RACSAM (2021) 115:90
https://doi.org/10.1007/s13398-021-01029-z

ORIGINAL PAPER

Bounded resolutions for spaces $C_p(X)$ and a characterization in terms of X

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Received: 5 February 2021 / Accepted: 15 March 2021

Abstract

An internal characterization of the Arkhangel'skiĭ-Calbrix main theorem from [4] is obtained by showing that the space $C_p(X)$ of continuous real-valued functions on a Tychonoff space X is K-analytic framed in $\mathbb{R}^X$ if and only if X admits a *nice framing*. This applies to show that a metrizable (or cosmic) space X is σ -compact if and only if X has a nice framing. We analyse a few concepts which are useful while studying nice framings. For example, a class of Tychonoff spaces X containing strictly Lindelöf Čech-complete spaces is introduced for which a variant of Arkhangel'skiĭ-Calbrix theorem for σ-boundedness of X is shown.

Keywords Cosmic space · K-analytic-framed space · Nice framing

Mathematics Subject Classification 54C30 · 46A03

The first named author is supported by Grant PGC2018-094431-B-I00 of Ministry of Science, Innovation & Universities of Spain. The second named author is supported by the GAČR project 20-22230L and RVO: 67985840. The authors thank to Arkady Leiderman for discussions and remarks concerning Example 35 and Taras Banakh for his contribution to the proof of Theorem 6.

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Published online: 26 March 2021

1 Introduction

A classic result of Christensen asserts that for a metric and separable space X the space $C_p(X)$ is analytic (i.e., is a continuous image of the Polish space ω^ω) if and only if X is σ-compact (see for example [23, Theorem 9.6] and references therein).

Calbrix proved [11] that the analyticity of $C_p(X)$ yields the σ-compactness of X for any Tychonoff space X. The converse fails in general. Nevertheless, as Okunev proved [27], if X is σ-bounded, $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$, i.e., there exists a K-analytic space Z such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. This latter result motivated paper [4], where Arkhangel'skiĭ and Calbrix characterized cosmic σ-compact spaces by showing

Theorem 1 *A cosmic space X is σ-compact if and only if $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$.*

The proof of Arkhangel'skiĭ-Calbrix theorem depends on a result of Christensen about fundamental compact resolutions in metric spaces and Okunev's [28] about projectively σ-compact spaces X. They asked if the same holds when X is just a Lindelöf space. The answer to this question could already be found in Leiderman's [24], and also recalled in [10, Remark 3.9]. We shall discuss again this example in a slightly stronger form. In [16] we extended the above mentioned Okunev's theorem by showing the following useful

Theorem 2 *$C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if it has a bounded resolution.*

We will show that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if X admits a *nice framing* (Theorem 13) if and only if X has a *fundamental resolution of functions* (Theorem 16, Corollary 14). The latter concept will be directly used to construct an (usc) map from ω^ω into the compact sets of Z with $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$ (showing K-analyticity of Z). Examples illustrating these results are presented in Sects. 3, 4 and 5 where we discuss some consequences of Theorem 13 for obtaining σ-compactness of X, and provide an alternative approach (independent of Arkhangel'skiĭ and Calbrix) to this fact. For example, we analyze situations when X admits a nice framing with a layer $\{U_{\alpha,n} : n \in \omega\}$ giving a sequence of compact sets covering X, and we introduce a class of spaces containing Lindelöf Čech-complete spaces for which a variant of the Arkhangel'skiĭ-Calbrix theorem is obtained (Theorem 32). Recall that if $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$, the space X is *projectively σ-compact*, i.e. every continuous separable and metrizable image of X is σ-compact, [4, Theorem 2.3]. This motivated us to examine some versions of projectively σ-compactness and show (Theorem 6) that if X is *projectively analytic* (i.e., if every continuous metrizable and separable image of X is analytic) then every continuous metrizable image of X is separable. This yields that a metrizable (or cosmic) space X is σ-compact if and only if X has a nice framing. However, this fact fails if X is only separable and strictly dominated by a metric space.

2 Bounded resolutions for spaces $C_p(X)$

A covering $\mathcal{F} = \{A_\alpha : \alpha \in \omega^\omega\}$ of a set X is called a *resolution* for X if $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ coordinatewise. If X is a topological space and the sets A_α are compact, the family $\mathcal{F}$ is called a *compact resolution*. Recall that a subset B of a locally convex space (lcs) E is said to be *bounded* if for each neighborhood of the origin U in E there exists $\lambda > 0$ such that $\lambda B \subseteq U$. A resolution $\{A_\alpha : \alpha \in \omega^\omega\}$ for E consisting of bounded sets is called a *bounded resolution*. If additionally every bounded set in E is contained in some A_α, we say that the family $\{A_\alpha : \alpha \in \omega^\omega\}$ is a *fundamental bounded resolution* for E. For a Tychonoff space X, by $C_p(X)$ and $C_k(X)$ we denote the space of continuous real-valued functions on X with the pointwise τ_p and the compact-open topology τ_k, respectively.

Proposition 3 *If $C_p(X)$ is a continuous linear image of a metrizable lcs F, then $C_p(X)$ admits a bounded resolution.*

Proof Let $T : F \to C_p(X)$ be a continuous *linear* surjection; let $\{U_n\}_{n=1}^\infty$ be a decreasing base of neighborhoods of zero in F. If for $\alpha = (\alpha(n)) \in \omega^\omega$ we set $A_\alpha = \bigcap_{n=1}^\infty \alpha(n) U_{\alpha(n)}$, the family $\{A_\alpha : \alpha \in \omega^\omega\}$ is a fundamental bounded resolution for F. Hence, $\{T(A_\alpha) : \alpha \in \omega^\omega\}$ is a bounded resolution for $C_p(X)$. □

We refer also the reader to [19, Proposition 22] for a sufficient condition for $C_p(X)$ to have a fundamental bounded resolution.

A topological space X is *pseudocompact* if $f(X) \subseteq \mathbb{R}$ is bounded, $f \in C(X)$; X is pseudocompact if and only if $C_p(X)$ does not contain a complemented copy of $\mathbb{R}^\omega$ (see [1, Section 4]); X is called *σ-bounded* if $X = \bigcup_n X_n$ and every X_n is *functionally bounded*, i.e., every $f \in C(X)$ is bounded on X_n. A special case of Theorem 2 is Proposition 4 (due to Uspenskiĭ, see [27, Theorem 3.1]). We provide a short alternative proof.

Proposition 4 *A Tychonoff space X is pseudocompact if and only if there exists a σ-compact space K with $C_p(X) \subseteq K \subseteq \mathbb{R}^X$.*

Proof If X is pseudocompact and $S = \{f \in C(X) : |f(x)| \leq 1, x \in X\}$, the sequence $\{nS\}_{n=1}^\infty$ covers $C_p(X)$. Hence, the closure of nS in $\mathbb{R}^X$ provides a sequence of compact sets in $\mathbb{R}^X$ whose union K contains $C(X)$. Conversely, if the conclusion holds, $C_p(X)$ is covered by a sequence of bounded sets. If X is not pseudocompact, $C_p(X)$ contains a complemented copy of $\mathbb{R}^\omega$, which is not covered by a sequence of functionally bounded sets. □

One can ask *whether a Tychonoff space X is σ-bounded if and only if $C_p(X)$ does not contain a copy of $\mathbb{R}^Y$ for some uncountable Y. The answer is negative*: $C_p(\mathbb{R}^\omega)$ does not contain a copy of $\mathbb{R}^Y$ for any set Y with $|Y| > \aleph_0$. Indeed, since the weak* dual of $C_p(\mathbb{R}^\omega)$ is separable, $C_p(\mathbb{R}^\omega)$ admits a weaker metrizable locally convex topology, but $\mathbb{R}^Y$ fails this property, whereas it is well known that $C_p(\omega) = \mathbb{R}^\omega$ is not σ-bounded. Nevertheless, the 'only if' part is true in general. In fact, if $\{B_n\}_{n=1}^\infty$ is a sequence of functionally bounded sets covering X, the sets $A_\alpha = \{f \in C(X) : \sup_{x \in B_n} |f(x)| \leq \alpha(n)\ \forall n \in \omega\}$ for $\alpha \in \omega^\omega$ compose a bounded resolution for $C_p(X)$. If $C_p(X)$ contains a copy of $\mathbb{R}^Y$ then this latter Baire space also admits a bounded resolution. So, according to [23, Proposition 7.1], Y must be countable.

