RACSAM (2021) 115:27
https://doi.org/10.1007/s13398-020-00967-4

ORIGINAL PAPER

Distinguished $C_p(X)$ spaces

J. C. Ferrando[1] · J. Kąkol[2,3] · A. Leiderman[4] · S. A. Saxon[5]

Received: 4 November 2020 / Accepted: 11 November 2020

Abstract
We continue our initial study of $C_p(X)$ spaces that are distinguished, equiv., are large subspaces of $\mathbb{R}^X$, equiv., whose strong duals $L_\beta(X)$ carry the strongest locally convex topology. Many are distinguished, many are not. All $L_\beta(X)$ spaces are, as are all metrizable $C_p(X)$ and $C_k(X)$ spaces. To prove a space $C_p(X)$ is not distinguished, we typically compare the character of $L_\beta(X)$ with $|X|$. A certain covering for X we call a *scant cover* is used to find distinguished $C_p(X)$ spaces. Two of the main results are: (i) $C_p(X)$ is distinguished if and only if its bidual E coincides with $\mathbb{R}^X$, and (ii) for a Corson compact space X, the space $C_p(X)$ is distinguished if and only if X is scattered and Eberlein compact.

Keywords Distinguished space · Bidual space · Eberlein compact space · Fréchet space · strongly splittable space · Fundamental family of bounded sets · Point-finite family · G_δ-dense subspace

Mathematics Subject Classification 54C35 · 46A03

The first named author is supported by Grant PGC2018-094431-B-I00 of the Ministry of Science, Innovation and Universities of Spain. The research for the second named author is supported by the GAČR Project 20-22230L and RVO: 67985840.

✉ J. C. Ferrando
jc.ferrando@umh.es

J. Kąkol
kakol@amu.edu.pl

A. Leiderman
arkady@math.bgu.ac.il

S. A. Saxon
stephen_saxon@yahoo.com

1 Centro de Investigación Operativa, Universidad Miguel Hernández, 03202 Elche, Spain

2 Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznan, Poland

3 Institute of Mathematics Czech Academy of Sciences, Prague, Czechia

4 Department of Mathematics, Ben-Gurion University of Negev, Beer Sheva, Israel

5 Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

Published online: 25 November 2020

1 Introduction

Recall that a locally convex space E (embedded in its bidual E'' by means of the evaluation map) is *semi-reflexive* if E'' coincides algebraically with E, *reflexive* if it is semi-reflexive and the original locally convex topology of E coincides with $\beta(E'', E')$, and *distinguished* if its strong dual E'_β is barrelled. Clearly, each reflexive space is semi-reflexive, and each semi-reflexive space is distinguished [26, 23.3 (4)]. In fact, (alternate definition [26, 23.7]), E is distinguished if and only if E is a large subspace of $(E'', \sigma(E'', E'))$. Recall that a subspace F of a locally convex space G is a *large subspace* of G if every bounded set in G is contained in the closure in G of a bounded set in F, [32, Definition 8.3.22]. If X is a Tychonoff space and $C_p(X)$ denotes the linear space $C(X)$ of all real-valued continuous functions defined on X equipped with the pointwise topology, it can be easily seen that $C_p(X)$ is semi-reflexive if and only if $C_p(X)$ is reflexive, if and only if X is discrete. The first statement follows directly from [26, 23.5 (1)] and [25, 11.3 Corollary]. The second is consequence of [13, Corollary 3.4]. Thus we readily resolve the (semi-)reflexive problem for $C_p(X)$. The simplest examples of distinguished $C_p(X)$ spaces which are not semi-reflexive are those with X any countable nondiscrete Tychonoff space (see [13] and below). The more general distinguished problem remains (see "Addendum"):

Problem 1 *Characterize those Tychonoff spaces X such that $C_p(X)$ is distinguished.*

We confront this abstract problem by finding a number of concrete spaces $C_p(X)$ which are (or are not) distinguished. We simplify by noting that $C_p(X)$ is distinguished if and only if it is a large subspace of $\mathbb{R}^X$.

2 Preliminaries

Distinguished locally convex spaces were introduced by J. Dieudonné and L. Schwartz. This class of locally convex spaces has attracted the attention of several specialists. For example, A. Grothendieck showed that a metrizable locally convex space E is distinguished if and only if E'_β is bornological, and Heinrich [23] observed that each metrizable quasinormable locally convex space [32, Definition 8.3.34] satisfies the density condition [7] which, as we shall see below, implies that every metrizable quasinormable locally convex space is distinguished. In particular, the strong dual of a distinguished Fréchet space can be described as a regular (LB)-space [32, Observation 8.5.14 (e)]. The reader may consult references [24, 3.16], [25, 13.4], [26, 23.7, 29.3] and [32, 8.3] for further information about distinguished spaces.

As a general fact, a Fréchet space E is distinguished if and only if its strong dual E_β has countable tightness [18, Corolary 4] . The most celebrated example of a nondistinguished Fréchet space is Köthe's echelon space $\lambda[\mathfrak{T}]$, [26, 31.7]. In [18, Example 4] it is shown that the tightness of the strong dual $\lambda_\beta^\times$ of $\lambda[\mathfrak{T}]$ is exactly $\mathfrak{d}$, the *dominating cardinal*. Other examples of non-distinguished Fréchet spaces can be found in [40].

Unless otherwise stated, X will stand for an infinite Tychonoff space. As usual, we shall denote by υX the *Hewitt realcompactification* of X. We shall assume that all linear spaces are over the field $\mathbb{R}$ of real numbers and all locally convex spaces are Hausdorff. As mentioned above, $C_p(X)$ stands for the ring $C(X)$ of real-valued continuous functions on X endowed with the pointwise topology τ_p. The topological dual of $C_p(X)$ will be denoted by $L(X)$, or by $L_p(X)$ when equipped with the weak* topology. We shall designate by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology τ_k.

Let us recall that a space X is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space, and Corson compact if it is homeomorphic to a compact subset of a Σ-product of real lines. Every Eberlein compact space is Corson compact. A topological space X is called *scattered* if every closed non-empty subspace Y of X has an isolated point in the relative topology.

A locally convex space E is called *quasinormable* [32, Definition 8.3.34] if for every absolutely convex neighborhood of zero U in E there exists an absolutely convex neighborhood of zero $V \subseteq U$ such that for every $\lambda > 0$ there exists a bounded set B in E with $V \subseteq B + \lambda U$. The class of quasinormable spaces contains, for example, the (DF)-spaces, the $C_k(X)$ spaces over a Tychonoff space X , the spaces $C^n(\Omega)$ for $n \in \mathbb{N}$, Ω being an open subset of $\mathbb{R}^{\mathbb{N}}$, as well as all Fréchet-Montel spaces (see [25,31]).

A metrizable locally convex space E with a decreasing base $\{U_n : n \in \mathbb{N}\}$ of absolutely convex neighborhoods of zero satisfies the *density condition* if there is a double sequence $\left\{B_{n,k} : n, k \in \mathbb{N}\right\}$ of bounded sets in E such that for each $n \in \mathbb{N}$ and each bounded set $C \subseteq E$ there is $k \in \mathbb{N}$ with $C \subseteq B_{n,k} + U_n$, [8, Theorem 9]. The following general results hold true.

Theorem 2 [8] *For a metrizable locally convex space E the following are equivalent.*

(1) *E satisfies the density condition.*
(2) *Every bounded set in E'_β is metrizable.*
(3) *The space $\ell_1(E)$ is distinguished.*

Nevertheless, we have the following.

Theorem 3 [14] *A metrizable locally convex space E is distinguished if and only if every bounded set in the strong dual of E has countable tightness.*

Consequently, if E is a metrizable locally convex space that satisfies the density condition, Theorem 2 ensures that every bounded set in the strong dual of E is metrizable, which allows Theorem 3 to guarantee that E is distinguished, but the converse fails, [9]. We also refer the reader to [8], where some results about distinguished Köthe echelon and co-echelon spaces are presented. Another characterization of the density condition for Fréchet spaces is given in [12, Theorem 4].

