

$$\tilde{g}(X, Y) - \eta(X)\eta(Y) = \tilde{g}(\phi X, \phi Y). \tag{3.4}$$

The skew-symmetric condition implies

$$0 = \tilde{g}(\phi(w), e_0) + \tilde{g}(\phi(e_0), w) = -\tau^2 \tilde{g}(D^s(w), v) - \tilde{g}(D^a(v), w) = -\tilde{g}(D(v), w)$$

for all w in $\mathfrak{g}$, giving $D(v)=0$. In addition,

$$\begin{aligned}0&=\tilde g(\phi(w),u)+\tilde g(\phi(u),w)\\&=g(\operatorname{ad}(v)^sw,u)+g(\operatorname{ad}(v)^su,w)+\frac12 g((\operatorname{ad}w)^*v,u)\\&\quad+\frac12 g((\operatorname{ad}u)^*v,w)+ag(D^s(w),u)+ag(D^s(u),w)\\&=2g(\operatorname{ad}(v)^sw,u)+2ag(D^s(w),u),\end{aligned}$$

giving $\operatorname{ad}(v)^s+aD^s=0$ and

$$\phi(w)=\frac12(\operatorname{ad}w)^*(v)-\tau g(D^s(v),w)e_0=\frac12(\operatorname{ad}w)^*(v)+\tau g(D^a(v),w)e_0.$$

Evaluating (3.4) on w,e_0 we get

$$\begin{aligned}-a\tau g(v,w)&=\tilde g(w,e_0)-\eta(w)\eta(e_0)=\tilde g(\phi(w),\phi(e_0))\\&=\tilde g(\frac12(\operatorname{ad}w)^*(v)+\tau g(D^a(v),w)e_0,-D^a(v))\\&=g(\frac12(\operatorname{ad}w)^*v+\tau g(D^a(v),w)e_0,-D^a(v))\\&=-\frac12 g((\operatorname{ad}w)^*v,D^a(v))\\&=-\frac12 g(v,[w,D^a(v)]=\frac12 g(w,(\operatorname{ad}D^a(v))^*v).\end{aligned}$$

This holds for all w if and only if $(\operatorname{ad}D^a(v))^*v=-2a\tau v$. Since $\mathfrak{g}$ is nilpotent, the operator $\operatorname{ad}D^a(v)$ and its transpose are nilpotent, so $a=0$ and $(\operatorname{ad}D^a(v))^*v=0$. Therefore, $\xi=v$, $b=D^a(v)$ and $(\operatorname{ad}b)^*v=0$, showing that ϕ takes the form (3.1) and ξ satisfies (3.2). Evaluating (3.4) on w,u gives

$$\begin{aligned}g(w,u)-g(w,\xi)g(u,\xi)&=\tilde g(\phi(w),\phi(u))\\&=g(\frac12(\operatorname{ad}w)^*\xi+\tau g(b,w)e_0,\frac12(\operatorname{ad}u)^*\xi+\tau g(b,u)e_0)\\&=\frac14 g((\operatorname{ad}w)^*\xi,(\operatorname{ad}u)^*(\xi))+\tau g(b,w)g(b,u),\end{aligned}$$

proving (3.3).

Lastly, evaluating (3.4) on e_0,e_0 we get

$$\tau=\tilde g(e_0,e_0)-\eta(e_0)\eta(e_0)=\tilde g(-b,-b)=g(b,b);$$

however, this is a redundant condition, for $g(b,\xi)=g(D^a(\xi),\xi)=0$, so (3.3) and (3.2) imply $g(b,u)=\tau g(b,b)g(b,u)$ for all u, which is equivalent to $g(b,b)=\tau$.

The converse is proved in the same way. □

Now observe that we can write

$$g((\operatorname{ad}w)^*(v),u)=g(v,[w,u])=-dv^\flat(w,u)=-g((w\lrcorner dv^\flat)^\sharp,u),$$

so $(\operatorname{ad} w)^*(\xi) = -(w \lrcorner d\eta)^\sharp$. Recall that d denotes the Chevalley-Eilenberg operator on $\mathfrak{g}$, not $\tilde{\mathfrak{g}}$.

Lemma 3.2. *Let g be a metric on a Lie algebra $\mathfrak{g}$, let Φ be a 2-form. Then*

$$\nabla_x \Phi = \frac{1}{2}\mathcal{L}_x \Phi - \frac{1}{2}(\operatorname{ad} x)^* \Phi + \frac{1}{2}\alpha_x^\Phi,$$

where

$$\alpha_x^\Phi(u,w) = \Phi(\operatorname{ad}(u)^*(x), w) - \Phi(\operatorname{ad}(w)^*(x), u).$$

Proof. Using (2.2) we have:

$$\begin{aligned}
&\nabla_x \Phi(u,w)\\
={}& -\Phi(\nabla_x u, w) - \Phi(u, \nabla_x w)\\
={}& \frac{1}{2}\big(\Phi((\operatorname{ad} x)^* u + (\operatorname{ad} u)x + (\operatorname{ad} u)^* x, w) - \Phi((\operatorname{ad} x)^* w + (\operatorname{ad} w)x + (\operatorname{ad} w)^* x, u)\big)\\
={}& -\frac{1}{2}(\operatorname{ad} x)^* \Phi(u,w) - \frac{1}{2}\Phi(\mathcal{L}_x u, w) + \frac{1}{2}\Phi(\mathcal{L}_x w, u) + \frac{1}{2}\alpha_x^\Phi(u,w)\\
={}& -\frac{1}{2}(\operatorname{ad} x)^* \Phi(u,w) + \frac{1}{2}\mathcal{L}_x \Phi(u,w) + \frac{1}{2}\alpha_x^\Phi(u,w).
\end{aligned}$$

□

Proposition 3.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation and $\tau = \pm 1$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a Sasaki structure $(\phi, \xi, \eta, \tilde{g})$ such that $\tilde{g} = g + \tau e^0 \otimes e^0$ if and only if for some $\xi \in \mathfrak{g}$, $b = D^a(\xi)$, $\eta = \xi^\flat$, writing*

$$\alpha_x(u,w) = d\eta(\operatorname{ad}(u)^*(x), w) - d\eta(\operatorname{ad}(w)^*(x), u),$$

the following hold for $x, y \in \mathfrak{g}$:

$$D(\xi) = 0, \qquad (\operatorname{ad}\xi)^s = 0, \qquad (\operatorname{ad} b)^*(\xi) = 0, \tag{3.5}$$

$$D^a(d\eta) = 0, \qquad D^a(b) = -\tau\xi, \tag{3.6}$$

$$\eta \wedge x^\flat = \frac{1}{4}\alpha_x - \frac{1}{4}(\operatorname{ad} x)^*(d\eta) + \frac{1}{4}d(\mathcal{L}_x \eta) + \tau b^\flat \wedge D^s(x)^\flat, \tag{3.7}$$

$$D^s(x) \lrcorner d\eta + x \lrcorner db^\flat + b \lrcorner dx^\flat + [x,b]^\flat = 0. \tag{3.8}$$

Then ϕ is given by

$$\phi(w) = \frac{1}{2}(\operatorname{ad} w)^*(\xi) + \tau g(b,w)e_0, \qquad \phi(e_0) = -b, \quad w \in \mathfrak{g}.$$