Recall that X is a *μ-space* if every functionally bounded set in X is relatively compact.

Theorem 5 *([22]) A Tychonoff space X is σ-compact if and only if X is a μ-space and there exists a metrizable locally convex topology ξ on $C(X)$ such that $\tau_p \leq \xi \leq \tau_k$.*

Recall that X is *projectively analytic* if each continuous metrizable and separable image of X is analytic. The space X is said to have the *Discrete Countable Chain Condition* (DCCC) if every discrete family of open sets is countable, which is equivalent to require that each continuous metrizable image of X is separable.

Theorem 6 *If an infinite Tychonoff space X is projectively analytic, then it has the DCCC.*

Proof Assume there exists a continuous surjective map $h : X \to Z$ and Z is metrizable but not separable. Choose a closed discrete set D in Z with $|D| = \aleph_1$. Such set exists since $d(Z) = w(Z) = e(Z)$, where $e(Z)$ means the extent of Z, see [13]. Then there exists a continuous one-to-one map $f : D \to Y$ onto a metrizable and separable space Y, which is not analytic. Indeed, such Y can be obtained as follows. Under (CH) we know that $\mathbb{R}$ contains $2^{\mathfrak{c}}$ subsets of the cardinality continuum, but only a continuum number of analytic subsets. So, one of those $2^{\mathfrak{c}}$ subsets Y is not analytic. Under $(\neg CH)$, take a subset $Y \subseteq \mathbb{R}$ of cardinality $\aleph_1$. Then it is not analytic. Indeed, every uncountable analytic subset of $\mathbb{R}$ contains a copy of the Cantor set and hence has cardinality $\mathfrak{c}$.

The map f admits a (canonical) extension $Pf : PD \to PY$ to spaces of finitely supported maps, where PY is the space of finitely supported probability measures endowed with the weak* topology determined by the subspace $C^b(X)$ of $C(Y)$ consisting of bounded functions. It turns out that PY is a separable and metrizable convex set by Prokhorov-Wasserman-Kantorovich metric, see [7, Lemma 4.3]. As follows from the proof of [2, 0.5.9 Proposition], the $y \mapsto \delta_y$ copy of Y in $L(Y)$ (the dual of $C_p(Y)$) is closed in $L(Y)$ when the latter linear space is provided with the weak topology of the dual pair $\langle L(Y), C^b(Y)\rangle$. Hence Y is closed in PY. Since PY is a convex metrizable subset of a lcs, $f : D \to Y \subseteq PY$ admits a continuous extension $\bar{f} : Z \to PY$ by Dugundji theorem [14, page 185]. $\bar{f}(Z)$ in PY is not analytic since it contains a closed subset Y which is not analytic. Then $\bar{f} \circ h$ has a non analytic (separable) metrizable image, a contradiction. □

A Tychonoff space X is called *strongly projectively σ-compact* if every continuous metrizable image of X is σ-compact.

Corollary 7 *Let X be an infinite Tychonoff space. Then X is projectively σ-compact if and only if X is strongly projectively σ-compact.*

Theorem 6 and Okunev's [27, Theorem 1.3] yield the following

Corollary 8 *A metrizable space X is analytic if and only if every continuous metrizable and separable image of X is analytic.*

Corollary 9 *A paracompact Čech-complete space X is σ-compact if and only if $C_p(X)$ has a bounded resolution.*

Indeed, if $C_p(X)$ has a bounded resolution, X is strongly projectively σ-compact by [4, Theorem 2.3] and Corollary 7. Since X is mapped onto a completely metrizable space Y by a perfect map T, see [13, 5.5.9(a)], the space Y is σ-compact. Hence X is σ-compact (since T is perfect). The converse implication is clear.

From [4, p. 5200] the one-point Lindeöfication of an uncountable discrete space X is projectively σ-compact but is not σ-bounded. Even more can be shown.

Example 10 $C_p(X)$ is not K-analytic framed in $\mathbb{R}^X$ for the one-point Lindeöfication X of an uncountable discrete space.

Indeed, X is an ω-space, i. e. every continuous metrizable separable image of X is countable, see [3] or [2]. So, X is projectively σ-compact. Since X is a P-space, $C_p(X)$ is a Baire lcs. So, if $C_p(X)$ admits a bounded resolution, [23, Proposition 7.1] ensures that the space $C_p(X)$ is metrizable. This X must be countable, a contradiction.

3 A characterization in terms of *X*

This section deals with the following

Problem 11 *Characterize Tychonoff spaces X such that $C_p(X)$ has a bounded resolution.*

According to [15] a family $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ of closed subsets of X is called a *framing* if (i) for each $\alpha \in \omega^\omega$ the *layer* $\{U_{\alpha,n} : n \in \omega\}$ is an increasing covering of X, and (ii) for every $n \in \omega$ one has that $U_{\beta,n} \subseteq U_{\alpha,n}$, $\alpha \le \beta$.

Lemma 12 *([15, Lemma 104]) A set $A \subseteq C_p(X)$ is bounded if and only if there is an increasing covering $\{V_n : n \in \omega\}$ of X by closed sets such that $\sup_{f\in A} |f(x)| \le n$, $x \in V_n$.*

Theorem 13 *The space $C_p(X)$ has a bounded resolution if and only if there exists a framing $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ in X enjoying the property that if $f \in C(X)$ there exists $\gamma \in \omega^\omega$ such that $|f(x)| \le n$ for each $x \in U_{\gamma,n}$ and $n \in \omega$.*

Proof If there is a framing $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ of the aforementioned characteristics, the sets

$$A_\alpha := \left\{ f \in C(X) : \sup_{x\in U_{\alpha,n}} |f(x)| \le n \ \forall n \in \omega \right\}$$

compose a bounded resolution for $C(X)$. Indeed, each set A_α is pointwise bounded by virtue of Lemma 12, since $\{U_{\alpha,n} : n \in \omega\}$ is an increasing covering of X by closed sets such that $\sup_{f\in A_\alpha} |f(x)| \le n$ for all $x \in U_{\alpha,n}$. Moreover, $A_\alpha \subseteq A_\beta$ if $\alpha \le \beta$ since $U_{\beta,n} \subseteq U_{\alpha,n}$. If $f \in C(X)$, by the statement of the theorem there exists $\gamma \in \omega^\omega$ such that $|f(x)| \le n$ for each $x \in U_{\gamma,n}$ and all $n \in \omega$. Hence $f \in A_\gamma$, so $\{A_\alpha : \alpha \in \omega^\omega\}$ covers $C(X)$. Conversely, assume $C_p(X)$ has a bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$. If $V_{\alpha,n} = \{x \in X : \sup_{f\in B_\alpha} |f(x)| \le n\}$, then $\{V_{\alpha,n} : n \in \omega\}$ is an increasing covering of X by closed sets for each $\alpha \in \omega^\omega$ with $V_{\beta,n} \subseteq V_{\alpha,n}$ whenever $\alpha \le \beta$, $n \in \omega$. If $f \in C(X)$ there is $\delta \in \omega^\omega$ such that $f \in B_\delta$. Hence $|f(x)| \le n$ for each $x \in V_{\delta,n}$ $n \in \omega$, so $\{V_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is a framing satisfying the required property. □

We say that X has a *nice framing* if X admits a framing as stated in Theorem 13.