Since, as mentioned earlier, each metrizable $C_k(X)$ space is quasinormable (see [25, 10.8.2 Theorem]), it turns out that

Proposition 4 *Every metrizable $C_k(X)$ space is distinguished.*

The converse fails, since if X is discrete $C_k(X) = \mathbb{R}^X$ is distinguished but not metrizable when $|X| > \aleph_0$. The case with spaces $C_p(X)$ is totally different from spaces $C_k(X)$ and others mentioned above. Let us review/refine our early theory with different proofs in a broader context. First let us mention the following fundamental facts about metrizability of bounded sets proved in [13, Corollary 2.2 and Theorem 2.5]. *Every bounded set in $C_p(X)$ or $C_k(X)$ is metrizable if and only if $C_p(X)$ or $C_k(X)$ is metrizable, respectively.*

For exploring the dual of spaces $C_p(X)$ we need some additional concepts. We recall a Hausdorff locally convex space E is *feral* if every bounded subset is finite-dimensional, *flat* if $E' = E^*$, and *fit* if it has a dense (linear) subspace whose codimension equals the dimension of E (see [36,38]). Each flat space is feral, but not conversely. Indeed, every $L_\beta(X)$ space is feral ([15, page 392] or [13, Theorem 3.1]). But often $L_\beta(X)$ is fit, not flat; e. g., when X is cosmic with $|X| = \mathfrak{c}$ (Corollary 31, below). Only the 0-dimensional linear space is both fit and flat. Non-trivial fit spaces are the extreme opposite of flat spaces. The former have dense subspaces of every possible (including infinite) codimension, the latter have no proper dense subspaces at all.

If A is a Hamel basis for E, let B be all points in A whose corresponding coefficient functionals are discontinuous, set $C := A \backslash B$, and let E_B and E_C be the linear span of B and C, respectively. Clearly, E splits algebraically into the direct sum of E_B and E_C. We say A is a *continuous basis* if $B = \emptyset$. The splitting theorem [35] says that if E is barrelled, then E is *topologically* the direct sum of E_B and E_C and, moreover, the component E_B is fit, while E_C bears the strongest locally convex topology (see [38]). Equivalently E_C is barrelled and flat. This augments standard exercises, proving

Theorem 5 *These five assertions about a locally convex space E are equivalent.*

(1) *E has its strongest locally convex topology.*
(2) *E is the strong dual of the product of* $\dim E$*-many lines.*
(3) *Every absolutely convex absorbing set is a zero-neighborhood in E.*
(4) *E is barrelled and flat.*
(5) *E is barrelled and admits a continuous basis* [35].

The *character* $\chi(E)$ of a locally convex space E is the smallest cardinality for a base of zero-neighborhoods in E.

Theorem 6 [36, Theorem 1] *A locally convex space E is fit if* $\chi(E) \leq \dim E$.

Strong duals of metrizable spaces are prototypical (DF)-spaces. We contribute

Theorem 7 *The strong dual of a metrizable locally convex space E is either fit or flat.*

Proof I. Suppose no infinite-dimensional subspace of E admits a continuous norm, i.e., E has its weak topology $\sigma\left(E, E'\right)$. Then E is (isomorphically) a dense subspace of the product G of $\dim E'$-many lines, and metrizability implies $\dim E' \leq \aleph_0$. Thus G is separable and metrizable, so E is a large subspace of G [32, 8.3.23 (b)] and has the same strong dual as G. Hence the strong dual E'_β of E has its strongest locally convex topology and therefore is flat (and barrelled).

II. Suppose some infinite-dimensional subspace of E admits a continuous norm. Then $\dim E' \geq \mathfrak{c}$ by the theorems of Mazur and Hahn–Banach. Metrizability and [37, Theorem 1] yield a fundamental family $\mathcal{B}$ of bounded sets in E with $|\mathcal{B}| \leq \mathfrak{d}$, with $\mathfrak{d}$ the dominating cardinal. Therefore $\chi(E'_\beta) \leq \mathfrak{d} \leq \mathfrak{c} \leq \dim E'_\beta$, so that E'_β is fit by Theorem 6. □

Part I is also a simpler proof of the basic

Theorem 8 [13, Theorem 3.3 (a)] *If X is countable Tychonoff, $C_p(X)$ is distinguished.*

For a third proof note that $L_\beta(X)$, being countable-dimensional, is countably tight, then apply [18, Corollay 3] and [13, Corollary 3.2]. Part I, again, shows that when X is countable, $L_\beta(X)$ has its strongest locally convex topology and thus admits a continuous basis. Surprisingly, *all* $L_\beta(X)$ spaces admit continuous bases.

Theorem 9 *The homeomorphic copy of X in $L_p(X)$ is a continuous basis for $L_\beta(X)$.*

Proof The set A of functions g in $C(X)$ such that $|g(X)| \leq 1$ for all $x \in X$ is pointwise bounded, so the polar A^0 is a zero-neighborhood in $L_\beta(X)$. Tychonoff extension theory ensures A^0 is just the absolutely convex hull of X. To see this, let $y := \sum_{x \in \triangle} a_x \cdot x$ be a finite linear combination from X, choose $g \in A$ with $g(X) = \operatorname{sgn} a_x$ for each $x \in \triangle$, and note that $|\langle g, y\rangle| = \sum_{x \in \triangle} |a_x| \leq 1$ if and only if $y \in A^0$. Thus the coefficient functionals for the basis X are each numerically bounded by 1 on A^0 , so are continuous on $L_\beta\,(X)$. □

We arrive at the foundational.

Theorem 10 *For a Tychonoff space X these three assertions are equivalent.*

(1) $C_p(X)$ *is distinguished.*
(2) $C_p(X)$ *is a large subspace of* $\mathbb{R}^X$.
(3) $L_\beta(X)$ *has its strongest locally convex topology, [13].*

Proof Theorems 5 and 9 prove (1) $\Leftrightarrow$ (3), which is just [13, Corollary 3.4]. Since $C_p(X)$ is a dense subspace of the product $\mathbb{R}^X$, we may algebraically identify $L(X)$ as their common dual by means of restriction. Then $L_\beta(X)$ is their common *strong* dual if and only if $C_p(X)$ is large in $\mathbb{R}^X$, by the bipolar theorem. Theorem 5 ensures (2) $\Leftrightarrow$ (3). □

Theorem 11 [17] *If $C_p(X)$ is distinguished, then $\chi(L_\beta(X)) > |X|$.*

Proof Otherwise Theorem 6 implies $L_\beta(X)$ is fit, not flat, contradicting Theorem 10. □

For the next theorem, if $Q \subseteq L(X)$ has infinite support $S = \bigcup\{\operatorname{supp}\mu : \mu \in Q\}$, there are a discrete sequence $\{x_n\}_{n=1}^\infty$ in X contained in S, a sequence $\{\mu_n\}_{n=1}^\infty$ in Q and a bounded sequence $\{f_n\}_{n=1}^\infty$ in $C_p(X)$ with $x_n \in \operatorname{supp}\mu_n$ and $\langle \mu_n, f_n\rangle = n$ for all $n \in \mathbb{N}$, [25, 11.7.2 Theorem]. The proof of the first statement of the next result is similar to that of [25, 10.8.2 Theorem]. The second statement can be found in [25, 11.7.3 Corollary].

Theorem 12 *The space $C_p(X)$ is always quasinormable and quasibarrelled.*

Proof Let us sketch the proofs of the two affirmations. If $\Delta \subseteq X$ is finite and $\epsilon > 0$, let $U = \{f \in C(X) : \sup_{x\in\Delta} |f(X)| < \epsilon\}$. With $B := \{f \in C(X) : \sup_{x\in X} |f(X)| \le 2\epsilon\}$, a similar argument to the proof of [25, 10.8.2 Theorem] shows that $U \subseteq B + \lambda U$ for $0 < \lambda \le 1$, so for all $\lambda > 0$ since U is absolutely convex. For the second statement, if Q is a strongly bounded set in $L(X)$, the above observation implies that $\bigcup\{\operatorname{supp}\mu : \mu \in Q\}$ is finite. So Q lives in a finite-dimensional subspace of $L(X)$. This means that the polar Q^0 of Q in $C(X)$ is a neighborhood of zero in $C_p(X)$. □

Since every metrizable quasinormable locally convex space is distinguished, we see once more that for countable X the space $C_p(X)$ is distinguished.