Proof. Suppose $(\phi, \xi, \eta, \tilde{g})$ is a Sasaki structure as in the hypothesis. Since Sasaki structures satisfy $\widetilde{\nabla}_X \xi = -\phi(X)$, by Lemma 3.1 equations (3.1), (3.2), (3.3) hold. By Proposition 1.4, the Sasaki condition implies

$$\eta \wedge X^\flat = \widetilde{\nabla}_X \Phi. \tag{3.9}$$

We have

$$\Phi(u,w) = \tilde{g}(u, \phi(w)) = \frac{1}{2}g(u, (\operatorname{ad} w)^*(\xi)) = -\frac{1}{2}g([u,w], \xi),$$

$$\Phi(e_0,w)=\tilde g(e_0,\phi(w))=g(b,w).$$

Thus, (3.9) for $X=e_0$ implies

$$\begin{aligned}
0&=(\widetilde\nabla_{e_0}\Phi)(u,w)\\
&=-\Phi(\widetilde\nabla_{e_0}u,w)-\Phi(u,\widetilde\nabla_{e_0}w)\\
&=-\Phi(D^a(u),w)-\Phi(u,D^a(w))\\
&=\frac12 g([D^a(u),w],\xi)+\frac12 g([u,D^a(w)],\xi)\\
&=-\frac12 d\eta(D^a(u),w)-\frac12 d\eta(u,D^a(w))\\
&=\frac12(D^a d\eta)(u,w).
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\tau g(w,\xi)=(\widetilde\nabla_{e_0}\Phi)(e_0,w)&=-\Phi(e_0,\widetilde\nabla_{e_0}w)\\
&=-\Phi(e_0,D^a(w))=-g(b,D^a(w))=g(D^a(b),w),
\end{aligned}$$

i.e., $D^a(b)=-\tau\xi$.

Then, (3.9) for $X=x\in\mathfrak g$ gives

$$\begin{aligned}
&g(u,\xi)g(x,w)-g(x,u)g(\xi,w)\\
&=(\widetilde\nabla_x\Phi)(u,w)=-\Phi(\widetilde\nabla_x u,w)-\Phi(u,\widetilde\nabla_x w)\\
&=\Phi(\mathrm{ad}(u)^s(x)+\frac12(\mathrm{ad}\,x)^*(u)-\tau g(D^s(u),x)e_0,w)\\
&\quad-\Phi(\mathrm{ad}(w)^s(x)+\frac12(\mathrm{ad}\,x)^*(w)-\tau g(D^s(w),x)e_0,u)\\
&=-\frac12 g\Big([\mathrm{ad}(u)^s(x)+\frac12(\mathrm{ad}\,x)^*(u),w]-[\mathrm{ad}(w)^s(x)+\frac12(\mathrm{ad}\,x)^*(w),u],\xi\Big)\\
&\quad-\tau g(b,w)g(D^s(x),u)+\tau g(b,u)g(D^s(x),w)\\
&=-\frac14 g\Big(\big[[u,x]+(\mathrm{ad}\,u)^*x+(\mathrm{ad}\,x)^*u,w\big]\\
&\qquad\qquad-\big[[w,x]+(\mathrm{ad}\,w)^*x+(\mathrm{ad}\,x)^*w,u\big],\xi\Big)\\
&\quad+\tau(b^\flat\wedge D^s(x)^\flat)(u,w)\\
&=-\frac14 g\Big(\big[(\mathrm{ad}\,u)^*x+(\mathrm{ad}\,x)^*u,w\big]-\big[(\mathrm{ad}\,w)^*x+(\mathrm{ad}\,x)^*w,u\big]\\
&\qquad\qquad+[[u,w],x],\xi\Big)+\tau(b^\flat\wedge D^s(x)^\flat)(u,w)\\
&=\frac14 d\eta(\mathrm{ad}(u)^*x+(\mathrm{ad}\,x)^*u,w)-\frac14 d\eta(\mathrm{ad}(w)^*x+(\mathrm{ad}\,x)^*w,u)
\end{aligned}$$

$$- \frac{1}{4}d\eta(x,[u,w]) + \tau(b^\flat \wedge D^s(x)^\flat)(u,w)$$
$$= \frac{1}{4}\alpha_x(u,w) - \frac{1}{4}(\operatorname{ad} x)^*(d\eta)(u,w) + \frac{1}{4}d(\mathcal{L}_x\eta)(u,w) + \tau(b^\flat \wedge D^s(x)^\flat)(u,w)$$

so

$$\eta \wedge x^\flat = \frac{1}{4}\alpha_x - \frac{1}{4}(\operatorname{ad} x)^*(d\eta) + \frac{1}{4}d(\mathcal{L}_x\eta) + \tau(b^\flat \wedge D^s(x)^\flat).$$

Finally,

$$0 = (\widetilde{\nabla}_x\Phi)(e_0,w) = -\Phi(\widetilde{\nabla}_x e_0, w) - \Phi(e_0, \widetilde{\nabla}_x w) = \Phi(D^s(x),w) - \Phi(e_0,\nabla_x w)$$
$$= \frac{1}{2}g([w,D^s(x)],\xi) - g(b,\nabla_x w)$$
$$= \frac{1}{2}g(D^s(x),(\operatorname{ad} w)^*(\xi)) + g(b,\operatorname{ad}(w)^s(x) + \frac{1}{2}(\operatorname{ad} x)^*(w))$$
$$= -\frac{1}{2}d\eta(w,D^s(x)) + \frac{1}{2}g\big(b,\operatorname{ad}(w)(x) + (\operatorname{ad} w)^*(x) + (\operatorname{ad} x)^*(w)\big).$$

Equivalently,

$$0 = -d\eta(w,D^s(x)) + g(b,\operatorname{ad}(w)(x) + (\operatorname{ad} w)^*(x) + (\operatorname{ad} x)^*(w))$$
$$= -d\eta(w,D^s(x)) + db^\flat(x,w) + dx^\flat(b,w) + g([x,b],w)$$
$$= (D^s(x) \lrcorner\, d\eta + x \lrcorner\, db^\flat + b \lrcorner\, dx^\flat + [x,b]^\flat)(w).$$

Conversely, define $(\phi,\xi,\eta,\tilde{g})$ as in the statement, and assume that (3.5)–(3.8) hold. Since $\operatorname{ad}\xi$ is antisymmetric,

$$\operatorname{ad}\xi = -(\operatorname{ad}\xi)^*, \qquad \xi \lrcorner\, d\eta = -(\operatorname{ad}\xi)^*(\xi)^\flat = (\operatorname{ad}\xi)(\xi)^\flat = 0.$$

Evaluating (3.7) on u, ξ, one obtains

$$g(u,\xi)g(x,\xi) - g(x,u)$$
$$= \frac{1}{4}d\eta(\operatorname{ad}(u)^*x + (\operatorname{ad} x)^*u, \xi) - \frac{1}{4}d\eta(\operatorname{ad}(\xi)^*x + (\operatorname{ad} x)^*\xi, u)$$
$$- \frac{1}{4}d\eta(x,[u,\xi]) + \tau(b^\flat \wedge D^s(x)^\flat)(u,\xi)$$
$$= -\frac{1}{4}d\eta(-[\xi,x],u) - \frac{1}{4}d\eta(x,[u,\xi])$$
$$- \frac{1}{4}d\eta((\operatorname{ad} x)^*\xi, u) + \tau g(b,u)g(D^s(x),\xi)$$
$$= -\frac{1}{4}\eta([\xi,[u,\xi]]) + \frac{1}{4}(u \lrcorner\, d\eta)((\operatorname{ad} x)^*\xi) + \tau g(b,u)g(x,D^s\xi)$$
$$= -\frac{1}{4}g((\operatorname{ad} u)^*\xi, (\operatorname{ad} x)^*\xi) - \tau g(b,u)g(x,b),$$

which is equivalent to (3.3). Since (3.5) is assumed to hold and ϕ is defined so as to satisfy (3.1), Lemma 3.1 implies that $(\phi,\xi,\eta,\tilde{g})$ is an almost contact metric structure. In order to prove that it is Sasaki, one only needs to verify that (3.9) holds, which follows from the computations above. □

Remark 3.4. The 2-form α_x of Proposition 3.3 corresponds to the 2-form α_x^Φ of Lemma 3.2 with Φ equal to $d\eta$.