The following concept can also be used when studying the role of framings, see also [10], where a similar concept was fixed for uniform spaces X with uniformly continuous functions f_α. A Tychonoff space X admits a *fundamental resolution of functions* if there exists on X a family of nonnegative real-valued functions $\{f_\alpha : \alpha \in \omega^\omega\}$ such that $f_\alpha \le f_\beta$ for $\alpha \le \beta$ and for each $f \in C(X)$ there exists $\alpha \in \omega^\omega$ with $|f| \le f_\alpha$.

Corollary 14 *A Tychonoff space X has a fundamental resolution of functions if and only if $C_p(X)$ has a bounded resolution, if and only if X has a nice framing.*

Indeed, if $\{A_\alpha : \alpha \in \omega^\omega\}$ is a bounded resolution on $C_p(X)$, then $f_\alpha(x) = \sup\{|f(x)| : f \in A_\alpha\}$ form a fundamental resolution of functions, and if (f_α) is a fundamental resolution of functions, sets $A_\alpha = \{f \in C_p(X) : |f| \le f_\alpha\}$ form a bounded resolution on $C_p(X)$. The last statement follows from Theorem 13.

To keep the paper self-contained we apply this concept to present a short proof of Theorem 1 if X is metrizable (or cosmic), although the main idea remains similar (see also [6, Proof of Theorem 2.2] for a similar argument). Nevertheless, theorem fails if X is only separable with a stronger metric topology, Example 37.

Theorem 15 *A metrizable space X is σ-compact if and only if X admits a nice framing. The same statement holds if X is cosmic.*

Proof Assume first that X is metrizable and separable, with a nice framing. Let $\{f_\alpha : \alpha \in \omega^\omega\}$ be a fundamental resolution of functions on X (we apply Corollary 14). Let $\overline{X}$ be a metric compactification (see [20] for details). For $\alpha \in \omega^\omega$ set $K_\alpha = \bigcap_{y \in X}(\overline{X} \setminus B(y, \exp(-f_\alpha(y)))$, where $B(y, r)$ is the open ball at y and radius r. Clearly each K_α is a compact subset of $\overline{X} \setminus X$ and $K_\alpha \subseteq K_\beta$, if $\alpha, \beta \in \omega^\omega$ with $\alpha \leq \beta$. Let $K \subseteq \overline{X} \setminus X$ be compact. For $h(y) = |\ln d(K, y)|$, $y \in X$, there exists $\sigma \in \omega^\omega$ with $h \leq f_\sigma$. Hence $d(K, y) \geq \exp(-f_\sigma(y))$, and then $K \subseteq \overline{X} \setminus B(y, \exp(-f_\sigma(y)))$ for every $y \in X$; so $K \subseteq K_\sigma$. Thus $\{K_\alpha : \alpha \in \omega^\omega\}$ is a fundamental compact resolution for the metrizable and separable space $\overline{X} \setminus X$. By Christensen's [23, Theorem 6.1] $\overline{X} \setminus X$ is Polish, so X is σ-compact.

Next, assume that X is metrizable and contains a nice framing. By Corollary 14 the space $C_p(X)$ has a bounded resolution. Assume that X is continuously mapped on a metrizable and separable space Y. Since $C_p(Y)$ is isomorphic to a subspace of $C_p(X)$, the space $C_p(Y)$ has a bounded resolution; consequently the metrizable and separable space Y admits a nice framing. By the first case we derive that Y is σ-compact. Now, Corollary 7 applies to get that X is σ-compact. The converse follows from the fact, mentioned earlier, that if $\{B_n\}_{n=1}^\infty$ is a sequence of functionally bounded sets covering X, the sets $A_\alpha = \{f \in C(X) : \sup_{x \in B_n} |f(x)| \leq \alpha(n)\ \forall n \in \omega\}$ for $\alpha \in \omega^\omega$ compose a bounded resolution for $C_p(X)$.

Finally, assume that X is cosmic with a nice framing, and let Y be a continuous metrizable and separable image of X. By the previous argument Y is σ-compact. So, according to [27, Theorem 1.5], the space X is σ-compact. The converse is clear. □

A regular space X is *angelic* if every relatively countably compact subset A of X is relatively compact and for every $x \in \overline{A}$ there exists a sequence in A which converges to x. The concept of a *fundamental resolution of functions* will be directly used to define an (usc) map F from ω^ω into compact subsets of some space Z where $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$.

Theorem 16 *If X has a nice framing, $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ and angelic.*

We provide two proofs of Theorem 16. For the first one we need the following two simple technical lemmas (which might be already known).

Lemma 17 *Each increasing function $\varphi : \omega^\omega \to [0, \infty)$ is bounded on some non-empty open subset of ω^ω.*

Proof Suppose, by contrary, that φ is unbounded on every non-empty open subset of ω^ω. Let $\beta^1 = (\beta^1_n) \in \omega^\omega$ with $\varphi(\beta^1) \geq 1$. Let $\gamma^1 \in \{\beta^1_1\} \times \omega^\omega$ with $\varphi(\gamma^1) \geq 2$. Put $\beta^2 = (\beta^2_n) = \max\{\beta^1, \gamma^1\}$; then $\varphi(\beta^2) \geq 2$, $\beta^2 \geq \beta^1$ and $\beta^2_1 = \beta^1_1$. Let $\gamma^2 \in \{(\beta^1_1, \beta^2_2)\} \times \omega^\omega$ with $\varphi(\gamma^2) \geq 3$. Put $\beta^3 = (\beta^3_n) = \max\{\beta^2, \gamma^2\}$; then $\varphi(\beta^3) \geq 3$, $\beta^3 \geq \beta^2$ and $\beta^3_1 = \beta^1_1$, $\beta^3_2 = \beta^2_2$. Following this procedure we get an element $\beta = (\beta^n_n) \in \omega^\omega$ and an increasing sequence $(\beta^k) \subseteq \omega^\omega$ such that $\varphi(\beta^k) \geq k$ and $\beta^k_i = \beta^i_i$ for all $1 \leq i \leq k$, $k \in \omega$. Then $\beta^k \leq \beta$ and $k \leq \varphi(\beta^k) \leq \varphi(\beta) < +\infty$ for any $k \in \omega$, a contradiction. □

Lemma 18 *(1) Each increasing function $\varphi : \omega^\omega \to [0, \infty)$ is locally bounded, i.e. each point $x \in \omega^\omega$ has an open neighborhood U such that $\varphi(U)$ is bounded. (2) For every locally bounded function $\varphi : \omega^\omega \to [0, \infty)$ there exists a locally constant function $g : \omega^\omega \to [0, \infty)$ with $g \geq \varphi$; in particular, g is continuous.*

Proof (1) Assume the claim fails. Then there exists $\alpha \in \omega^\omega$ such that φ is unbounded on $\{(\alpha_1, \dots, \alpha_m)\} \times \omega^\omega$ for every $m \in \omega$. Hence for every $\beta \geq \alpha$ the function φ is unbounded on $\{(\beta_1, \dots, \beta_m)\} \times \omega^\omega$ for every $m \in \omega$.