3 General facts on distinguished $C_p(X)$ spaces

To be distinguished, $C_p(X)$ must be a large subspace of $\mathbb{R}^X$ (Theorem 10), so that its bidual E coincides with $\mathbb{R}^X$. Surprisingly, the necessary coincidence is also sufficient. For each $g \in \mathbb{R}^X$ define $P_g := \{f \in \mathbb{R}^X : |f| \le |g|\}$, a bounded set in $\mathbb{R}^X$. For each bounded set A in $\mathbb{R}^X$ define $\phi_A \in \mathbb{R}^X$ by writing $\phi_A(x) := \sup_{f\in A} |f(X)|$ for each $x \in X$. For each bounded set B in $C_p(X)$, define another bounded set B^+ in $C_p(X)$ consisting of those h in $C(X)$ for which there exists a finite set $F \subseteq B$ such that $|h(x)| \le \max_{f\in F} |f(X)|$ for all $x \in X$.

Lemma 13 *If a subset A of $\mathbb{R}^X$ lies in the closure $\overline{B}$ in $\mathbb{R}^X$ of a bounded set B in $C_p(X)$, then $P_{\phi_A} \subseteq \overline{B^+}$. In particular, $P_{\phi_B} \subseteq \overline{B^+}$ always holds.*

Proof Given $f_0 \in P_{\phi_A}$, a finite set Δ in X, and $\epsilon > 0$, choose by definition of ϕ_A a finite set G in A with $|G| = |\Delta|$ such that, for every $x \in \Delta$,

$$\phi_A(X) < \max_{g\in G} |g(X)| + \epsilon.$$

Since $G \subseteq \overline{B}$ there exists $F \subseteq B$ with $|F| = |G|$ such that, for every $x \in \Delta$,

$$|f_0(X)| \leq \phi_A(X) < \max_{f \in F} |f(X)| + \epsilon.$$

For each $x \in \Delta$ choose a_x with $|a_x| \leq \max_{f \in F} |f(X)|$ and $|f_0(X) - a_x| < \epsilon$. As X is Tychonoff, there exists $h \in C(X)$ such that $h(X) = a_x$ for all $x \in \Delta$ and $|h(X)| \leq \max_{f \in F} |f(X)|$ for all $x \in X$. Therefore, h is in B^+ and within ϵ of f_0 on Δ, so $f_0 \in \overline{B^+}$. □

Theorem 14 *The space $C_p(X)$ is distinguished if and only if the strong dual $L_\beta(X)$ is flat; equivalently, the strong bidual E is the product space $\mathbb{R}^X$.*

Proof of sufficiency The strong bidual E is a subspace of $\mathbb{R}^X$ which, according to Theorem 9, always contains the space E_0 of functions on X having finite support. Let A be a bounded set in $\mathbb{R}^X$. By hypothesis, $\phi_A \in E$; i. e., $\phi_A \in \overline{B}$, the closure in $\mathbb{R}^X$ of some bounded set B in $C_p(X)$. If $S := \{\phi_A\}$, then $\phi_S = \phi_A$, and Lemma 13 ensures $A \subseteq P_{\phi_A} = P_{\phi_S} \subseteq \overline{B^+}$, which proves $C_p(X)$ is large in $\mathbb{R}^X$. □

Corollary 15 *The space $C_p(X)$ is distinguished if and only if its strong bidual E is quasi-complete.*

Proof As $\mathbb{R}^X$ is the quasi-completion of E_0, Theorem 14 applies. □

Given $Y \subseteq X$, a continuous (linear) map $\varphi : C_p(Y) \to C_p(X)$ is called a *continuous* (*linear*) *extender* for $C_p(Y)$ if $\varphi(f)|_Y = f$ for every $f \in C(X)$. We denote by $\mathcal{P}_Y$ the canonical projection from $\mathbb{R}^X$ onto the subspace of functions whose support lies in Y.

Theorem 16 *Let $\mathcal{F}$ be a finite family of sets covering X.*

(1) *If $C_p(X)$ is distinguished and $Y \subseteq X$, then $C_p(Y)$ is also distinguished.*
(2) *The space $C_p(X)$ is distinguished if and only if for each bounded set A in $\mathbb{R}^X$ and $Y \in \mathcal{F}$ there is a bounded set B in $C_p(X)$ such that $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$.*
(3) *Assume that $C_p(Y)$ is distinguished for each $Y \in \mathcal{F}$. If for each bounded set D in $C_p(Y)$ there is a bounded set B in $C_p(X)$ with $B|_Y = D$ (e. g., if each $C_p(Y)$ admits a continuous linear extender), then $C_p(X)$ is distinguished.*

Proof If A is bounded in $\mathbb{R}^X$ and $Y \subseteq X$, then $\mathcal{P}_Y(A)$ is bounded in $\mathbb{R}^X$. If $C_p(X)$ is distinguished, Theorem 10 implies $\mathcal{P}_Y(A) \subseteq \overline{B}$ for some bounded set B in $C_p(X)$. Applying the idempotent $\mathcal{P}_Y$, we get $\mathcal{P}_Y(A) \subseteq \mathcal{P}_Y\left(\overline{B}\right) \subseteq \overline{\mathcal{P}_Y(B)}$. The inclusion $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}$ just means the restriction sets $A|_Y$ and $B|_Y$ satisfy $A|_Y \subseteq \overline{B|_Y}$, closure in $\mathbb{R}^Y$, where $B|_Y$ is bounded in $C_p(Y)$ and $A|_Y$ could be any bounded set in $\mathbb{R}^Y$. This simultaneously proves (1) via Theorem 10 and necessity of the condition in (2).

We prove sufficiency. Let A be bounded in $\mathbb{R}^X$. For each $Y \in \mathcal{F}$ the condition posits a bounded set B_Y in $C_p(X)$ such that $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B_Y)}$. Thus if $x \in Y \in \mathcal{F}$, then

$$\phi_A(X) = \sup_{f \in A} |f(X)| \leq \sup_{h \in B_Y} |h(X)| \leq \sup_{h \in B} |h(X)| = \phi_B(x),$$

where the finite union $B := \bigcup_{Y \in \mathcal{F}} B_Y$ of bounded sets is bounded in $C_p(X)$. Since $\mathcal{F}$ covers X, the inequality $\phi_A(X) \leq \phi_B(X)$ holds for all $x \in X$. Appealing to Lemma 13, we have $A \subseteq P_{\phi_A} \subseteq P_{\phi_B} \subseteq \overline{B^+}$. Thus $C_p(X)$ is large in $\mathbb{R}^X$; the proof of (2) is complete.

Now (3) follows from (2). Indeed, if A is bounded in $\mathbb{R}^X$, so is $A|_Y$ in $\mathbb{R}^Y$. Largeness produces a bounded set D in $C_p(Y)$ with $A|_Y \subseteq \overline{D}^{\mathbb{R}^Y}$. By hypothesis there is a bounded set B in $C_p(X)$ with $B|_Y = D$. Therefore $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$ and (2) applies. □

The converse of the first statement of the previous theorem fails, see Example 38.

Corollary 17 *If E is a nontrivial topological vector space, the space $C_p(E)$ is not distinguished. In particular, $C_p\left(C_p(X)\right)$ and $C_p\left(\mathbb{R}^X\right)$ are never distinguished.*

Proof Since E contains a copy of $\mathbb{R}$ and $C_p(\mathbb{R})$ is not distinguished [17, Corollary 2.5], Theorem 16 ensures that $C_p(E)$ is not distinguished. □

Recall that a subset X of a Polish space is called an *analytic set* if X is a continuous image of a Polish space, or equivalently, a continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$.