Remark 3.5. Using Lemma 3.2, we see that (3.7) can be rewritten as

$$\eta \wedge x^\flat = \frac{1}{2}\nabla_x d\eta + \tau b^\flat \wedge D^s(x)^\flat. \tag{3.10}$$

Using equation (2.2), we can read condition (3.8) as:

$$D^s(x) \lrcorner\, d\eta = \nabla_x b.$$

Remark 3.6. It is well known that on a Sasaki Lie algebra $\tilde{\mathfrak{g}}$ the center is contained in $\operatorname{Span}\{\xi\}$; indeed, any element of the center satisfies $v \lrcorner\, d\eta = 0$, so it is a multiple of ξ.

If $\tilde{\mathfrak{g}}$ has nontrivial center, then $\mathfrak{z}(\tilde{\mathfrak{g}}) = \operatorname{Span}\{\xi\}$ and the quotient $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{\xi\}$ has an induced pseudo-Kähler structure $(\hat{g}, J, \omega)$ by Proposition 1.5.

Remark 3.7. The equations of Proposition 3.3 simplify if we assume that the center is nontrivial, because then $\operatorname{ad}\xi = 0$. However, the center may be trivial on a Sasaki Lie algebra, see e.g. Example 2.1. It is noteworthy that Example 2.1 is isometric to a standard Lie algebra with nontrivial center (see Example 2.4).

4. $\mathfrak{z}$-Standard Sasaki structures

In this section we study the particular case where the vector b of Proposition 3.3 is central in $\mathfrak{g}$. More precisely, we say that a Sasaki structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on a Lie algebra $\tilde{\mathfrak{g}}$ is $\mathfrak{z}$-*standard* if there is a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ with $b = -\phi(e_0)$ in the center of $\mathfrak{g}$ and $\tilde{g} = g + \tau e^0 \otimes e^0$, with $\tau = \pm 1$.

We will start by giving a geometric interpretation of this condition; to that end, we will need to recall a well-known construction. Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a Sasaki structure $(\tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{\phi})$. Let X be a nonzero vector in $\tilde{\mathfrak{g}}$. The associated, left-invariant Sasaki structure on the connected, simply connected group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ is invariant under the left action of the group $\{\exp tX\}$. The fundamental vector field X^* is defined by

$$X^*_g = \frac{d}{dt}(\exp tX)g,$$

so identifying $T_g\tilde{G}$ with $\tilde{\mathfrak{g}}$ by left-translation we get

$$L_{g^{-1}*}X^*_g = \frac{d}{dt}g^{-1}(\exp tX)g = \operatorname{Ad}(g^{-1})X.$$

The moment map $\mu\colon \tilde{G} \to \mathbb{R}$ is by definition

$$\mu(g) = \eta(\operatorname{Ad}(g^{-1})X).$$

Therefore,

$$d\mu_g(L_{g*}v) = \frac{d}{dt}|_{t=0}\mu(g\exp tv)$$

$$= \frac{d}{dt}|_{t=0}\eta(\operatorname{Ad}(\exp -tv)\operatorname{Ad}(g^{-1})X) = -\eta([v, \operatorname{Ad}(g^{-1})X]).$$

Now if $\mu(g) = 0$, $\operatorname{Ad}(g^{-1})X \in \ker\eta$. This implies that $\operatorname{Ad}(g^{-1})X \lrcorner\, d\eta$ is nonzero, i.e., there is some v such that $\eta([v, \operatorname{Ad}(g^{-1})X]) \neq 0$. Thus, 0 is a regular value and $\mu^{-1}(0)$ is a hypersurface.

Since X^* is nowhere zero, the action of $\{\exp tX\}$ is well defined on $\mu^{-1}(0)$. Therefore, the quotient

$$\tilde{G}//\{\exp tX\} = \mu^{-1}(0)/\{\exp tX\}$$

is well defined (locally), and it has an induced Sasaki structure.

$\mathfrak{z}$-standard Sasaki structures can be characterized as follows:

Lemma 4.1. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a Sasaki structure $(\phi, \xi, \eta, \tilde{g})$. The following are equivalent*:

(i) *there is a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ with $\phi(e_0)$ in the center of* $\mathfrak{g}$;

(ii) *$\tilde{\mathfrak{g}}$ contains a vector X with $\tilde{g}(X, X) \neq 0$ such that its centralizer $\mathfrak{z}(X)$ is a nilpotent ideal of codimension one*;

(iii) *the simply connected Lie group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ has a one-parameter subgroup $\{\exp tX\}$ such that*
- $\tilde{g}(X, X) \neq 0$;
- *the zero set of the moment map is a normal nilpotent subgroup G; and*
- $\{\exp tX\}$ *commutes with G.*

Proof. If (i) holds, observe that e_0 is not a multiple of ξ by Proposition 3.3; thus, $X = -\phi(e_0)$ has centralizer equal to $\mathfrak{g}$. This implies (ii).

Now assume that (ii) holds; then $\tilde{\mathfrak{g}}$ is solvable, as it contains a codimension one nilpotent ideal. The zero level set of the moment map $\{g \mid \eta(\operatorname{Ad}(g^{-1})X) = 0\}$ is the connected subgroup with Lie algebra $\mathfrak{z}(X)$, giving (iii).

Finally, suppose that (iii) holds. Since $\mu^{-1}(0)$ is a normal nilpotent subgroup, its Lie algebra is the nilpotent ideal

$$\mathfrak{g} = \ker X \lrcorner\, d\eta.$$

In addition, $\mu^{-1}(0)$ contains the identity, so $\eta(X) = 0$. This implies that $\mathfrak{g}$ has codimension one. By construction, $e_0 = \phi(X)$ is orthogonal to $\mathfrak{g}$. Since X is not lightlike, the restriction of the metric to $\mathfrak{g}$ is definite; hence we have a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$. By construction, $\phi(e_0) = -X$, so it is central in $\mathfrak{g}$, giving (i). □

Given a $\mathfrak{z}$-standard Sasaki structure, Lemma 4.1 implies that $\{\exp tX\}$ is central in G, so the right action of $\{\exp tX\}$ preserves the Sasaki structure and the quotient $G/\exp\{tX\}$ is a Lie group with Lie algebra $\mathfrak{z}(X)/\operatorname{Span}\{X\}$, which is Sasaki by construction. Conversely, we can express $\mathfrak{z}(X)$ as a central extension of X, and then express $\mathfrak{g}$ as a standard extension of $\mathfrak{z}(X)$.

Example 4.2. In Example 2.4, $\{\exp te_2\}$ satisfies the conditions of Lemma 4.1; the three-dimensional quotient in this case is the Heisenberg algebra, with its Sasaki structure.

In the language of Proposition 3.3, we can express this as follows:

Corollary 4.3. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with a pseudo-Riemannian metric* g*,* D *a derivation and* $\tau = \pm 1$*. Assume* $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ *has a* $\mathfrak{z}$*-standard Sasaki structure* $(\phi, \xi, \eta, \tilde{g})$*. Then the following hold for* $x \in \mathfrak{g}$:

$$D(\xi) = 0, \qquad D(b) = -2\tau\xi + hb, \quad h \in \mathbb{R}, \quad b, \xi \in \mathfrak{z}(\mathfrak{g}),$$
$$D^a(d\eta) = 0, \qquad D(d\eta) = 2db^\flat,$$
$$\eta \wedge x^\flat = \frac{1}{2}\nabla_x d\eta + \tau b^\flat \wedge D^s(x)^\flat,$$
$$d\eta(D^s(x), y) = d\eta(x, D^s(y)).$$

Furthermore, ϕ *is given by*

$$\phi(w) = \frac{1}{2}(\operatorname{ad} w)^*(\xi) + \tau g(b, w)e_0, \qquad \phi(e_0) = -b, \quad w \in \mathfrak{g}.$$