Set $\psi : \omega^\omega \to [0, +\infty)$, $\psi((\beta_n)) = \varphi((\beta_n + \alpha_n))$. Then ψ is increasing and unbounded on any non-empty open subset of ω^ω. Indeed, let $\beta = (\beta_n) \in \omega^\omega$ and $m \in \omega$. Let $\gamma_i = \beta_i + \alpha_i$ for $1 \leq i \leq m$ and $A = \{(\lambda_i) \in \omega^\omega : \lambda_i > \alpha_{i+m}, i \in \omega\}$. Then $\psi(\{(\beta_1, \dots, \beta_m)\} \times N^N) = \varphi(\{(\gamma_1, \dots, \gamma_m)\} \times A)$ and for any $(\lambda'_i) \in \omega^\omega$ we have $\varphi((\gamma_1, \dots, \gamma_m, \lambda'_1, \lambda'_2, \dots)) \leq \varphi((\gamma_1, \dots, \gamma_m, \lambda'_1 + \alpha_{m+1}, \lambda'_2 + \alpha_{m+1}, \dots))$ and $(\gamma_1, \dots, \gamma_m, \lambda'_1 + \alpha_{m+1}, \lambda'_2 + \alpha_{m+1}, \dots) \in \{(\gamma_1, \dots, \gamma_m)\} \times A$. Thus $\psi(\{(\beta_1, \dots, \beta_m)\} \times \omega^\omega) = \varphi(\{(\gamma_1, \dots, \gamma_m)\} \times A)$ is unbounded, since $\varphi(\{(\gamma_1, \dots, \gamma_m)\} \times \omega^\omega)$ is unbounded. It follows that ψ is unbounded on any non-empty open subset of ω^ω, a contradiction with Lemma 17.

(2) For $\alpha \in \omega^\omega$ let $m(\alpha)$ be the least integer such that φ is bounded on $\{(\alpha_1, \dots, \alpha_{m(\alpha)})\} \times \omega^\omega$. Put $V_\alpha = \{(\alpha_1, \dots, \alpha_{m(\alpha)})\} \times \omega^\omega$ for any $\alpha \in \omega^\omega$. Clearly, $\bigcup\{V_\alpha : \alpha \in \omega^\omega\} = \omega^\omega$. For all $\alpha, \beta \in \omega^\omega$ we have $V_\alpha = V_\beta$ or $V_\alpha \cap V_\beta = \emptyset$. Indeed, if $m(\alpha) = m(\beta)$ and $\alpha_i = \beta_i$ for $1 \leq i \leq m(\alpha)$, then $V_\alpha = V_\beta$; if $m(\alpha) = m(\beta)$ and $\alpha_i \neq \beta_i$ for some $1 \leq i \leq m(\alpha)$, then $V_\alpha \cap V_\beta = \emptyset$; if $m(\alpha) \neq m(\beta)$, then $\alpha_i \neq \beta_i$ for some $1 \leq i \leq \min\{m(\alpha), m(\beta)\}$ and $V_\alpha \cap V_\beta = \emptyset$. Thus for some $W \subseteq \omega^\omega$ the family $\{V_\alpha : \alpha \in W\}$ is a partition of ω^ω on non-empty clopen subsets such that φ is bounded on V_α for every $\alpha \in W$. Let $t_\alpha = \sup \varphi(V_\alpha)$ for $\alpha \in W$. Let $g : \omega^\omega \to [0, +\infty)$ be the function such that $g(\beta) = t_\alpha$ for any $\beta \in V_\alpha, \alpha \in W$. Then $g \geq \varphi$ and g is locally constant, so it is continuous. □

First proof of Theorem 16 By Corollary 14 fix a fundamental resolution of functions $\{f_\alpha : \alpha \in \omega^\omega\}$ for X. Let $x \in X$. Then $\varphi_x : \omega^\omega \to [0, +\infty), \alpha \to f_\alpha(x)$ is increasing. By Lemma 18 there exists a locally constant function $g_x : \omega^\omega \to [0, +\infty)$ with $g_x \geq \varphi_x$. Let $g : \omega^\omega \times X \to [0, +\infty)$, $g(\alpha, x) = g_x(\alpha)$. Clearly, for any $x \in X$ the function $\omega^\omega \to [0, +\infty), \alpha \to g(\alpha, x)$ is locally constant. Moreover for any function $f \in C_p(X)$ there is an $\alpha \in \omega^\omega$ with $|f(x)| \leq f_\alpha(x) = \varphi_x(\alpha) \leq g_x(\alpha) = g(\alpha, x)$ for every $x \in X$. For any $\alpha \in \omega^\omega$ the set $F_\alpha = \prod_{x \in X}[-g(\alpha, x), g(\alpha, x)]$ in $\mathbb{R}^X$ is compact. Put $Z = \bigcup\{F_\alpha : \alpha \in \omega^\omega\}$. Then $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. Using the continuity of g with respect to the first variable it is easy to see that $F : \alpha \mapsto F_\alpha$ is an upper semi-continuous (usc) set-valued map from ω^ω with compact values in Z. Thus $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. □

We propose another proof of Theorem 16, which uses an idea included in the proof of [2, Proposition IV 9.3]. First we prove the following

Lemma 19 *If X has a nice framing, there exists a countable nice framing $\{W_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ for X.*

Proof Let $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ be a nice framing for X. For each $(\alpha, n) \in \omega^\omega \times \omega$, define the closed set

$$W_{\alpha,n} = \bigcap \left\{U_{\beta,n} : \beta \in \omega^\omega, \beta(i) = \alpha(i), 1 \leq i \leq n\right\}.$$

Observe that $W_{\alpha,n} \subseteq W_{\alpha,n+1}$ for each $\alpha \in \omega^\omega$ and $W_{\beta,n} \subseteq W_{\alpha,n}$ for each $n \in \omega$ whenever $\alpha \leq \beta$. We claim that $\bigcup_{n\in\omega} W_{\alpha,n} = X$ for each $\alpha \in \omega^\omega$. Indeed, suppose otherwise that there exists $x \notin \bigcup_{n\in\omega} W_{\alpha,n}$ for some $\alpha \in \omega^\omega$. For every $n \in \omega$ choose $\beta_n \in \omega^\omega$ with $\beta_n(i) = \alpha(i)$ for $1 \leq i \leq n$ such that $x \notin U_{\beta_n,n}$. Put $\gamma := \sup\{\beta_n : n \in \omega\}$. Then, for every $n \in \omega$, $\beta_n \leq \gamma$ and hence $x \notin U_{\gamma,n}$ since $U_{\gamma,n} \subseteq U_{\beta_n,n}$ by the definition of framing. Hence $x \notin \bigcup_{n\in\omega} U_{\gamma,n} = X$, a contradiction. All this means that $\{W_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is a framing for X. Note that the family $\{W_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is countable since $W_{\alpha,n}$ depends only on $\alpha(1), \dots, \alpha(n)$. Finally, if $f \in C(X)$, by Theorem 13 there is $\gamma \in \omega^\omega$ such that $|f(x)| \leq n$ for every $x \in U_{\gamma,n}$ and all $n \in \omega$. So $|f(x)| \leq n$, $x \in W_{\gamma,n}$, $n \in \omega$ □