Corollary 18 *If X is an analytic set (in particular, a Polish space), then $C_p(X)$ is distinguished if and only if X is countable.*

Proof Sufficiency is a consequence of Theorem 8. If X is uncountable, by the classical Alexandroff–Hausdorff theorem X contains a homeomorphic copy of the Cantor set Δ. So, $C_p(X)$ is not distinguished by the first statement of Theorem 16. □

Proposition 19 *Let $\{X_i : i \in I\}$ be a family of Tychonoff spaces and let $\oplus_{i\in I} X_i$ denote their topological sum. If each $C_p(X_i)$ is distinguished, then $C_p(\oplus_{i\in I} X_i)$ is distinguished.*

Proof As the locally convex space $C_p(\oplus_{i\in I} X_i)$ is a (linearly homeomorphic) copy of the product space $\prod_{i\in I} C_p(X_i)$, its strong dual is a copy of the direct sum $\oplus_{i\in I} L_\beta(X_i)$ of barrelled spaces. □

Proposition 20 *The space $C_p(X)$ is distinguished if and only if the subspace $C_p^b(X)$ of $C_p(X)$ consisting of all bounded functions is distinguished.*

Proof Since each pointwise bounded set A in $C(X)$ is contained in the closure in $C_p(X)$ of a pointwise bounded subset of $C^b(X)$, one has that $C_p^b(X)$ is a large subspace of $C_p(X)$. So, both strong duals are isomorphic. □

4 The borne number

In the search for nondistinguished spaces $C_p(X)$ the character $\chi\left(L_\beta(X)\right)$ of the strong dual $L_\beta(X)$ of $C_p(X)$ proves to be particularly useful. For emphasis and convenience, in place of $\chi\left(L_\beta(X)\right)$ we often simply write $bn(X)$, calling it the *borne number* of X. Thus the borne number $bn(X)$ of a Tychonoff space X is the least cardinality of a fundamental family of bounded sets of $C_p(X)$. For each non-normable metrizable locally convex space E, we know the dominating cardinal $\mathfrak{d}$ is the smallest size for a fundamental family of bounded sets in E [37, Proposition 1] (see also [19, Theorem 2.3.2]). Therefore, when $|X| = \aleph_0$ we have

$$\chi\left(L_\beta(X)\right) = bn(X) = \mathfrak{d}.$$

It is well known that $\aleph_0 < \mathfrak{d} \le \mathfrak{c}$ and one may assume that $\mathfrak{d} = \aleph_1$, or that $\mathfrak{d} > \aleph_1$; either choice is ZFC-consistent.

Theorem 21 *If E denotes the bidual of $C_p(X)$, then $\mathfrak{d} \le bn(X) \le |E|$.*

Proof Since X is an infinite Tychonoff space, there is a sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty pairwise disjoint open sets in X and a sequence $\{f_n : n \in \mathbb{N}\}$ of functions in $C(X)$ such that each f_n vanishes off U_n but is not identically 0. If F denotes the linear span of the f_n with the topology induced by $C_p(X)$, it is clear that F is a non-normable metrizable space, a copy of a dense subspace of $\mathbb{R}^{\mathbb{N}}$. By intersecting F with members of a fundamental family $\mathcal{G}$ of bounded sets in $C_p(X)$, one obtains a fundamental family $\mathcal{H}$ in F with $|\mathcal{G}| \geq |\mathcal{H}|$. But $|\mathcal{H}| \geq \mathfrak{d}$ by [37, Proposition 1], as noted above. Thus $|\mathcal{G}| \geq \mathfrak{d}$, and since $\mathcal{G}$ is arbitrary, $bn(X) \geq \mathfrak{d}$. If B is bounded in $C_p(X)$, Lemma 13 implies $\phi_B \in \overline{B^+}$. Hence $\phi_B \in E$. Moreover, $B \subseteq P_{\phi_B} \cap C(X)$. Thus the intersections form a fundamental family $\mathcal{F}$ of bounded sets indexed by all the ϕ_B, a subset of E, so that $|\mathcal{F}| \leq |E|$. □

Recall that a *fundamental bounded resolution* for a locally convex space E is a fundamental family $\mathcal{F} = \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded sets in E with $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$. Since the dominating cardinal $\mathfrak{d}$ is also *strongly* dominating [37, p.138], there is a fundamental subfamily $\mathcal{F}_0$ with $|\mathcal{F}_0| \leq \mathfrak{d}$. This, combined with Theorem 21, yields

Proposition 22 *If $C_p(X)$ has a fundamental bounded resolution, then $bn(X) = \mathfrak{d}$.*

In [11, Theorem 3.3] it is shown that $C_p(X)$ has a fundamental bounded resolution if and only if X is countable, which provides an independent proof.

Observe the relation $bn(X) \leq 2^{|C(X)|}$ always holds, since $2^{|C(X)|}$ is the number of subsets of $C(X)$. So, if X is separable we have in particular $bn(X) \leq 2^{\mathfrak{c}}$.

A subspace Y of a topological space X is called a *retract of* X if there is a *retraction* φ from X onto Y, i.e., a surjective continuous map φ such that $\varphi(y) = y$ for all $y \in Y$.

Proposition 23 *If X is a Tychonoff space and Y is a retract of X then $bn(Y) \leq bn(X)$.*

Proof Let $\mathcal{F}$ be a fundamental family of bounded sets in $C_p(X)$. Denote by $S : C_p(X) \to C_p(Y)$ the restriction map $Sf = f|_Y$ and by $T : C_p(Y) \to C_p(X)$ the embedding map $Tg = g \circ \varphi$. Observe that $\mathcal{B} = \{S(Q) : Q \in \mathcal{F}\}$ is a fundamental family of bounded sets in $C_p(Y)$. In fact, if P is a bounded set in $C_p(Y)$ then $T(P)$ is a bounded set in $C_p(X)$. Hence, there is $Q \in \mathcal{F}$ such that $T(P) \subseteq Q$. Since $\varphi|_Y = \mathrm{id}_Y$, we get

$$S(Q) \supseteq S(T(P)) = S(\{g \circ \varphi : g \in P\}) = P,$$

which shows that the family $\mathcal{B}$ of bounded sets of $C_p(Y)$ swallows the bounded sets in $C_p(Y)$. So $bn(Y) \leq |\mathcal{B}| \leq |\mathcal{F}|$, which implies that $bn(Y) \leq bn(X)$. □

A subspace Y of a space X is said to be G_δ*-dense* in X if each nonempty G_δ-set in X meets Y. The following result will be used in Theorem 41 and Example 62 below.

Proposition 24 *Let Y be a G_δ-dense subspace of a Tychonoff space X. A set A in $C_p(X)$ is bounded if and only if A is bounded at the points of Y. If Y is C-embedded in X, then $bn(X) = bn(Y)$.*

Proof If A is a bounded set in $C_p(X)$, clearly A is pointwise bounded on Y. If A is unbounded at some point x in X, choose a sequence $\{f_n\}_{n=1}^{\infty}$ in A with each $|f_n(X)| > n$. The G_δ-set $G := \bigcap_{n=1}^{\infty} f_n^{-1}(\{r \in \mathbb{R} : |r| > n\})$ contains x, thus is nonempty and meets Y at some point y. But $f_n(y) > n$ by definition of G, so A is unbounded at the point $y \in Y$. This proves (the contrapositive of) the first part. For the last part note that the (unique) extensions of bounded sets in $C_p(Y)$ are bounded in $C_p(X)$. □

Corollary 25 *If Y is dense and C-embedded in a P-space X, then $bn(X) = bn(Y)$.*

Proof In a P-space each nonempty G_δ-set is open, so dense subspaces are G_δ-dense. □

We shall require the following well-known fact, of which we include a proof.

Lemma 26 *Let X be a dense subset of a Tychonoff space T. If X is C-embedded, then X is G_δ-dense in T.*

Proof If not, there exist U_n open in T and $t \in G := \bigcap_{n=1}^{\infty} U_n$ with $G \cap X = \emptyset$. We may assume $\overline{U_{n+1}} \subseteq U_n$. Choose $f_n \in C(T)$ with $0 \le f_n \le 1$ such that $f_n(t) = 1$ and $f_n(y) = 0$ if $y \in T \backslash U_n$. For each $x \in X$, the sequence $\{f_n(x) : n \in \mathbb{N}\}$ is eventually 0 since $x \notin U_n$ for sufficiently large n. Setting $h(X) := \sum_{n=1}^{\infty} f_n(X)$ for each $x \in X$, note that $h \in C(X)$. This follows from the facts that (i) on the open set $X \backslash \overline{U_{n+1}}$ in X, our function h agrees with the continuous $\sum_{i=1}^{n} f_i$, and (ii) the sets $X \backslash \overline{U_{n+1}}$ cover X, since smaller sets $X \backslash U_n$ cover. By hypothesis, h admits an extension $\overline{h} \in C(T)$. For each $n \in \mathbb{N}$ there is a neighborhood V_n of t in T such that if $s \in V_n$ and $1 \le k \le n+1$, then $f_k(s) > \frac{n}{n+1}$. If U is any neighborhood of t in T, density provides $x_n \in V_n \cap U \cap X$, so that $h(x_n) \ge \sum_{k=1}^{n+1} f_k(x_n) > \frac{(n+1)n}{n+1} = n$. Therefore $\overline{h}$ is unbounded on the arbitrary neighborhood U, which means $\overline{h}$ cannot be continuous at t, a contradiction. □

Conversely, each G_δ-dense subspace X of a perfectly κ -normal space T (i. e., such that the closure of each open subset of T is a G_δ-set in T) is C-embedded in T, [41, Theorem 2]. Since υX is the largest topological subspace of the Stone-Čech compactification βX of X in which X is C-embedded, Lemma 26 and Proposition 24 yield

Corollary 27 *If X is a Tychonoff space, then $bn(\upsilon X) = bn(X)$.*

5 Nondistinguished $C_p(X)$ spaces

We exhibit spaces $C_p(X)$ which are not distinguished, often via the contrapositive of our Theorem 11, conveniently stated here.