In addition, $\mathfrak{g}/\operatorname{Span}\{b\}$ *has a Sasaki structure* $(\check{\phi}, \check{\xi}, \check{\eta}, \check{g})$ *induced by the identification* $\operatorname{Span}\{e_0, b\}^\perp \cong \mathfrak{g}/\operatorname{Span}\{b\}$*; at the level of the corresponding Lie groups, this amounts to taking the Sasaki reduction by the left action of the one-parameter subgroup* $\{\exp tb\}$*.*

Proof. We specialize Proposition 3.3 with $b = -\phi(e_0)$ central. Then $(\operatorname{ad} b)^*$ and $b \lrcorner dx^\flat$ are zero. In particular, from (3.8), we get

$$D^s(x) \lrcorner d\eta + x \lrcorner db^\flat = 0. \tag{4.1}$$

For $x = b$, this implies $D^s(b) \lrcorner d\eta = 0$. Since $d\eta$ is nondegenerate on $\operatorname{Span}\{b, \xi\}^\perp$, this implies that $D^s(b) \in \operatorname{Span}\{b, \xi\}$. Furthermore, we have

$$g(D^s(b), \xi) = g(b, D^s(\xi)) = g(b, -b) = -\tau,$$

so $D^s(b) = -\tau\xi + hb$ for some real constant h. Therefore,

$$D(b) = -2\tau\xi + hb.$$

Since D is a derivation, we have

$$0 = D[b, x] = [D(b), x] + [b, D(x)] = -2\tau[\xi, x].$$

Therefore ξ is in the center of $\mathfrak{g}$.

By (3.6), $D^a(d\eta) = 0$, so we observe that

$$\begin{aligned} D^s d\eta(x, y) &= Dd\eta(x, y) = -d\eta(Dx, y) - d\eta(x, Dy) \\ &= \eta([Dx, y] + [x, Dy]) = \eta(D[x, y]) \\ &= -2g(b, [x, y]) = 2db^\flat(x, y). \end{aligned}$$

Therefore, $D(d\eta) = 2db^\flat$ and (4.1) becomes equivalent to

$$0 = d\eta(D^s(x), y) + \frac{1}{2}(D^s d\eta)(x, y) = \frac{1}{2}\big(d\eta(D^s(x), y) - d\eta(x, D^s(y))\big).$$

For the last part, observe that $\mathfrak{g}$ is the centralizer of b in $\tilde{\mathfrak{g}}$, and apply the observation before the statement. The fact that $(\check{\phi}, \check{\xi}, \check{\eta}, \check{g})$ is Sasaki can be seen from $\eta \wedge x^\flat = \frac{1}{2}\check{\nabla}_x d\eta$. □

We can describe the situation of Corollary 4.3 in terms of the Kähler quotient as follows:

Corollary 4.4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, D a derivation and $\tau = \pm 1$. Assume $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a $\mathfrak{z}$-standard Sasaki structure $(\phi, \xi, \eta, \tilde{g})$. Then ξ is central in $\mathfrak{g}$ and there is $h \in \mathbb{R}$ such that*

1. *$g(\xi, \xi) = 1$, $g(b, b) = \tau$, $g(b, \xi) = 0$;*
2. *the quotient $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{b, \xi\}$ has a pseudo-Kähler structure $(\check{g}, J, \omega)$ with $(\mathfrak{g}, g) \to (\check{\mathfrak{g}}, \check{g})$ a Riemannian submersion, $\omega = \frac{1}{2}d\eta$ and $\check{D}(\omega) = db^\flat$;*
3. *relative to the splitting $\operatorname{Span}\{b, \xi\}^\perp \oplus \operatorname{Span}\{b\} \oplus \operatorname{Span}\{\xi\}$, D takes the form*

$$D = \begin{pmatrix} \check{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix};$$

- $[J, \check{D}] = 0$;
- *$\check{D}$ is a derivation and $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$.*

Proof. Define $b = \phi(e_0)$, hence $g(\xi, \xi) = 1$ by definition of Sasaki and

$$g(b, \xi) = \tilde{g}(b, \xi) = -\tilde{g}(e_0, \phi(\xi)) = 0, \quad g(b, b) = \tilde{g}(e_0, e_0) = \tau$$

give the first condition.

Let $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{b, \xi\}$. Then arguing as in Proposition 1.5 we see that $\check{\nabla} d\eta$ is the projection of $\nabla d\eta$; projecting the equation (3.10), we see that $d\eta$ is $\check{\nabla}$-parallel. Furthermore, for x orthogonal to b, ξ, we get by taking the interior product of (3.10) with ξ that

$$x^\flat = \frac{1}{2}\xi \lrcorner \nabla_x d\eta - g(D^s(x), \xi)\tau b^\flat = \frac{1}{2}\xi \lrcorner \nabla_x d\eta;$$

using Lemma 3.2, we get

$$x^\flat = \frac{1}{4}\xi \lrcorner (\alpha_x - (\operatorname{ad} x)^* d\eta + \mathcal{L}_x d\eta) = \frac{1}{4}(\operatorname{ad} x)^* \xi \lrcorner d\eta. \tag{4.2}$$

This implies that $d\eta$ is nondegenerate. Now set

$$J(x) = -\frac{1}{2}(x \lrcorner d\eta)^\sharp.$$

Then in $\operatorname{Span}\{b,\xi\}^{\perp}$ equation (4.2) reads

$$x^\flat = -\frac{1}{4}(x \lrcorner d\eta)^\sharp \lrcorner d\eta = \frac{1}{2}J(x)\lrcorner d\eta = -\big(J\circ J(x)\big)^\flat = -\big(J^2(x)\big)^\flat;$$

therefore, J is an almost complex structure, and $(\check{g}, J, d\eta)$ is a pseudo-Kähler structure. In particular, we can write

$$d\eta(x,y) = 2g(x, Jy).$$

Now from Corollary 4.3 write

$$d\eta(D^s(x), y) = d\eta(x, D^s(y))$$

as

$$g(JD^s(x), y) = g(Jx, D^s(y)) = -g(x, JD^s(y)),$$

i.e., $JD^s = -(JD^s)^* = D^sJ$. In addition, $D^a d\eta = 0$ can be rewritten as

$$\begin{aligned} 0 = D^a d\eta(x,y) &= d\eta(D^a x, y) + d\eta(x, D^a y)\\ &= 2g(D^a x, JY) + 2g(x, JD^a y) = 2g(x, [J, D^a]y). \end{aligned}$$

This shows that J and D commute.

The Lie bracket on $\check{\mathfrak{g}}$ and the Lie bracket on $\mathfrak{g}$ are related by

$$[x,y] = [x,y]_{\check{\mathfrak{g}}} - \tau db^\flat(x,y)b - d\eta(x,y)\xi;$$

b, ξ are in the center for $\mathfrak{g}$. Relative to the splitting $\operatorname{Span}\{b,\xi\}^\perp \oplus \operatorname{Span}\{b\} \oplus \operatorname{Span}\{\xi\}$, D takes the form

$$D = \begin{pmatrix} \check{D} & 0 & 0\\ 0 & h & 0\\ 0 & -2\tau & 0 \end{pmatrix}. \tag{4.3}$$

A linear map D of the form (4.3) automatically satisfies $D[x,y] = [Dx,y] + [x,Dy]$ when x lies in $\operatorname{Span}\{b,\xi\}$; therefore, D is a derivation if and only if for x,y in $\operatorname{Span}\{b,\xi\}^\perp$ one has

$$\begin{aligned} 0 = D[x,y] - [Dx,y] - [x,Dy] = \check{D}[x,y]_{\check{\mathfrak{g}}} &- \tau db^\flat(x,y)(hb - 2\tau\xi)\\ &- [\check{D}x, y]_{\check{\mathfrak{g}}} + \tau db^\flat(\check{D}x, y)b + d\eta(\check{D}x,y)\xi\\ &- [x, \check{D}y]_{\check{\mathfrak{g}}} + \tau db^\flat(x, \check{D}y)b + d\eta(x, \check{D}y)\xi. \end{aligned}$$

Thus, D is a derivation if and only if $\check{D}$ is a derivation of $\check{\mathfrak{g}}$ and

$$\begin{aligned} hdb^\flat(x,y) &= db^\flat(\check{D}x, y) + db^\flat(x, \check{D}y),\\ -2db^\flat(x,y) &= d\eta(\check{D}x, y) + d\eta(x, \check{D}y), \end{aligned}$$

where the latter is again $2db^\flat = \check{D}d\eta$.