Second proof of Theorem 16 By Lemma 19 let $\mathcal{F} = \{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ be a countable nice framing for X. First we prove that $C_p(X)$ is Lindelöf Σ-framed in $\mathbb{R}^X$. Let us say that a function $f \in \mathbb{R}^X$ is $\mathcal{F}$*-bounded* if for each $x \in X$ there exists $(\alpha, n) \in \omega^\omega \times \omega$ such that $x \in U_{\alpha,n}$ and $f(U_{\alpha,n}) \subseteq [-n, n]$. Let us denote by Z the subset of $\mathbb{R}^X$ consisting of all $\mathcal{F}$-bounded functions on X. We claim that $C(X) \subseteq Z$. Indeed, if $f \in C(X)$ there exists $\delta \in \omega^\omega$ such that $f(U_{\delta,n}) \subseteq [-n, n]$ for every $n \in \omega$. Since $\{U_{\delta,n} : n \in \omega^\omega\}$ covers X, given $x \in X$ there exists $m \in \mathbb{N}^\mathbb{N}$ with $x \in U_{\delta,m}$ and $f(U_{\delta,m}) \subseteq [-m, m]$, which shows that $f \in Z$. Thus $C(X) \subseteq Z$, as stated. Now we prove that Z is a Lindelöf Σ-space. If $\overline{\mathbb{R}}$ designates the usual two points compactification of $\mathbb{R}$, then $\overline{\mathbb{R}}^X$ is a compactification of Z. For $(\alpha, n) \in \omega^\omega \times \omega$ define

$$L_{\alpha,n} = \{f \in \overline{\mathbb{R}}^X : f(U_{\alpha,n}) \subseteq [-n, n]\}.$$

The sets $L_{\alpha,n}$ are compact since they are closed in $\overline{\mathbb{R}}^X$, and compose a countably family because the framing $\mathcal{F}$ is countable. Choose $f \in Z$ and $g \in \overline{\mathbb{R}}^X \setminus Z$. As $g \in \overline{\mathbb{R}}^X \setminus Z$, there exists $y \in X$ such that $g(U_{\alpha,n}) \nsubseteq [-n, n]$ for each $(\alpha, n) \in \omega^\omega \times \omega$ for which $y \in U_{\alpha,n}$. Due to $f \in Z$ there is $(\gamma, m) \in \omega^\omega \times \omega$ with $y \in U_{\gamma,m}$ and $f(U_{\gamma,m}) \subseteq [-m, m]$, so $f \in L_{\gamma,m}$. On the other hand $g \notin L_{\gamma,m}$ since $g(U_{\gamma,m}) \nsubseteq [-m, m]$ because $y \in U_{\gamma,m}$. Since $\overline{\mathbb{R}}^X$ is a compactification of Z, [2, 4.9.2 Proposition] applies to get that Z is a Lindelöf Σ-space.

Next we show that $C_p(X)$ if K-analytic-framed in $\mathbb{R}^X$. Indeed, for each $\alpha \in \omega^\omega$ we set $A_\alpha := \{f \in C(X) : \sup_{x \in U_{\alpha,n}} |f(x)| \leq n\ \forall n \in \omega\}$ and put $B_\alpha = \overline{A_\alpha}$, where the closure is in $\mathbb{R}^X$. Note that B_α is a compact set in $\mathbb{R}^X$. We claim that $B_\alpha \subseteq Z$. Indeed, if $f \in B_\alpha$ there is a net $\{f_d : d \in D\}$ in A_α such that $f_d(x) \to f(x)$ for every $x \in X$. So, given $n \in \omega$, one has in particular $f_d(x) \to f(x)$ for every $x \in U_{\alpha,n}$, which implies that $\sup_{x \in U_{\alpha,n}} |f(x)| \leq n$. Hence $f(U_{\alpha,n}) \subseteq [-n, n]$, so that $f \in Z$, and $B_\alpha \subseteq Z$.

Define $Y = \bigcup\{B_\alpha : \alpha \in \omega^\omega\}$ and note that, as a consequence of the previous claim, $Z \subseteq Y$. Since Y quasi-Suslin [23, Proposition 3.11], there is a set-valued map $T : \omega^\omega \to 2^Y$ with $\bigcup\{T(\alpha) : \alpha \in \omega^\omega\} = Y$ and if $\alpha_n \to \alpha$ in ω^ω and $x_n \in T(\alpha_n)$ for all $n \in \omega$ the sequence $\{x_n\}_{n=1}^\infty$ has a cluster point $x \in T(\alpha)$. By a result of Cascales, we may assume $T(\alpha) \subseteq T(\beta)$ whenever $\alpha \leq \beta$ (see [23, Theorem 3.1]). Define $S : \omega^\omega \to 2^Y$ by $S(\alpha) = \overline{T(\alpha)}$, closure in Z, and put $\Omega = \bigcup\{S(\alpha) : \alpha \in \omega^\omega\}$. Then, the fact that Y is quasi-Suslin implies that $T(\alpha)$ is countably compact, hence functionally bounded in Y, so $S(\alpha)$ is functionally bounded in Z. Since Z is Lindelöf, $S(\alpha)$ compact. So, the map S is compactly-valued. If $\alpha_n \to \alpha$ in ω^ω and $z_n \in S(\alpha_n)$, we may proceed as in the proof of [15, Theorem 57] to show that $\{z_n\}_{n=1}^\infty$ has a cluster point $z \in S(\alpha)$. This proves that Ω is K-analytic. Since $C_p(X) \subseteq \Omega \subseteq \mathbb{R}^X$, the space $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. *Proof that $C_p(X)$ is angelic:* By Okunev's [27, Theorem 3.5] the space υX is a Lindelöf Σ-space, and then by Orihuela's angelic theorem [23, Theorem 4.5] the space $C_p(\upsilon X)$ is angelic, and the same holds also for $C_p(X)$, see [23, Lemma 9.2]. □

Corollary 20 *([16, Theorem 1]) $C_p(X)$ has a bounded resolution if and only if $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$.*

Corollary 21 *If $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, X has a nice framing if and only if Y has a nice framing.*

Corollary 22 *([5]) Let X be σ-bounded and Y metric, and assume that there exists a continuous linear surjection from $C_p(X)$ onto $C_p(Y)$. Then Y is σ-compact.*

Remark 23 A countable infinite product X of metrizable non-compact spaces X_n each with a nice framing does not have a nice framing, since each X_n is σ-compact but X is not σ-compact (as X contains a closed copy of ω^ω).

4 Strong framings, σ-compactess

One may expect that *each* nice framing for a separable and metrizable X should contain a layer consisting of compact sets, so providing a σ-compact cover of X. We prove however the following

Theorem 24 *Let $\mathfrak{M}$ be the class of metrizable and separable spaces with a nice framing.*

(1) *If $X \in \mathfrak{M}$, then X admits a nice framing such that for each $\alpha \in \omega^\omega$ the layer $\{U_{\alpha,n} : n \in \omega\}$ consists of compact sets.*
(2) *If $X \in \mathfrak{M}$ is non-Polish, then X admits also a nice framing such that for each $\alpha \in \omega^\omega$ there exists $n \in \omega$ such that $U_{\alpha,n}$ is not compact.*
(3) *There exists a countable Polish space $\Gamma \in \mathfrak{M}$ with the conclusion like in item (2).*

First we show some auxiliary results. The first one, when dealing with $X \in \mathfrak{M}$, asserts that X admits a nice framing each layer $\{U_{\alpha,n} : n \in \omega\}$ consists of compact sets. We need also the following concept. For $\alpha, \beta \in \omega^\omega$ *we write $\alpha \preceq \beta$, if there exists $m \in \omega$ such that $\alpha_n \leq \beta_n$ for every $n \geq m$*. A nice framing is said to be *a strong framing*, if for all $\alpha, \beta \in \omega^\omega$ with $\alpha \preceq \beta$ there exists $p \in \omega$ such that $U_{\beta,n} \subseteq U_{\alpha,n}$ for every $n \geq p$.