Theorem 28 [17, Theorem 2.1] *If $bn(X) \le |X|$, then $C_p(X)$ is not distinguished.*

Lemma 29 *If each closed bounded set in $C_p(X)$ is separable, then $bn(X) \le |C(X)|^{\aleph_0}$.*

Proof The number of sequences in $C_p(X)$ cannot exceed $|C_p(X)|^{\aleph_0}$, where $|C_p(X)| \ge \mathfrak{c}$. Given that every closed bounded set in $C_p(X)$ is separable, each closed bounded set in $C_p(X)$ is the closure of a sequence in $C_p(X)$. So, the number of closed bounded sets does not exceed the number of sequences. As the family $\mathcal{F}$ of closed bounded sets in $C_p(X)$ is a fundamental family of bounded sets, it follows that $|\mathcal{F}| \le |C_p(X)|^{\aleph_0}$. Therefore $bn(X) \le |\mathcal{F}| \le |C(X)|^{\aleph_0}$. □

Corollary 30 *Let X be a separable Tychonoff space with $|X| = \mathfrak{c}$. If $C_p(X)$ is hereditarily separable, then $C_p(X)$ is not distinguished.*

Proof The separability of X yields $|C(X)| = 2^{\aleph_0}$, so $|C(X)|^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = |X|$. Since $C_p(X)$ is hereditarily separable, Lemma 29 ensures that $bn(X) \le |C_p(X)|^{\aleph_0} = |X|$. Thus, Theorem 28 yields the conclusion. □

Recall that a family $\mathcal{N}$ of subsets of a Tychonoff space X is a *network* of X if for any $x \in X$ and any open set U in X with $x \in U$ there is some $P \in \mathcal{N}$ such that $x \in P \subseteq U$. The *network weight* $nw(X)$ of X is the least cardinality of a network of X, and a space X is called *cosmic* if $nw(X) = \aleph_0$. In particular, each metrizable separable space is cosmic. In what follows we denote by $d(X)$ the *density character* of X.

Corollary 31 [17, Corollary 2.4] *If X is cosmic with $|X| = \mathfrak{c}$, then $C_p(X)$ is not distinguished.*

Proof If X has a countable network, $C_p(X)$ has a countable network [29]. Since each space with a countable network is hereditarily separable and inequality $d(X) \leq nw(X)$ always holds true, X is separable. So, Corollary 30 yields the conclusion. □

Since $C_p(X)$ is cosmic if and only if X is cosmic [29, Proposition 10.5], and each cosmic space is hereditarily separable, Corollary 30 is more general than Corollary 31 . Apart from cosmic spaces X, the Velichko-Zenor theorem ([3, 2.5.10 Theorem] or [44,45]) tells us that $C_p(X)$ is hereditarily separable if X^n is hereditarily Lindelöf for each $n \in \mathbb{N}$.

Example 32 Nondistinguished hereditarily separable $C_p(X)$ spaces. *Under CH there is a strong S-space Y with $|Y| = \omega_1$, where ω_1 denotes the first uncountable ordinal (see* [42, Problem 098]). *So, Y^n is hereditarily separable for each $n \in \mathbb{N}$ but Y is not hereditarily Lindelöf, in fact even not Lindelöf. Then, by the Velichko–Zenor theorem, the space $C_p(Y)^n$ is hereditarily Lindelöf for each $n \in \mathbb{N}$. Let D be a countable dense subset of $\mathbb{R}^{\omega_1}$ and consider the space $X := D \cup C_p(Y)$. Note that X is separable, since $D \subseteq X \subseteq \mathbb{R}^{\omega_1}$ and D is dense in $\mathbb{R}^{\omega_1}$, and X^n is clearly hereditarily Lindelöf for each $n \in \mathbb{N}$. So, again by the Velichko–Zenor theorem $C_p(X)$ is hereditarily separable. Observe that X is not a cosmic space, otherwise $C_p(Y)$ would be cosmic, as well as $C_p(Y)^n$, so that $C_p(Y)^n$ would be hereditarily separable for each $n \in \mathbb{N}$. Hence $C_p(C_p(Y))$ would be hereditarily Lindelöf. But then Y, which is homeomorphically embedded in $C_p(C_p(Y))$, would be a Lindelöf space, which is not true. Note that $|X| = \mathfrak{c}$ and, since X is separable, Corollary 30 asserts that $C_p(X)$ is not distinguished. In addition $C_p(X)$ is not cosmic, since X is not cosmic either.*

Since each compact cosmic space X is metrizable [2], the next theorem also extends Corollary 31 for compact X, which corresponds to the case $d(X) = \aleph_0$ and $|X| = 2^{\aleph_0}$. If X is a scattered compact then $|X| = d(X)$ (see [39]), so we always have $|X| < 2^{d(X)}$. Consequently, then next theorem makes sense for non scattered compact spaces.

Theorem 33 *Let X be a compact space such that $d(X) = d(C_k(X))$. If $|X| \geq 2^{d(X)}$, then $C_p(X)$ is not distinguished.*

Proof Let $\mathcal{F}$ be a dense subspace of $C_k(X)$ with $|\mathcal{F}| = d(C_k(X))$. Then for each $f \in C(X)$ and $0 < \epsilon < 1$ there is $g_f \in \mathcal{F}$ such that $\|f - g_f\| < \epsilon$, where $\|\cdot\|$ is the supremum norm of the Banach space $C_k(X)$. If A is a bounded set in $C_p(X)$ we claim that $Q_A := \{g_f \in \mathcal{F} : f \in A\} \subseteq \mathcal{F}$ is a bounded set in $C_p(X)$. Otherwise there is $x \in X$ and a sequence $\{g_{f_n}\}_{n=1}^{\infty} \in Q_A$ with $|g_{f_n}(X)| \geq n + 1$ for every $n \in \mathbb{N}$. Consequently

$$|f_n(X)| \geq |g_{f_n}(x)| - \|f_n - g_{f_n}\| \geq n + 1 - \epsilon \geq n$$

for each $n \in \mathbb{N}$, which contradicts the fact that A is pointwise bounded. Moreover, by the definition of Q_A one has $A \subseteq \overline{Q_A}^{\tau_k}$, where the closure is in $C_k(X)$.

If Bound $(C_p(X))$ is the family of all bounded sets in $C_p(X)$, we have seen that for each $A \in$ Bound $(C_p(X))$ there is $Q_A \subseteq \mathcal{F}$ such that $A \subseteq \overline{Q_A}^{\tau_p}$. Since $\overline{Q_A}^{\tau_p} \in$ Bound $(C_p(X))$, this shows that there is a fundamental bounded family of closed bounded sets in $C_p(X)$ of cardinality at most $2^{|\mathcal{F}|} = 2^{d(X)}$. Hence $bn(X) \leq 2^{d(X)} \leq |X|$. Therefore $C_p(X)$ is not distinguished by virtue of Theorem 28. □

Recall that the *Sorgenfrey line* $\mathbb{S}$ is the space $(\mathbb{R}, \tau)$, where a base for the topology τ is the family $\{[a, b) : a, b \in \mathbb{R}, a < b\}$.

Example 34 The space $C_p(\mathbb{S})$ is not distinguished.