Then using $[J, D] = 0$,

$$\begin{aligned} db^\flat(x,y) = -\frac{1}{2}\check{D}d\eta(x,y) &= \frac{1}{2}d\eta(\check{D}x,y) + \frac{1}{2}d\eta(x,\check{D}y)\\ &= g(\check{D}x, Jy) + g(x, J\check{D}y) = g(x, (\check{D}^*J + J\check{D})y) \end{aligned}$$

$$= g(x, (\check{D}^s J - \check{D}^a J + J\check{D})y) = 2g(x, \check{D}^s Jy).$$

Thus

$$\begin{aligned} 2hg(x, \check{D}^s Jy) &= hdb^\flat(x,y) = db^\flat(\check{D}x, y) + db^\flat(x, \check{D}y) \\ &= 2g(\check{D}x, \check{D}^s Jy) + 2g(x, \check{D}^s J\check{D}y) \\ &= 2g(x, (\check{D}^s - \check{D}^a)\check{D}^s Jy) + 2g(x, \check{D}^s \check{D} Jy). \end{aligned}$$

Therefore,

$$h\check{D}^s J = (\check{D}^s - \check{D}^a)\check{D}^s J + \check{D}^s \check{D} J = 2(\check{D}^s)^2 J + [\check{D}^s, \check{D}^a]J,$$

i.e.,

$$h\check{D}^s - 2(\check{D}^s)^2 = [\check{D}^s, \check{D}^a]. \qquad \square$$

In the situation of Corollary 4.4, we will say that the pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ is the *Kähler reduction* of the $\mathfrak{z}$-standard Sasaki structure of $\tilde{\mathfrak{g}}$. Notice that $\check{\mathfrak{g}}$ is indeed a Kähler reduction in the sense of symplectic geometry, arising from the action of $\{\exp tb\}$ on the pseudo-Kähler nilmanifold $\tilde{\mathfrak{g}}/\operatorname{Span}\{\xi\}$.

Example 4.5. In Example 2.4, we have

$$\check{\mathfrak{g}} = \operatorname{Span}\{e_3, e_4\}, \qquad \check{D} = I, \qquad b = -e_2, \qquad h = 2, \qquad \tau = -1,$$
$$\omega = e^{34}, \qquad db^\flat = de^2 = -2e^{34}, \qquad d\eta = 2e^{34}.$$

Corollary 4.3 has a Kähler analogue, which can be viewed as a consequence of Corollary 4.4, using the fact that any pseudo-Kähler Lie algebra yields a Sasaki Lie algebra by taking a central extension. Notice that this construction only works one way in general, i.e., it is not generally true that a Sasaki Lie algebra is a central extension of a pseudo-Kähler Lie algebra. This only occurs when ξ is central, which happens to be true in the situation of Corollary 4.4.

Proposition 4.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation and $\tau = \pm 1$. Suppose that $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a pseudo-Kähler structure $(\tilde{J}, \tilde{g}, \tilde{\omega})$ such that $\tilde{g} = g + \tau e^0 \otimes e^0$, with $b = -\tilde{J}e_0$ in the center of $\mathfrak{g}$. Then*

1. *the quotient $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{b\}$ has a pseudo-Kähler structure $(\check{g}, \check{J}, \check{\omega})$ with $\pi\colon (\mathfrak{g}, g) \to (\check{\mathfrak{g}}, \check{g})$ a Riemannian submersion, $\pi^*\check{\omega} = \tilde{\omega}|_{\mathfrak{g}}$ and $D(\omega) = db^\flat$;*
2. *relative to the splitting $\operatorname{Span}\{b\}^\perp \oplus \operatorname{Span}\{b\}$, D takes the form*
$$D = \begin{pmatrix} \check{D} & 0 \\ 0 & h \end{pmatrix};$$
3. $[\check{J}, \check{D}] = 0$;
4. *$\check{D}$ is a derivation and $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$.*

Proof. Write $\check{\mathfrak{g}} = \operatorname{Span}\{b\}^\perp$ in $\mathfrak{g}$, and let ω be the restriction of $\tilde{\omega}$ to $\check{\mathfrak{g}}$. Then $\tilde{\omega} = \omega - \tau b \wedge e^0$.

Let $\mathfrak{h} = \mathfrak{g} \oplus \operatorname{Span}\{\xi\}$ be the central extension of $\mathfrak{g}$ by the cocycle 2ω, $\check{\mathfrak{h}}$ the quotient $\mathfrak{h}/\operatorname{Span}\{b\}$, and $\tilde{\mathfrak{h}}$ the semidirect product $\mathfrak{h} \rtimes_{D'} \operatorname{Span}\{e_0\}$, where D' is defined by

$$D'v = Dv, \quad v \in \check{\mathfrak{g}}, \qquad D'\xi = 0, \qquad D'b = Db - 2\tau\xi.$$

We can summarize the situation as follows

$$\check{\mathfrak{h}} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{\xi\}, \qquad \mathfrak{h} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi\}, \qquad \tilde{\mathfrak{h}} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi, e_0\}.$$

We can view equivalently $\tilde{\mathfrak{h}}$ as the central extension of $\tilde{\mathfrak{g}}$ by $2\tilde{\omega}$. In particular, $\tilde{\mathfrak{h}}$ has a Sasaki metric $(\tilde{\phi}, \xi, \tilde{h}, \tilde{\eta})$ induced by the pseudo-Kähler metric of $\tilde{\mathfrak{g}}$ (see [11]). Explicitly, $\tilde{\eta}$ is the 1-form on $\tilde{\mathfrak{h}}$ that vanishes on $\tilde{\mathfrak{g}}$, with $\tilde{\eta}(\xi) = 1$, so that $d\eta = 2\tilde{\omega}$, we have

$$\tilde{h} = \tilde{g} + \tilde{\eta} \otimes \tilde{\eta}, \qquad \tilde{\phi} = \tilde{J}.$$

Since b is central in $\mathfrak{h}$, we can apply Corollary 4.4. Then $(\check{g}, \check{J}, \check{\omega})$ is pseudo-Kähler, and $\check{D}\omega = db^\flat$,

$$D' = \begin{pmatrix} \check{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix},$$

proving (1) and (2). (3) and (4) follow directly from Corollary 4.4. □

5. Construction of $\mathfrak{z}$-standard Sasaki structures

In this section we invert the reduction process of Corollary 4.4 and describe a constructive way of obtaining $\mathfrak{z}$-standard Sasaki structures. We also classify $\mathfrak{z}$-standard Sasaki structures of dimension ≤ 7 whose Kähler reduction is abelian.