Proposition 25 *For a topological space X the following statements are equivalent:*

(1) *X is σ-bounded.*
(2) *X admits a strong framing such that for each $\alpha \in \omega^\omega$ the layer $\{U_{\alpha,n} : n \in \omega\}$ consists of functionally bounded sets.*
(3) *X admits a strong framing such that there exists $\alpha \in \omega^\omega$ for which $\{U_{\alpha,n} : n \in \omega\}$ consists of functionally bounded sets.*
(4) *X admits a nice framing such that for each $\alpha \in \omega^\omega$ the layer $\{U_{\alpha,n} : n \in \omega\}$ consists of functionally bounded sets.*

Proof Only (1) $\Rightarrow$ (2) needs to be shown. Let X be a σ-bounded space with an increasing cover $(X_n)_{n\geq 0}$ of functionally bounded (closed) sets, $X_0 = \emptyset$. Then X has a strong framing $\left\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\right\}$ such that $\{U_{\alpha,n} : n \in \omega\} = \{X_n : n \geq 0\}$ for every $\alpha \in \omega^\omega$.

Indeed, let $\alpha = (\alpha_n) \in \omega^\omega$. Let $\alpha_0 = 0$ and $\hat{\alpha}_n = n + \max\{k\alpha_k : 0 \leq k \leq n\}$ for $n \geq 0$. Clearly $\hat{\alpha}_0 = 0$ and $\hat{\alpha}_n < \hat{\alpha}_{n+1}$ for every $n \geq 0$. Let $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \ldots)$. Clearly, $\hat{\alpha} \in \omega^\omega$.

Let $\alpha, \beta \in \omega^\omega$. If $\alpha \leq \beta$, then $\hat{\alpha} \leq \hat{\beta}$. Moreover, if $\alpha \preceq \beta$, then $\hat{\alpha} \preceq \hat{\beta}$. In fact, there exists $m \in \omega$ such that $\alpha_n \leq \beta_n$ for every $n \geq m$. Put $A = \max\{k\alpha_k : 0 \leq k \leq m\}$, $B = \max\{k\beta_k : 0 \leq k \leq m\}$ and $C = \max\{A, B\}$. Let $n \in \omega$ with $n \geq C$. Then $\hat{\alpha}_n = n + \max(\{A\} \cup \{k\alpha_k : m < k \leq n\}) = n + \max\{k\alpha_k : m < k \leq n\} \leq n + \max\{k\beta_k : m < k \leq n\} = n + \max(\{B\} \cup \{k\beta_k : m < k \leq n\}) = \hat{\beta}_n$. Thus $\hat{\alpha}_n \leq \hat{\beta}_n$ for every $n \geq C$, so $\hat{\alpha} \preceq \hat{\beta}$. Let $n \in \omega$. Then there exists $m \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$. Put $U_{\alpha,n} = X_m$. Then $U_{\alpha,n} \subseteq U_{\alpha,n+1}$ for all $n \in \omega$ and $\bigcup_{n=1}^\infty U_{\alpha,n} = \bigcup_{n=0}^\infty X_n = X$.

Let $f \in C_p(X)$. Then there exists $\alpha = (\alpha_n) \in \omega^\omega$ such that $\|f|X_k\|_\infty \leq \alpha_k$ for every $k \in \omega$. Let $n \in \omega$. Then there exists $m \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$ and $\|f|U_{\alpha,n}\|_\infty = \|f|X_m\|_\infty \leq \alpha_m \leq \hat{\alpha}_m < n$. Thus

$$\forall f \in C_p(X)\exists\alpha \in \omega^\omega\forall n \in \omega : \|f|U_{\alpha,n}\|_\infty \leq n.$$

Let $\alpha, \beta \in \omega^\omega$ with $\alpha \leq \beta$. Let $n \in \omega$. Then there exist $m, k \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$ and $\hat{\beta}_k < n \leq \hat{\beta}_{k+1}$. Clearly $\hat{\alpha} \leq \hat{\beta}$, so $m \geq k$. Thus $U_{\alpha,n} = X_m \supseteq X_k = U_{\beta,n}$.

Let $\alpha, \beta \in \omega^\omega$ with $\alpha \preceq \beta$. Then $\hat{\alpha} \preceq \hat{\beta}$, so there exists $v \in \omega$ such that $\hat{\alpha}_n \leq \hat{\beta}_n$ for every $n \geq v$. Let $p = \hat{\beta}_v + 1$. Let $n \geq p$. Then there exist $m, k \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$ and $\hat{\beta}_k < n \leq \hat{\beta}_{k+1}$. Since $n > \hat{\beta}_v$ we infer that $k \geq v$. Hence $\hat{\alpha}_k \leq \hat{\beta}_k < n$, so $k \leq m$. Thus $U_{\beta,n} = X_k \subseteq X_m = U_{\alpha,n}$, so $U_{\beta,n} \subseteq U_{\alpha,n}$ for every $n \geq p$. □

Fact 26 A countable metrizable space X is scattered if and only if X is Polish. *Indeed, if X is scattered, it is Polish by* [26, Lemma 8.1, Theorem 1.3]. *Conversely, if X is not scattered, it contains a closed copy of rationals $\mathbb{Q}$, so X is not Polish. This applies to illustrate the following example which will be used in the sequel.*

Example 27 There exists a countable Polish subspace Γ of $\mathbb{R}$ which is not open in its completion $\hat{\Gamma}$ and admits a nice framing such that for every $\alpha \in \omega^\omega$ there exists $n \in \omega$ such that $U_{\alpha,n}$ is not functionally bounded.

Proof Let $x_{n,k} = 2^{-n}(1 + 2^{-k})$ for all $n, k \in \omega$. Set $X_n = \{x_{n,k} : k \in \omega\}$ for $n \in \omega$. The set $\Gamma = \bigcup_{n=1}^{\infty} X_n \cup \{0\}$ endowed with the topology induced from $\mathbb{R}$ is a metrizable and separable space. For any $n \in \omega$ the set X_n is infinite, discrete and closed, so it is not functionally bounded in Γ. Note that Γ is a Polish space by applying Fact 26.

Let $A_0 = \emptyset$ and $A_m = \{x_{n,k} : 1 \leq n, k \leq m\}$ for $m \in \omega$. Then $\bigcup_{m=0}^{\infty} A_m = \bigcup_{n=1}^{\infty} X_n$. Let $\alpha = (\alpha_n) \in \omega^\omega$. Let $\alpha_0 = 0$ and $\hat{\alpha}_m = \sum_{j=0}^{m} \alpha_j$ for $m \geq 0$. Let $n \in \omega$. Then there exists $m \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$. Put $U_{\alpha,n} = A_0$ if $n < \alpha_1$ and $U_{\alpha,n} = A_m \cup Y_{\alpha_1}$ if $n \geq \alpha_1$. Clearly $U_{\alpha,n}$ is not functionally bounded in Γ, if $n \geq \alpha_1$, since $U_{\alpha,n} \supseteq X_n$, $n \geq \alpha_1$.