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence of distinct points in $\mathbb{S}$. Define $h \in \mathbb{R}^{\mathbb{S}}$ so that $\lim_{n\to\infty} h(x_n) = \infty$. If the closure $\overline{A}$ in $\mathbb{R}^{\mathbb{S}}$ of a set A in $C_p(\mathbb{S})$ contains h, choose $n_1 \in \mathbb{N}$ such that $h\left(x_{n_1}\right) > 1$ and set $a_1 = x_{n_1}$. Since $h \in \overline{A}$, there exists $f_1 \in A$ with $f_1(a_1) > 1$. Continuity at a_1 yields $b_1 > a_1$ such that $f_1([a_1, b_1]) \subseteq (1, +\infty)$. Set $I_1 = [a_1, b_1]$. Since infinitely many x_n lie in the open interval (a_1, b_1), there exists $x_{n_2} \in (a_1, b_1)$ with $h\left(x_{n_2}\right) > 2$. For $a_2 := x_{n_2}$ there exists $f_2 \in A$ such that $f_2(a_2) > 2$. Continuity provides $b_2 > a_2$ such that $f_1([a_2, b_2]) \subseteq (2, +\infty)$, and we may assume $b_2 \leq b_1$, so that $I_2 := [a_2, b_2] \subseteq I_1$. Continuing inductively, we obtain a sequence of functions $f_n \in A$ and nested non-degenerate closed intervals I_n of finite length such that $f_n(y) > n$ for all $y \in I_n$. Clearly, then, A is not bounded in $C_p(\mathbb{S})$ since it is unbounded at the point(s) in the nonempty intersection of the intervals I_n. This proves that h is not in the bidual E of $C_p(\mathbb{S})$. Thus $E \neq \mathbb{R}^{\mathbb{S}}$ and $C_p(\mathbb{S})$ is not distinguished. □

Corollary 35 *If D denotes the double arrow space, then $C_p(D)$ is not distinguished.*

Proof As is well known D contains two homeomorphic copies of the Sorgenfrey line $\mathbb{S}$, say the subspaces $S_0 = (0, 1) \times \{0\}$ and $S_1 = (0, 1) \times \{1\}$. If $C_p(D)$ were distinguished, then $C_p(S_1)$ would be distinguished by the first part of Theorem 16. Consequently $C_p(\mathbb{S})$ would be distinguished, which is not true. Hence, $C_p(D)$ is not distinguished. □

Since the topology τ on $\mathbb{R}$ is stronger than the usual, the previous example also shows that $C_p(\mathbb{R})$, where here $\mathbb{R}$ is equipped with the usual topology, is not distinguished. Of course, this is also consequence of Corollary 31.

Example 36 $C_p(H)$ over the Helly space H is not distinguished. *Helly's space H is the closed set of the compact space $[0, 1]^{[0,1]}$ consisting of all non-decreasing functions equipped with the relative product topology. It is separable and not metrizable. The constant functions compose a homeomorphic copy of the interval $[0, 1]$ (even Hilbert's parallelotope $P = [0, 1]^{\aleph_0}$ is embedded in H, as noted in [33]). So, Corollary 31 together with Theorem 16 ensure that $C_p(H)$ is not distinguished. Similarly, since the long line L is locally homeomorphic to $\mathbb{R}$, neither is $C_p(L)$ a distinguished space.*

Regarding the next proposition, recall that for a topological space X the inequality $|X| \leq 2^{2^{d(X)}}$ holds in general [34], so for separable X one has $|X| \leq 2^{2^{\aleph_0}}$.

Proposition 37 *Let X be an infinite Tychonoff space. If $|X| = 2^{2^{d(X)}}$, then $C_p(X)$ is not distinguished.*

Proof Set $\kappa := d(X)$. Clearly $|C(X)| \leq \mathfrak{c}^{\kappa} = \left(2^{\aleph_0}\right)^{\kappa} = 2^{\kappa}$, since $\aleph_0 \leq \kappa$. So, there are at most $2^{2^{\kappa}}$ bounded subsets in $C(X)$. Hence $bn(X) \leq 2^{2^{\kappa}} = |X|$, and again Theorem 28 yields the conclusion. □

Example 38 If X is an infinite discrete space, then $C_p(\beta X)$ is not distinguished. *Note that $d(\beta X) = |X| \geq \aleph_0$. Since $|\beta X| = 2^{2^{|X|}}$, we apply the previous proposition. The fact that $C_p(\beta\mathbb{N})$ is nondistinguished was noticed in* [17, Corollary 2.2].

Corollary 39 *If X is separable with $|X| = 2^{\mathfrak{c}}$, then $C_p(X)$ is not distinguished.*

Proof If X is separable, $d(X) = \aleph_0$. As $|X| = 2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$, Proposition 37 applies. □

Example 40 Both $C_p(\beta\mathbb{Q})$ and $C_p(\beta\mathbb{R})$ are not distinguished. *According to* [20, 9.3], *one has* $|\beta\mathbb{Q}| = |\beta\mathbb{R}| = 2^{\mathfrak{c}}$. *Since* $\beta\mathbb{Q}$ *and* $\beta\mathbb{R}$ *are separable, the previous corollary applies.*

Proposition 41 *Let* X *be an infinite Tychonoff space and let* Y *be a dense and* C*-embedded subset of* X. *If* $|X| \geq 2^{|Y|}$, *then* $C_p(X)$ *is not distinguished.*

Proof If E denotes the bidual of $C_p(X)$, Theorem 21, Proposition 24 and Lemma 26 yield $bn(X) = bn(Y) \leq |E| \leq 2^{|Y|} \leq |X|$. So, Theorem 28 does the job. □

Corollary 42 *If* $|\upsilon X| \geq 2^{|X|}$, *then* $C_p(\upsilon X)$ *is not distinguished.*

Example 43 *Haydon's third example* $\mathbb{H}$ *is a pseudocompact space of cardinal* $\mathfrak{c}$ *which contains* $\mathbb{N}$ *so that* $\upsilon\mathbb{H} = \beta\mathbb{H} = \beta\mathbb{N}$ [22]. *Hence* $|\upsilon\mathbb{H}| = 2^{\mathfrak{c}} = 2^{|\mathbb{H}|}$ *and Corollary* 42 *ensures that* $C_p(\upsilon\mathbb{H})$ *is not distinguished. Nonetheless, since* $\upsilon\mathbb{H} = \beta\mathbb{N}$ *this is also a consequence of* [17, Corollary 2.2]. *The same can be said of the first Marciszewski example* [27].

6 Distinguished $C_p(X)$ spaces

The Lindelöfication $L_{\mathfrak{m}}$ of an uncountable discrete space $D(\mathfrak{m})$ is a nondiscrete P-space such that $C_p(L_{\mathfrak{m}})$ is distinguished (see [13, Example 3.8] for $\mathfrak{m} = \aleph_1$). Let us show that there are also distinguished $C_p(X)$ spaces over uncountable *compact* X. First we introduce some useful machinery.

Definition 44 We say that a family $\{\mathcal{N}_x : x \in X\}$ of subsets of a Tychonoff space X is a *scant cover* for X if each $\mathcal{N}_x$ is a neighborhood of x and for each $u \in X$ the set $X_u := \{x \in X : u \in \mathcal{N}_x\}$ is finite.

Example 45 Each countable Tychonoff space $X = \{x_n : n \in \mathbb{N}\}$ admits a scant cover. *Simply set* $\mathcal{N}_{x_n} = \{x_i : i \geq n\}$ *for each* $n \in \mathbb{N}$.

Theorem 46 *If* X *admits a scant cover* $\{\mathcal{N}_x : x \in X\}$ *then* $C_p(X)$ *is distinguished.*

Proof Let A be bounded in $\mathbb{R}^X$. For each $y \in X$ choose $0 \leq f_y \in C(X)$ such that f_y vanishes off $\mathcal{N}_y$; $f_y(y) = \phi_A(y)$; and $f_y(x) \leq \phi_A(y)$ for all $x \in X$. Since a given x is in $\mathcal{N}_y$ for only finitely many $y \in X$, the supremum $\sup_{y \in X} |f_y(x)|$ cannot exceed the maximum of finitely many numbers of the form $\phi_A(y)$. Hence $B := \{f_y : y \in X\}$ is a bounded set in $C_p(X)$. Lemma 13 implies $P_{\phi_B} \subseteq \overline{B^+}$. But $\phi_A \leq \phi_B$, so $A \subseteq P_{\phi_A} \subseteq P_{\phi_B} \subseteq \overline{B^+}$. Therefore $C_p(X)$ is large in $\mathbb{R}^X$. □

Corollary 47 *If* X *has only finitely many non-isolated points, then* $C_p(X)$ *is distinguished.*

Proof If $X = \Gamma \cup \{u_1, \ldots, u_n\}$, where all points $x \in \Gamma$ are isolated in X, the family $\{\mathcal{N}_x : x \in X\}$ consisting of $\mathcal{N}_x = \{x\}$ if $x \in \Gamma$ and $\mathcal{N}_{u_i} = X$ if $1 \leq i \leq n$ is a scant cover for X. Theorem 46 applies. □

Corollary 48 *If* X *is a discrete space and* $\alpha(X)$ *stands for the one-point compactification or the one-point Lindelöfication of* X, *then* $C_p(\alpha(X))$ *is distinguished.*

A family $\mathcal{F}$ of subsets of X is called *point-finite* [1] if each $x \in X$ belongs at most to finitely many members of $\mathcal{F}$. It is called σ*-point-finite* if $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where each $\mathcal{F}_n$ is point-finite. Note that every scant cover of X is point-finite, but not every point-finite (even clopen) cover is scant (e. g., take X any infinite space and set each $\mathcal{N}_x = X$).