Proposition 5.1. *Let $(\check{\mathfrak{g}}, J, \omega)$ be a pseudo-Kähler nilpotent Lie algebra. Let $\check{D}$ be a derivation of $\check{\mathfrak{g}}$, $\tau = \pm 1$, and $\mathfrak{g} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi\}$ a central extension of $\mathfrak{g}$ with a metric of the form:*

$$g(x, y) = \check{g}(x, y), \qquad g(x, b) = 0 = g(x, \xi),$$
$$g(\xi, \xi) = 1, \qquad g(b, b) = \tau, \qquad g(b, \xi) = 0,$$

where $x, y \in \check{\mathfrak{g}}$. Assume furthermore

- *$d\xi^\flat = 2\omega$, where the right-hand-side is implicitly pulled back to $\mathfrak{g}$;*
- *$db^\flat = \check{D}\omega$, where the right-hand-side is implicitly pulled back to $\mathfrak{g}$;*
- *$[J, \check{D}] = 0$;*
- *$[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ for some constant h.*

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$, where

$$[e_0, x] = \check{D}x, \qquad [e_0, b] = hb - 2\tau\xi, \qquad [e_0, \xi] = 0.$$

Then $\tilde{\mathfrak{g}}$ has a $\mathfrak{z}$-standard Sasaki structure $(\phi, \eta, \xi, \tilde{g})$ given by

$$\tilde{g} = g + \tau e^0 \otimes e^0, \qquad \phi(x) = J(x) + \tau g(b, x)e_0, \qquad \phi(e_0) = -b, \quad x \in \mathfrak{g}.$$

Proof. The fact that $D = \check{D} + \tau b^\flat \otimes (hb - 2\tau\xi)$ is a derivation is proved as in Corollary 4.4.

Then we use Proposition 3.3. To prove (3.8), write

$$\begin{aligned} db^\flat(y,x) &= \check{D}\omega(y,x) = -\omega(\check{D}y,x) - \omega(y,\check{D}x) \\ &= -g(\check{D}y,Jx) - g(y,J\check{D}x) = -g(y,(\check{D}^*J + J\check{D})x) \\ &= -g(y,J(\check{D}+\check{D}^*)x) = -2\omega(y,\check{D}^s x) = -d\eta(y,\check{D}^s x); \end{aligned}$$

then $D^s(x) \lrcorner\, d\eta + x \lrcorner\, db^\flat = 0$, which is equivalent to (3.8) since b is central.

To prove (3.10), notice that projecting this equation to $\Lambda^2\check{\mathfrak{g}}$ simply says that ω is parallel on $\check{\mathfrak{g}}$. The interior product with ξ yields (4.2), which holds by construction. Finally, taking interior product of (3.10) with b and using the fact that $D^s(b) \in \operatorname{Span}\{b,\xi\}$, we compute

$$\begin{aligned} 0 &= \frac{1}{4} b \lrcorner\, (\alpha_x - (\operatorname{ad} x)^* d\eta + \mathcal{L}_x d\eta) + D^s(x)^\flat \\ &= \frac{1}{4}((\operatorname{ad} x)^* b \lrcorner\, d\eta) + D^s(x)^\flat = \big(\frac{1}{2} J((\operatorname{ad} x)^* b) + D^s(x)\big)^\flat. \end{aligned}$$

We also have $\operatorname{ad}(x)^* b = \operatorname{ad}(D^s(x))^*\xi = -2J(D^s(x))$. Therefore, this equation reduces to $J^2(D^s(x)) = -D^s(x)$, which is automatically satisfied.

The other hypotheses of Proposition 3.3 are trivially satisfied; therefore, $\tilde{\mathfrak{g}}$ has a Sasaki structure with

$$\phi(w) = \frac{1}{2}(\operatorname{ad} w)^*\xi + \tau g(b,w)e_0 = -w \lrcorner\, \omega + \tau(g,b,w)e_0 = Jw + \tau(g,b,w)e_0. \qquad \square$$

Remark 5.2. It is no loss of generality to assume $h \geq 0$; indeed, changing the sign of $\check{D}$, e_0, b and h gives the same Sasaki Lie algebra up to isometric isomorphism.

Remark 5.3. The hypotheses of Proposition 5.1 are preserved if one rescales both h and $\check{D}$. This yields different metrics on $\tilde{\mathfrak{g}}$, which are however related by a $\mathcal{D}$-homothety (in particular, they have different curvature).

Accordingly, one can assume that either $h = 0$ or $h = 2$ up to $\mathcal{D}$-homothety. The condition $h = 0$ implies that $\operatorname{tr}(\check{D}^s)^2 = 0$. If $\check{\mathfrak{g}}$ is Riemannian, $\check{D}^s$ is diagonalizable, so $h = 0$ implies that $\check{D}$ is skew-symmetric.

Remark 5.4. One can always reverse the sign of the metric $\check{g}$ and the 2-form ω and obtain a different Sasaki metric on an isomorphic Lie algebra $\tilde{\mathfrak{g}}'$; the isomorphism is realized by the mapping $b \mapsto -b'$, $\xi \mapsto -\xi'$.

Let $(\check{\mathfrak{g}}_0, J_0, g_0, \omega_0)$, $(\check{\mathfrak{g}}_1, J_1, g_1, \omega_1)$ be pseudo-Kähler Lie algebras, with $\mathfrak{g}_1$ abelian. Let $\rho\colon \check{\mathfrak{g}}_0 \to \mathfrak{gl}(\check{\mathfrak{g}}_1)$ be a representation such that

$$\rho(X)\omega_1 = 0, \qquad [J_1, \rho(X)] + [\rho(J_0X), J_1]J_1 = 0. \tag{5.1}$$

Then $\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1$ has an almost Hermitian structure (g, J, ω), with $g = g_0 + g_1$, $\omega = \omega_0 + \omega_1$, and $J = \left(\begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix}\right)$. It is straightforward to check that ω is closed and J integrable, i.e., $\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1$ is pseudo-Kähler. In addition, the projection π_1

on the factor $\check{\mathfrak{g}}_1$ is a derivation, giving a one-parameter family of derivations $\check{D} = \frac{h}{2}\pi_1$ that satisfy the hypotheses of Proposition 5.1. The resulting Sasaki extension $\tilde{\mathfrak{g}}$ takes the form

$$\begin{aligned}&(\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1 \oplus \operatorname{Span}\{b, \xi\}) \rtimes \operatorname{Span}\{e_0\}, \qquad d\xi^\flat = 2\omega, \qquad db^\flat = -h\omega,\\ &[e_0, X_0] = 0, \qquad [e_0, X_1] = \frac{h}{2}X_1, \qquad [e_0, b] = hb - 2\tau\xi, \qquad [e_0, \xi] = 0,\end{aligned} \tag{5.2}$$

where X_0 denotes the generic element of $\check{\mathfrak{g}}_0$ and X_1 the generic element of $\check{\mathfrak{g}}_1$.

Proposition 5.5. *In the hypotheses of Proposition 5.1, if $\check{D}^s$ is a derivation and $[\check{D}^s, \check{D}^a] = 0$, we can assume up to isometry that $\check{\mathfrak{g}}$ is a semidirect product $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$, where $\check{\mathfrak{g}}_0, \check{\mathfrak{g}}_1$ are pseudo-Kähler with $\check{\mathfrak{g}}_1$ abelian, $\check{D} = \frac{h}{2}\pi_1$ and $\tilde{\mathfrak{g}}$ takes the form* (5.2)*.*

Proof. Write $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$, where $\operatorname{ad}(e_0) = \check{D} + hb^* \otimes (hb - 2\tau\xi)$. Then define

$$\chi\colon \operatorname{Span}\{e_0\} \to \operatorname{Der}\mathfrak{g}, \qquad \chi(e_0) = \check{D}^s + hb^* \otimes (hb - 2\tau\xi).$$

Then $\chi(e_0)^s = \operatorname{ad}(e_0)^s$ and $[\chi(e_0), \operatorname{ad} e_0] = 0$. Thus, the Lie algebra $\mathfrak{g} \rtimes_\chi \operatorname{Span}\{e_0\}$ is isometric to the Lie algebra $\tilde{\mathfrak{g}}$ constructed in Proposition 5.1. In other words, replacing $\check{D}$ with $\check{D}^s$ gives the same metric $\tilde{g}$ up to isometry. In addition, $\check{D}\omega = \check{D}^s\omega$, so $db^\flat$ is unchanged.