Let $f \in C_p(\Gamma)$. Since $\sup Y_n = \sup X_n = 3 \cdot 2^{1-n} \to_n 0$, there exists $s \in \omega$ with $\|f|Y_s\|_\infty < |f(0)| + 1 < s$. Let $\alpha = (\alpha_n) \in \omega^\omega$ with $\alpha_1 \geq s$ and $\alpha_k \geq \|f|A_k\|_\infty$ for $k \in \omega$. If $n < \alpha_1$, then $U_{\alpha,n} = \emptyset$, so $\|f|U_{\alpha,n}\|_\infty = 0 < n$. Let $n \geq \alpha_1$. Then $\|f|A_m\|_\infty \leq \alpha_m \leq \hat{\alpha}_m < n$ and $\|f|Y_{\alpha_1}\|_\infty \leq \|f|Y_s\|_\infty < s \leq \alpha_1 \leq n$. Hence $\|f|U_{\alpha,n}\|_\infty \leq n$. Clearly, $U_{\alpha,n} \subseteq U_{\alpha,n+1}$ for all $\alpha \in \omega^\omega$, $n \in \omega$ and $\bigcup_{n=1}^{\infty} U_{\alpha,n} = \bigcup_{n=1}^{\infty} X_n \cup \{0\} = \Gamma$ for all $\alpha \in \omega^\omega$. Let $\alpha, \beta \in \omega^\omega$ with $\alpha \leq \beta$. If $n < \beta_1$ then $U_{\alpha,n} \supseteq \emptyset = U_{\beta,n}$. Let $n \geq \beta_1$. Then there exist $m, k \geq 0$ such that $\hat{\alpha}_m < n \leq \hat{\alpha}_{m+1}$ and $\hat{\beta}_k < n \leq \hat{\beta}_{k+1}$. Clearly $\hat{\alpha} \leq \hat{\beta}$, so $m \geq k$. Thus $U_{\alpha,n} = A_m \cup Y_{\alpha_1} \supseteq A_k \cup Y_{\beta_1} = U_{\beta,n}$. Thus $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is a nice framing in Γ such that for every $\alpha \in \omega^\omega$ there exists $n \in \omega$ such that $U_{\alpha,n}$ is not functionally bounded. Note that $\hat{\Gamma} \setminus \Gamma = \{2^{-n} : n \in \omega\}$. □

Lemma 28 *If a metrizable space Z is not open in its completion $\hat{Z}$, then Z has a closed copy of Γ. Hence a separable metrizable non-Polish space contains a closed copy of Γ.*

Proof Assume Z is not open in $\hat{Z}$. Then there exist $z_0 \in Z$ and a sequence $(z_n)_n \subseteq \hat{Z} \setminus Z$ that is convergent to z_0 in $\hat{Z}$. We can assume that $z_n \neq z_m$, if $n \neq m$. Let $s_n = \inf_{m \neq n} d(z_n, z_m)$, $n \in \omega$, where d is the metric in $\hat{Z}$. Clearly, $s_n > 0$ for any $n \in \omega$. Let $(r_n)_n$ be a sequence of positive numbers that is convergent to 0 such that $r_n < 2^{-1} s_n$, $n \in \omega$. Clearly, the balls $K_{\hat{Z}}(z_n, r_n)$, $n \in \omega$, are pairwise disjoint. For every $n \in \omega$ there exists a sequence $(z_{n,m})_m \subseteq Z \cap K_{\hat{Z}}(z_n, r_n)$ which is convergent to z_n and such that $z_{n,m} \neq z_{n,k}$, if $m \neq k$. Set $Z_n = \{z_{n,m} : m \in \omega\}$ for $n \in \omega$ and $Z_0 = \bigcup_{n=1}^{\infty} Z_n \cup \{z_0\}$. Clearly Z_0 is a closed subspace of Z and the map $h : \Gamma \to Z_0$ such that $h(0) = z_0$ and $h(x_{n,m}) = z_{n,m}$ for all $n, m \in \omega$ is a homeomorphism. □

The next result follows from Lemma 28 and Example 27.

Proposition 29 *Let X be a metrizable space with a nice framing and which is not open in its completion $\hat{X}$. Then X admits a nice framing no layer of it forms a σ-compact cover. In particular, every separable metrizable space which is non-Polish enjoys this property.*

Proof of Theorem 24 $X \in \mathfrak{M}$ is σ-compact by Theorem 15. (1) follows from Proposition 25. (2) follows from Proposition 29 and (3) follows from Example 27. □

5 More about strong framings

We introduce a class of Tychonoff spaces containing the Lindelöf Čech-complete spaces which are naturally related to the subject of the previous section. One may define a cardinal function b on X as the least cardinality of a set A in $C(X)$ such that a set B in X is functionally bounded if $f(B)$ is bounded for every $f \in A$. We call this cardinal $b(X)$ the *functional boundedness* of X.

Definition 30 We say that a Tychonoff space X has countable functional boundedness if $b(X) = \aleph_0$, that is, if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ such that a set $B \subseteq X$ is functionally bounded if all f_n are bounded on B.

Clearly, $\mathbb{R}^{\omega}$ has countable functional boundedness, since a subset $B \subseteq \mathbb{R}^{\omega}$ is functionally bounded if and only if the canonical projections $\pi_n : \mathbb{R}^{\omega} \to \mathbb{R}$, $(x_1, x_2, x_3, \ldots) \to x_n$, are bounded on B . By Tietze-Urysohn's Theorem any closed subspace of a space that has countable functional boundedness, has countable functional boundedness. Hence each Polish space has countable functional boundedness, as it is homeomorphic to a closed subspace of $\mathbb{R}^{\omega}$. Recall (see [13, 5.5.9(a)]) that X is Lindelöf Čech-complete if and only if X can be mapped onto a Polish space under a perfect map. We prove the main result of this section.

Theorem 31 *X has countable functional boundedness if and only if there exists a continuous map T from X onto a Polish space Y such that $T^{-1}(A)$ is functionally bounded for each functionally bounded $A \subseteq Y$. Hence, if X is a μ-space, the following assertions are equivalent:*

(1) *X has countable functional boundedness.*
(2) *X is a Lindelöf Čech-complete space.*

If $C_p(X)$ is a μ-space, $C_p(X)$ has countable functional boundedness if and only if $C_p(X)$ is isomorphic to $\mathbb{R}^{\omega}$.

Claim (for $C_p(X)$) holds for example if X is metrizable [2, 3.4.12 Theorem], so a metric separable X has countable functional boundedness if and only if X is Polish.

Proof of Theorem 31 If X has countable functional boundedness, it admits a fundamental resolution consisting of functionally bounded sets. Indeed, set $K_{\alpha} = \{x \in X : |f_n(x)| \leq \alpha(n)\}$, $\alpha \in \omega^{\omega}$, where $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ is as in the definition. Define a map $T : X \to \mathbb{R}^{\omega}$, $T(x) = (f_n(x))_{n=1}^{\infty} \in \mathbb{R}^{\omega}$, $x \in X$. Let $A \subseteq T(X)$ be functionally bounded. By properties of T and X the set $T^{-1}(A)$ is functionally bounded. Hence, since $T(X)$ is metrizable and separable, the closure (in $T(X)$) of the sets $T(K_{\alpha})$ compose a fundamental compact resolution. By Christensen's theorem [23, Theorem 6.1] the image $Y = T(X)$ is Polish. The converse is clear since any Polish space has countable functional boundedness. If additionally X is a μ -space, then the preimage of any compact set of Y is compact in X, so T is perfect. Hence X

is a Lindelöf Čech-complete space. Each Lindelöf Čech-complete space has countable functional boundedness. Finally, recall that $C_p(X)$ is Čech-complete if and only if X is countable and discrete, see [30, S.265]. □

Next theorem characterizes those σ-bounded spaces that have countable functional boundedness. In contrast to nice framings, each strong framing in a space X with countable functional boundedness has a layer consisting of bounded sets.