Theorem 49 *Let X be a Corson compact space. The following are equivalent.*

(1) *X is scattered.*
(2) *X is a scattered Eberlein compact space.*
(3) *$C_p(X)$ is distinguished.*

Proof $1 \Rightarrow 2$. If X is a Corson scattered compact, then X is necessarily a scattered Eberlein compact space by Alster's theorem [1, Theorem].

$2 \Rightarrow 3$. If we regard the proof of [6, Lemma 1.1] with X an arbitrary scattered Eberlein compactum, for each $a \in X$ there is defined a clopen neighborhood V_a of a such that the family $\{V_a : a \in X\}$ is point-finite, with V_a and V_b (clearly) distinct for distinct $a, b \in X$. This shows that $\{V_a : a \in X\}$ is a scant cover for X, so Theorem 46 shows that $C_p(X)$ is distinguished.

$3 \Rightarrow 1$. Assume that $C_p(X)$ is distinguished but X is a non-scattered Corson compact space. According to the classical theorem of Pełczyński and Semadeni, there is a continuous surjection f from X onto the closed interval $[0, 1]$. We claim that there exists a compact set Y in X which is metrizable and $|Y| = \mathfrak{c}$. Indeed, fix any countable dense subset Q of $[0, 1]$ and choose a countable subset P in X such that $f(P) = Q$. Let Y be the closure of P in X. Clearly Y is metrizable, since it is a Corson separable compact space. In addition, Y must have the cardinality of continuum since by the density of P in Y and the density of Q in $[0, 1]$ one has $f(Y) = [0, 1]$. Since $C_p(Y)$ is not distinguished by Corollary 31, neither is $C_p(X)$ distinguished, by the first statement of Theorem 16. □

Remark 50 *Since $d(X) = d(C_p(X))$ for any Corson compact space with $d(X) \geq \aleph_0$, whereas in general $d(C_k(X)) = d(C_p(X))$ [28], then $d(X) = d(C_k(X))$ holds for Corson compact spaces. So, Theorem 33 works for a nonscattered Corson compact X. The previous result improves this fact.*

Let us recall that a family $\mathcal{U}$ of subsets of X is called *separating* if given any two distinct points $x, y \in X$, there is a member $U \in \mathcal{U}$ such that either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$. A known Rosenthal theorem says that a compact space X is an Eberlein compact if and only if X has a σ-point-finite separating family of open F_σ-subsets. The next theorem provides a concrete class of compact spaces for which Theorem 46 applies.

Theorem 51 *A compact space X admits a scant cover if and only if X is a scattered Eberlein compact.*

Proof If X is a scattered Eberlein compact, then X admits a scant cover by the argument of Theorem 49 (see $2 \Rightarrow 3$). Assume now that X admits a scant cover $\nu = \{\mathcal{N}_x : x \in X\}$. Let us refine ν . Fix any $u \in X$. By definition, there are only finitely many points $x_i \neq u$ in X such that $u \in \mathcal{N}_{x_i}$. Since X is a Tychonoff space we can choose an open F_σ-set $\mathcal{M}_u$ such that $u \in \mathcal{M}_u \subseteq \mathcal{N}_u$ and, yet, $\mathcal{M}_u$ is disjoint from all points x_i. Evidently, the new family $\mu = \{\mathcal{M}_x : x \in X\}$ is a scant cover for X because μ is a shrinking of ν. Let us show that μ separates points of X. Take distinct points $x, y \in X$. If $y \in \mathcal{M}_x$, by our construction we have $x \notin \mathcal{M}_y$. Hence μ is a point-finite separating family of open F_σ-subsets of X. Therefore, X is an Eberlein compact by Rosenthal's characterization. Further, $C_p(X)$ is distinguished by Theorem 46. So, X is scattered by Theorem 49. □

Corollary 52 *A non-compact locally compact space X admits a scant cover if and only if the one-point compactification of X is a scattered Eberlein compact.*

Proof Denote the one-point compactification of X by $\alpha(X)$. Observe that X is open in $\alpha(X)$. Let ν be a scant cover of X. It is easy to see that by adding to ν the set $\alpha(X)$ we obtain a scant cover of $\alpha(X)$. So, the compact space $\alpha(X)$ is a scattered Eberlein compact by Theorem 51. Conversely, it suffices to observe that if μ is a scant cover of $\alpha(X)$ then $\nu = \{U \cap X : U \in \mu\}$ is a scant cover of X. □

Remark 53 The one-point compactification of a non-compact locally compact space need not have a scant cover. *In fact, as follows from Corollary 52, if X is any compact space which is not a scattered Eberlein compact and we remove one point, say $y \in X$, we obtain a non-compact locally compact space $Y = X\backslash\{y\}$ which does not admit a scant cover. In particular, the interval* $(0, 1]$ *or the locally compact space of all countable ordinals ω_1 equipped with the order topology does not admit a scant cover.*

Let us recall that a space X is *strongly σ -discrete* if it is the union of countably many of its closed discrete subspaces (see [43, 1.5]). A topological space X is *strongly splittable* if each $f \in \mathbb{R}^X$ is the pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ in $C(X)$. Countable Tychonoff spaces and discrete spaces are strongly splittable, as well as every normal strongly σ-discrete space. Since each $C_p(X)$ over a P-space X is sequentially complete [16, Theorem 1.1], a P-space is strongly splittable if and only if it is discrete.

Proposition 54 *If X is a strongly splittable Tychonoff space, $C_p(X)$ is distinguished.*

Proof Cauchy sequences in $C_p(X)$ are bounded, so the bidual of $C_p(X)$ is $\mathbb{R}^X$ and the conclusion follows from Theorem 14. □

Remark 55 *The converse fails: the one-point compactification X of an uncountable discrete space is not strongly splittable* [43, U.417].

A similar result to Corollary 18 is valid for absolutely analytic metrizable topological spaces. Recall that a metrizable space Y is called *absolutely analytic* if Y is homeomorphic to a Souslin subspace of a complete metric space X (of an arbitrary weight), i.e., if Y is expressible as

$$Y = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} Y_{\alpha|n}$$

where each $Y_{\sigma|n}$ is a closed subset of X. Every Borel subspace of a complete metric space is an absolutely analytic space (see [21]).

Proposition 56 *For an absolutely analytic metrizable space X, these are equivalent.*

(1) *X is strongly σ-discrete.*
(2) *$C_p(X)$ is distinguished.*

Proof By the main result of [10], every absolutely analytic metrizable space X either contains a homeomorphic copy of the Cantor set Δ or it is strongly σ-discrete. Since $C_p(\Delta)$ is not distinguished, we have that $2 \Rightarrow 1$. Implication $1 \Rightarrow 2$ is an immediate consequence of Proposition 54. □

Next we establish a permanence property for distinguished $C_p(X)$ spaces when X is enlarged with a finite set of points of $\beta X\backslash X$.