By Proposition 5.1, the minimal polynomial of $\check{D}$ divides $p(t) = ht - 2t^2$. Thus $\check{D}$ is diagonalizable over $\mathbb{R}$, and takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{h}{2}I \end{pmatrix}$$

in some basis; since $\check{D}$ commutes with J, its eigenspaces are J-invariant. Since it is symmetric, they are orthogonal. Since a diagonalizable derivation defines a grading, we have $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$, the Kähler form splits as $\omega_0 + \omega_1$ and

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix}.$$

We have that $(\check{\mathfrak{g}}_0, J_0, \omega_0)$ is Kähler, $\check{\mathfrak{g}}_1$ is abelian, and (5.1) holds. ☐

Corollary 5.6. *In the hypotheses of Proposition 5.1, if $\check{D}^s$ is a derivation and it is diagonalizable over $\mathbb{C}$, then we can assume up to isometry that $\check{\mathfrak{g}}$ is a semidirect product $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$, where $\check{\mathfrak{g}}_0, \check{\mathfrak{g}}_1$ are pseudo-Kähler with $\check{\mathfrak{g}}_1$ abelian, $\check{D} = \frac{h}{2}\pi_1$ and $\tilde{\mathfrak{g}}$ takes the form* (5.2)*.*

Proof. Denote by $\check{\mathfrak{g}}^{\mathbb{C}}$ the complexification of $\check{\mathfrak{g}}$, with the scalar product obtained by complexifying the scalar product of $\check{\mathfrak{g}}$. The complexified endomorphisms $(\check{D}^s)^{\mathbb{C}}\colon \check{\mathfrak{g}}^{\mathbb{C}} \to \check{\mathfrak{g}}^{\mathbb{C}}$, $(\check{D}^a)^{\mathbb{C}}\colon \check{\mathfrak{g}}^{\mathbb{C}} \to \check{\mathfrak{g}}^{\mathbb{C}}$ are symmetric and antisymmetric, respectively. Furthermore, we get

$$[(\check{D}^s)^{\mathbb{C}}, (\check{D}^a)^{\mathbb{C}}] = h(\check{D}^s)^{\mathbb{C}} - 2((\check{D}^s)^{\mathbb{C}})^2. \tag{5.3}$$

By hypothesis, there exists an orthonormal basis of eigenvectors of $(\check{D}^s)^{\mathbb{C}}$. Then $(\check{D}^s)^{\mathbb{C}}$ is diagonal in this basis, and $(\check{D}^a)^{\mathbb{C}}$ has zero on the diagonal. Therefore, $[(\check{D}^s)^{\mathbb{C}}, (\check{D}^a)^{\mathbb{C}}]$ has zero on the diagonal, so (5.3) implies that it vanishes and we can apply Proposition 5.5. □

In particular, Corollary 5.6 classifies $\mathfrak{z}$-standard Sasaki structures that reduce to an abelian Kähler Lie algebra, as positive-definiteness of the metric implies that $\check{D}^s$ is automatically a diagonalizable derivation in this case.

The case of indefinite signature is more flexible, as we will see below. Notice that the signature of a pseudo-Kähler metric is necessarily of the form $(2p, 2q)$.

Theorem 5.7. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra of dimension 5 with a $\mathfrak{z}$-standard Sasaki structure. Then, up to isometry and $\mathcal{D}$-homothety, $\tilde{\mathfrak{g}}$ is one of*

$$(0,0,0,-2e^{12}-2\tau e^{35},0),$$
$$(0,0,2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0),$$
$$(e^{15},e^{25},2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0),$$

and the Sasaki structure is given by

$$\tilde{g} = \pm(e^1\otimes e^1+e^2\otimes e^2)+\tau e^3\otimes e^3+e^4\otimes e^4+\tau e^5\otimes e^5, \quad \xi = e_4, \quad \Phi = -e^{12}-\tau e^{35}.$$

Proof. The Kähler reduction $\check{\mathfrak{g}}$ is a nilpotent Lie algebra of dimension two, hence abelian. Assume first that $\check{\mathfrak{g}}$ has positive-definite signature. In some basis $\{e_1, e_2\}$, we have

$$\check{g} = e^1\otimes e^1 + e^2\otimes e^2, \qquad \omega = -e^{12}, \qquad J = e^1\otimes e_2 - e^2\otimes e_1.$$

Derivations that commute with J lie in Span $\{I, J\}$. In particular, $\check{D}^s$ commutes with $\check{D}^a$, so Proposition 5.5 implies that up to isometry we can assume $\check{D} = 0$ or $\check{D} = \frac{h}{2}I$.

Up to $\mathcal{D}$-homothety, we can assume that either $h = 0$ or $h = 2$.

For $h = 0$, (5.2) gives

$$\tilde{\mathfrak{g}} = (0,0,0,-2e^{12}-2\tau e^{35},0);$$

for $h = 2$, either $\check{D} = 0$ and

$$\tilde{\mathfrak{g}} = (0,0,2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0),$$

or $\check{D} = I$ and

$$\tilde{\mathfrak{g}} = (e^{15},e^{25},2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0).$$

In either case, the metric is

$$\tilde{g} = e^1\otimes e^1 + e^2\otimes e^2 + \tau e^3\otimes e^3 + e^4\otimes e^4 + \tau e^5\otimes e^5.$$

Taking into consideration the negative-definite metric on $\check{\mathfrak{g}}$ has the effect of adding the $\pm$ signs, as per Remark 5.4. □

Notice that the third Lie algebra appearing in Theorem 5.7 is Example 2.4.

We proceed to give a list of the 7-dimensional Lie algebras with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ up to isometry and $\mathcal{D}$-homothety. This list is given in Table 1, where we write the diagonal metric $\tilde{g}$ as a line vector with respect to the basis $\{e^1, \dots, e^7\}$, using the convention that $[1]_n$ is a vector of n elements, each equal to 1. For example $[1]_4 = (1,1,1,1)$ and $(\pm[1]_4, \tau, +1, \tau)$ represents the metric

$$\tilde{g} = \pm(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4) + \tau e^5 \otimes e^5 + e^6 \otimes e^6 + \tau e^7 \otimes e^7.$$

Table 1. 7-dimensional Lie algebras with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ up to isometry and $\mathcal{D}$-homothety.

$n.$	$\tilde{\mathfrak{g}}$	Metric $\tilde{g}$
1.	$0,0,0,0,0,-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
2.	$0,0,0,0,2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
3.	$0,0,e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
4.	$e^{17},e^{27},e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
5.	$0,0,0,0,0,-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
6.	$0,0,0,0,2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
7.	$0,0,e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
8.	$e^{17},e^{27},e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
9.	$\frac{1}{2}e^{17}+2\lambda e^{27}-\frac{1}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{1}{2}e^{27}+\lambda e^{37}-\frac{1}{2}e^{47},$ $\frac{1}{2}e^{17}+\lambda e^{27}-\frac{1}{2}e^{37},-\lambda e^{17}+\frac{1}{2}e^{27}-\frac{1}{2}e^{47},$ $-\tau e^{12}+\tau e^{14}-\tau e^{23}-\tau e^{34},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
10.	$\frac{1}{2}e^{17}+2\lambda e^{27}-\frac{3}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{1}{2}e^{27}+\lambda e^{37}-\frac{3}{2}e^{47},$ $-\frac{1}{2}e^{17}+\lambda e^{27}-\frac{1}{2}e^{37},-\lambda e^{17}-\frac{1}{2}e^{27}-\frac{1}{2}e^{47},$ $-\tau e^{12}+\tau e^{14}-\tau e^{23}-\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
11.	$\frac{3}{2}e^{17}+2\lambda e^{27}+\frac{1}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{3}{2}e^{27}+\lambda e^{37}+\frac{1}{2}e^{47},$ $\frac{3}{2}e^{17}+\lambda e^{27}+\frac{1}{2}e^{37},-\lambda e^{17}+\frac{3}{2}e^{27}+\frac{1}{2}e^{47},$ $-3\tau e^{12}+3\tau e^{14}-\tau e^{23}+\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$

Theorem 5.8. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra of dimension 7 with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$. Then, up to isometry and $\mathcal{D}$-homothety, the metric Lie algebra $(\tilde{\mathfrak{g}}, \tilde{g})$ is one of the Lie algebras appearing in Table 1 and the Sasaki structure is given by*

$$\xi = (e^6)^\flat = e_6, \qquad \eta = e^6, \qquad 2\Phi = d\eta = de^6$$

with respect to the basis $\{e^1, \dots, e^7\}$ of Table 1.