Theorem 32 *A space X with countable functional boundedness is σ-bounded if and only if it has a strong framing. If $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is a strong framing, there is $\alpha \in \omega^\omega$ with all $U_{\alpha,n}$ functionally bounded.*

Proof Since X has countable functional boundedness, there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq C(X)$ as mentioned in the definition. If $\{U_{\alpha,n} : (\alpha, n) \in \omega^\omega \times \omega\}$ is a strong framing, for any $k \in \omega$ there exists $\alpha^k \in \omega^\omega$ such that $\| f_k|U_{\alpha^k,n}\|_\infty \leq n$ for every $n \in \omega$. Let $\alpha_n = \max\{\alpha_n^k : 1 \leq k \leq n\}$ for $n \in \omega$. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots) \in \omega^\omega$ and $\alpha_n^k \leq \alpha_n$ for all $k, n \in \omega$ with $n \geq k$. Hence $\alpha^k \preceq \alpha$ for every $k \in \omega$. Thus for every $k \in \omega$ there exists $n_k \in \omega$ such that $U_{\alpha^k,n} \supseteq U_{\alpha,n}$ for every $n \geq n_k$. Hence for any $k \in \omega$ we have $\| f_k|U_{\alpha,n}\|_\infty \leq \| f_k|U_{\alpha^k,n}\|_\infty \leq n$ for every $n \geq n_k$. The sequence $(U_{\alpha,n})_{n=1}^\infty$ is increasing, so $\| f_k|U_{\alpha,n}\|_\infty < \infty$, $k, n \in \omega$. Thus $U_{\alpha,n}$, with $n \in \omega$, are functionally bounded. The converse follows from Proposition 25. □

A direct consequence of above Theorem 32 is Corollary 33. Note only that, by applying [22, Remark 3.1 (i)], paracompact X is Lindelöf if X has a nice framing.

Corollary 33 *A paracompact Čech-complete space X is σ-compact if and only if it has a strong framing.*

6 Around two problems

Being motivated by Proposition 3 one can formulate a natural question (*):

Is it true that $C_p(X)$ has a bounded resolution if and only if $C_p(X)$ admits a stronger metrizable locally convex topology?

This problem has been also posed in [17, Problem 9.3]. We show that this question has a negative solution by applying Example 35 below. Observe first that the following claims are equivalent.

(i) There exists a μ-space such that the space $C_p(X)$ admits a bounded resolution but does not admit a stronger metrizable locally convex topology.
(ii) There exists a μ-space space X such that $C_p(X)$ is K-analytic framed in $\mathbb{R}^X$ but X is not σ-compact.

Indeed, $(i) \Rightarrow (ii)$: We apply [16] (see Corollary 20) and Theorem 5 to get that X is not σ-compact. $(ii) \Rightarrow (i)$: Apply again Theorem 5.

The following problems have been posed in [4].

Problem 34 *([4]) Let X be a Tychonoff space.*

(1) *Is X σ-compact if X is Lindelöf and $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$?*
(2) *Let $C_p(X)$ be K-analytic-framed in $\mathbb{R}^X$. Is X a σ-bounded space?*
(3) *Let X be a Lindelöf space such that $C_p(X)$ is K-analytic. Is X a σ-compact space?*

Example 35, due to Leiderman [24], shows that the above problems (including question (*)) have negative solutions. Later on, Banakh and Leiderman recalled this again in [10, Proposition 3.8, Remark 3.9]. The present version of Example 35 provides a slightly stronger claim than the original one from [24].

Example 35 There is a Lindelöf Σ-space X with a unique non-isolated point and:

(1) $C_p(X)$ is K-analytic.
(2) X lacks a compact resolution, so X is not σ-compact and does not have countable functional boundedness.
(3) Every continuous metrizable image of X is countable.
(4) X has a nice framing; no nice framing has a layer with functionally bounded sets.

Remark 36 Leiderman's example [24] is based on Talagrand's paper [29] who constructed a space X with a unique non-isolated point which is a Lindelöf Σ but not K-analytic. Leiderman proved that $C_p(X)$ is K-analytic. (3) follows from: Every disjoint covering of X by G_δ-sets is countable. Item (2) follows from [22, Lemma 2.3]: X is K-analytic if and only if X is a μ-space and X has a compact resolution. Clearly X does not have countable functional boundedness by Theorem 31.

Note that if X is both separable and is a continuous image of a metrizable space, the conclusion in (1) of Problem 34 still may fails.

Example 37 There exists a separable Tychonoff space X not being a μ-space and

(1) X is a continuous compact-covering image of a metric space.
(2) X does not admit a compact resolution, in particular X is not σ-compact.
(3) There exists a σ-compact space L such that $C_p(X) \subseteq L \subseteq \mathbb{R}^X$ but $C_p(X)$ is not K -analytic. Hence X admits a nice framing.
(4) $C_p(X)$ admits a quotient map onto the σ-compact subspace $(\ell_\infty)_p = \{(x_n) \in \mathbb{R}^\omega : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^\omega$, but $C_p(X)$ is not projectively σ-compact.

Proof Denote the family of all infinite subsets of a countable set X by $[X]^\omega$. Set $\omega^* = \beta\omega \setminus \omega$. For each $A \in [\omega]^\omega$, choose an ultrafilter $u_A \in \omega^*$ in the closure of A in $\beta\omega$. Let $X = \omega \cup \{u_A : A \in [\omega]^\omega\}$ be topologized as a subspace of $\beta\omega$.

Proof of (1): It is known (Haydon [21]) that X is pseudocompact (separable) with cardinality of continuum and all compact subspaces of X are finite. Clearly, X is a continuous compact-covering image of a metrizable space by [25, Theorem 1.1].

Proof of (2): Assume X admits a compact resolution $\{K_\alpha : \alpha \in \omega^\omega\}$. Since X is uncountable, some K_α is infinite, [23, Proposition 3.7], a contradiction.

Proof of (3): By Proposition 4 the space $C_p(X)$ has the first property. For the next one, assume $C_p(X)$ is K-analytic. Then by [18, Corollary 3.4] the Banach space $C^b(X)$ of continuous bounded real-valued functions on X equipped with the Banach topology ξ generated by the norm $\|f\| = \sup_{x\in X}|f(x)|$ is weakly K-analytic, i. e., the weak topology σ of $C^b(X)$ is K-analytic. Hence the weak topology of $C^b(X)$ admits a compact resolution [23, Proposition 3.10]. Since X is separable, $C_p(X)$ admits a weaker metrizable topology. But then σ is analytic by [12, Theorem 15]. Hence $C^b(X) = C(\beta X)$ is separable, impossible as βX is non-metrizable. X is not a μ -space: Otherwise $C_p(X) = C_k(X)$ is barrelled by [23, Proposition 2.15], so by the closed graph theorem the identity map $I : C_k(X) \to (C(X), \xi)$ is continuous; hence X is compact, a contradiction.

Proof of (4): Since X is pseudocompact containing ω, C^*-embedded into X, we apply [9, Theorem 1] to get a quotient map from $C_p(X)$ onto the subspace $(\ell_\infty)_p$ of $\mathbb{R}^\omega$. Clearly

$(\ell_\infty)_p$ is covered by the sequence $[-n, n]^\omega$ of compact sets. By construction of X it is clear (by applying [3, Proposition 3.4]) that $C_p(X)$ is not projectively σ-compact. □

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