Proof. We first consider the case where $\check{g}$ is positive definite, applying Corollary 5.6 and proceeding as in the proof of Theorem 5.7.

If $h=0$, we get

$$(0,0,0,0,0,-2e^{12}-2e^{34}-2\tau e^{57},0);$$

for $h=2$, we have the three possibilities $\check{D}=0$, $\check{D}=e^3\otimes e_3+e^4\otimes e_4$, $\check{D}=I$, corresponding to

$$\begin{gathered}(0,0,0,0,2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0),\\ (0,0,e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0),\\ (e^{17},e^{27},e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0).\end{gathered}$$

The negative definite case gives rise to the same Lie algebras, with the restriction of the metric to $\check{\mathfrak{g}}$ of opposite sign.

In the neutral case, we can assume

$$\begin{gathered}\check{g}=e^1\otimes e^1+e^2\otimes e^2-e^3\otimes e^3-e^4\otimes e^4,\\ \omega=-e^{12}+e^{34},\\ J=e^1\otimes e_2-e^2\otimes e_1+e^3\otimes e_4-e^4\otimes e_3.\end{gathered}$$

If $\check{D}^s$ is diagonalizable, Corollary 5.6 applies and computations as above yield

$$\begin{gathered}(0,0,0,0,0,-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (0,0,0,0,2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (0,0,e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (e^{17},e^{27},e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0).\end{gathered}$$

If $\check{D}^s$ is not diagonalizable, we can exploit the $U(1,1)$ symmetry preserving the pseudo-Kähler structure of $\check{\mathfrak{g}}$. Indeed, a symmetric derivation commuting with J is effectively an element of $i\mathfrak{u}(1,1)$, with $U(1,1)$ acting on it by the adjoint action. Write $\check{D}^s=tI+\check{D}^s_0$, where $\check{D}^s_0$ is traceless. Then $\check{D}^s_0$ can therefore be viewed as an element of $i\mathfrak{su}(1,1)$. Now $\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ via the Cayley isomorphism

$$\mathrm{SL}(2,\mathbb{R})\ni g\mapsto CgC^{-1}\in\mathrm{SU}(1,1), \tag{5.4}$$

where $C=\left(\begin{smallmatrix}1&-i\\1&i\end{smallmatrix}\right)$. The action of $\mathrm{SL}(2,\mathbb{R})$ on its Lie algebra is conjugation, so any nondiagonalizable element of $\mathfrak{sl}(2,\mathbb{R})$ is in the $\mathrm{SL}(2,\mathbb{R})$-orbit of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. Reading this in $\mathfrak{su}(1,1)$ via (5.4) and multiplying by $-i$, we see that $\check{D}^s_0$ corresponds to the complex matrix $\left(\begin{smallmatrix}1/2&-1/2\\1/2&-1/2\end{smallmatrix}\right)$; writing it as a real matrix, we obtain

$$\check{D}^s=\begin{pmatrix}(t+\frac12)I & -\frac12 I\\ \frac12 I & (t-\frac12)I\end{pmatrix}.$$

A derivation $\check{D}$ that satisfies $[D, J] = 0$ and is not diagonalizable takes the form

$$\check{D} = \begin{pmatrix} x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\ -\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\ \lambda_5 & \lambda_6 & x-1 & \lambda_8 \\ -\lambda_6 & \lambda_5 & -\lambda_8 & x-1 \end{pmatrix}.$$

Now, thanks to Proposition 2.2, we can consider any

$$\check{D}' = \begin{pmatrix} y & \mu_2 & \mu_5 - 1 & -\mu_6 \\ -\mu_2 & y & \mu_6 & \mu_5 - 1 \\ \mu_5 & \mu_6 & y-1 & \mu_8 \\ -\mu_6 & \mu_5 & -\mu_8 & y-1 \end{pmatrix}$$

such that $[\check{D}', \check{D}] = 0$ and $\check{D}'^s = \check{D}^s$. This yields $y = x$, $\mu_5 = \lambda_5$, $\mu_6 = \lambda_6$ and $\mu_2 - \mu_8 = \lambda_2 - \lambda_8$, hence we can consider $\check{D}$ to be

$$\check{D} = \begin{pmatrix} x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\ -\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\ \lambda_5 & \lambda_6 & x-1 & 0 \\ -\lambda_6 & \lambda_5 & 0 & x-1 \end{pmatrix}.$$

Again we distinguish two cases depending on h.

If $h = 0$, then equation $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ yields

$$\check{D} = \begin{pmatrix} \frac{1}{2} & 2\lambda & -\frac{1}{2} & -\lambda \\ -2\lambda & \frac{1}{2} & \lambda & -\frac{1}{2} \\ \frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\ -\lambda & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}.$$

Hence we set $d\xi^\flat = -2e^{12} + 2e^{34}$, $db^\flat = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}$, and the first Lie algebra extension is

$$\mathfrak{g} = (0, 0, 0, 0, -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34}),$$

with metric

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + \tau b^\flat \otimes b^\flat + \xi^\flat \otimes \xi^\flat. \tag{5.5}$$

The Sasaki extension $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ is determined by

$$d\xi^\flat = -2e^{12} + 2e^{34}, \qquad db^\flat = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34},$$
$$[e_0, x] = \check{D}x, \qquad [e_0, \xi] = 0, \qquad [e_0, b] = -2\tau\xi;$$

hence the Lie algebra is

$$\begin{aligned}\tilde{\mathfrak{g}} = (&\frac{1}{2}e^{17} + 2\lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{47}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} - \frac{1}{2}e^{47}, \\ &\frac{1}{2}e^{17} + \lambda e^{27} - \frac{1}{2}e^{37}, -\lambda e^{17} + \frac{1}{2}e^{27} - \frac{1}{2}e^{47}, \\ &-\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0).\end{aligned}$$

If $h = 2$, then equation $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ yields two distinct solutions for $\check{D}$:

$$\check{D}_1 = \begin{pmatrix} \frac{1}{2} & 2\lambda & -\frac{3}{2} & -\lambda \\ -2\lambda & \frac{1}{2} & \lambda & -\frac{3}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\ -\lambda & -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{or} \quad \check{D}_2 = \begin{pmatrix} \frac{3}{2} & 2\lambda & \frac{1}{2} & -\lambda \\ -2\lambda & \frac{3}{2} & \lambda & \frac{1}{2} \\ \frac{3}{2} & \lambda & \frac{1}{2} & 0 \\ -\lambda & \frac{3}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

For $\check{D}_1$ we get $db^\flat = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}$, hence

$$\mathfrak{g} = (0, 0, 0, 0, -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34});$$

for $\check{D}_2$ we get $db^\flat = -3\tau e^{12} + 3\tau e^{14} - \tau e^{23} + \tau e^{34}$ and

$$\mathfrak{g} = (0, 0, 0, 0, -3\tau e^{12} + 3\tau e^{14} - \tau e^{23} + \tau e^{34}, -2e^{12} + 2e^{34}).$$

In both cases, the metric is given by (5.5). The resulting Lie algebras $\tilde{\mathfrak{g}}$ correspond to n. 10 and n. 11 in Table 1. □

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