Theorem 57 *Let $X = Y \cup Z$, where Y is finite and every $f \in C^b(Z)$ has an extension $\widetilde{f} \in C(X)$. Then $C_p(X)$ is distinguished if and only if $C_p(Z)$ is distinguished.*

Proof First statement of Theorem 16 proves one part. For the other, we assume $C_p(Z)$ is distinguished and use the second statement of Theorem 16. Finite induction reduces the proof to the case in which Y consists of a single point $y_0 \notin Z$. If A is bounded in $\mathbb{R}^X$, its bounded one-dimensional projection $\mathcal{P}_Y(A)$ easily lies in $\mathcal{P}_Y(B)$ for some aptly chosen one-dimensional set B. We still must find a suitable set B for the other projection $\mathcal{P}_Z(A)$. Since $C_p(Z)$ is distinguished, so is $C_p^b(Z)$. Hence there is a bounded set B_0 in $C_p^b(Z)$ whose closure in $\mathbb{R}^Z$ contains $A|_Z$. The extension hypothesis then provides a set B_1 in $C_p(X)$ such that $B_1|_Z = B_0$. Thus we have (i) $\mathcal{P}_Z(A) \subseteq \overline{\mathcal{P}_Z(B_1)}^{\mathbb{R}^X}$. If B_1 is bounded at y_0, then B_1 is bounded on all of X, and part (2) of Theorem 16 yields the conclusion with $B = B_1$. If B_1 is unbounded at y_0, we form a new set B in $C_p(X)$ by replacing each function f in B_1 with functions $f_1, f_2, \ldots$ such that each $f_n = f - a_n h_n$, where $h_n \in B_1$, $|a_n| \leq 1/n$, and $f(y_0) = a_n h_n(y_0)$. Now each $f_n(y_0) = 0$ and if $z \in Z$ we have

$$|f_n(z)| \leq |f(z)| + |h_n(z)| \leq 2 \sup_{g \in B_0} |g(z)|,$$

so the new set B is bounded at all points of X. Also, for each f in B_1 the f_n converge pointwise to $\mathcal{P}_Z(f)$. Consequently, we have (ii) $\mathcal{P}_Z(B_1) \subseteq \overline{\mathcal{P}_Z(B)}^{\mathbb{R}^X}$. Together, (i) and (ii) imply $\mathcal{P}_Z(A) \subseteq \overline{\mathcal{P}_Z(B)}^{\mathbb{R}^X}$. □

Corollary 58 *If $X = Y \cup Z$, where Y is a finite subset of $\beta Z \backslash Z$, then $C_p(X)$ is distinguished if and only if $C_p(Z)$ is distinguished.*

The next challenge could be the case where $|Y| = \aleph_0$. We note that in Michael's line (below) we have $|Y| = \aleph_0$ and Z is discrete, so that both $C_p(Y)$ and $C(Z)$ are distinguished, but $C_p(X)$ is not distinguished.

If we equip $\mathbb{Q}$ with the relative topology of $\mathbb{R}$ and $\mathbb{P}$ with the discrete topology and set $X := \mathbb{Q}$ (relative) $\oplus$ $\mathbb{P}$ (discrete), it follows from Proposition 19 that $C_p(X)$ is distinguished. However, since $C_p(\mathbb{P})$ is not distinguished when $\mathbb{P}$ has the relative topology of $\mathbb{R}$, if we retain on $\mathbb{P}$ the usual open sets of $\mathbb{R}$ and equip $\mathbb{Q}$ with the discrete topology, setting $X :=$ $\mathbb{Q}$ (discrete) $\cup$ $\mathbb{P}$ (relative), Theorem 16 assures that $C_p(X)$ is not distinguished.

In what follows we are going to consider topological spaces of Michael's type (see [30]). We start with a Tychonoff space Z and a proper topological subspace Y of Z. We retain on Y the original topology of Z and equip $Z \backslash Y$ with the discrete topology. Let us denote by X the space Z endowed with this stronger topology. If $x \in Z \backslash Y = X \backslash Y$ then $\{x\}$ is an open neighborhood of x in X, but if $x \in Y$ a basic open neighborhood of x in X is just an open neighborhood U of x in Z. Note that X is a Tychonoff space. Indeed, X is clearly T_2 and if $x \in X$ and Q is a closed set in X with $x \notin Q$, two cases are in order. If $x \in Y$ there is an open neighborhood U of x in Z with $U \cap Q = \emptyset$. So there is $f \in C(Z) \subseteq C(X)$ with $0 \leq f \leq 1$ such that $f(X) = 1$ and $f(y) = 0$ for every $y \in X \backslash U$, in particular $f(y) = 0$ for $y \in Q$. If $x \in X \backslash Y$ we choose $g \in \mathbb{R}^X$ defined by $g(X) = 1$ and $g(y) = 0$ if $y \neq x$. Clearly $g \in C(X)$, $0 \leq g \leq 1$, and $g(y) = 0$ for each $y \in Q$. Let us start with a negative example, namely with the original Michael line, where the rationals retain the original topology of $\mathbb{R}$ and the irrationals are declared to be isolated.

Example 59 If $\mathbb{M}$ denotes the Michael line, the space $C_p(\mathbb{M})$ is not distinguished.

Proof Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of the rationals. Since $\mathbb{M}$ and $\mathbb{R}$ produce the same neighborhood base at x_n, one repeats verbatim the inductive proof of Example 34 to see that the bidual of $C_p(\mathbb{M})$ does not contain some $h \in \mathbb{R}^{\mathbb{M}}$. □

Theorem 60 *Let Y be a nonempty subspace of a Tychonoff space Z. Let X be Z with the topology of Michael's type obtained by retaining on Y the open sets of Z and declaring isolated the points of $Z \backslash Y$. Then $C_p(X)$ is distinguished if and only if for each bounded set A in $\mathbb{R}^X$ there is a bounded set B in $C_p(X)$ with $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$.*

Proof Let A be bounded in $\mathbb{R}^X$. For each $f \in A$ and each finite set Δ in $X \backslash Y$, let f_Δ agree with f on Δ and vanish on $X \backslash \Delta$. Clearly, $f_\Delta \in C(X)$ and the collection B of all such f_Δ is bounded in $C_p(X)$ since A is bounded. In addition $\mathcal{P}_{X \backslash Y}(A) \subseteq \overline{\mathcal{P}_{X \backslash Y}(B)}^{\mathbb{R}^X}$. Now with $\mathcal{F} := \{Y, X \backslash Y\}$, apply Part (2) of Theorem 16. □

Recall that a subspace $Y \subseteq Z$ is l-embedded in Z if there exists a continuous linear extender $\varphi : C_p(Y) \to C_p(Z)$, [4]. In particular, each metrizable compact subspace Y is l-embedded in Z [5].

Corollary 61 *Let Z be a Tychonoff space and Y be a subspace of Z. Denote by X the space Z with the topology of Michael's type obtained by retaining on Y the original topology and by declaring isolated the points of $Z \backslash Y$. If either* (i) *Y is dense and C-embedded in Z or* (ii) *Y is l-embedded in Z, then $C_p(X)$ is distinguished if and only if $C_p(Y)$ is distinguished.*

Proof Assume that $C_p(Y)$ is distinguished. In the first case, if A is bounded in $\mathbb{R}^X$ then, since Y is C-embedded, there is a set B in $C_p(X)$ such that B is bounded at the points of Y with $A|_Y \subseteq \overline{B|_Y}^{\mathbb{R}^Y}$, i.e., with $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$. Proposition 24 and Lemma 26 imply B is bounded in $C_p(X)$, so Theorem 60 assures $C_p(X)$ is distinguished. In the second case, if A is a bounded set in $\mathbb{R}^X$, there is a bounded set Q in $C_p(Y)$ whose closure in $\mathbb{R}^Y$ contains $A|_Y$. Now, if $\varphi : C_p(Y) \to C_p(Z)$ is a continuous linear extender, $B := \varphi(Q)$ is bounded in $C_p(X)$ and again $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$. □

Corollary 62 *Let Y be a Tychonoff space. On υY consider the stronger topology τ that coincides with the original on Y but the points of $\upsilon Y \backslash Y$ are declared to be isolated. If $X := (\upsilon Y, \tau)$, then $C_p(X)$ is distinguished if and only if $C_p(Y)$ is distinguished.*

We conclude by posing two questions which we haven't been able to solve.

Problem 63 *Is the space $C_p(\omega_1)$ distinguished?*

Theorem 28 seemingly cannot provide a *negative* answer, since (i) *bn* $(\omega_1) \geq \mathfrak{d}$ (Theorem 21), and (ii) one may assume $\mathfrak{d} > |\omega_1|$. On the other hand, scant covers cannot provide a *positive* answer since, according to Remark 53, there is no scant cover for ω_1.

Problem 64 *Let X be a compact space. Is it true that if $C_p(X)$ is distinguished, then X is scattered?*

Addendum. Problem 1 has recently been solved independently by Ferrando/Saxon and Kąkol/Leiderman. Problems 63 and 64 have been solved by Kąkol/Leiderman. All these solutions should appear soon.

Acknowledgements We wish to thank Vladimir Tkachuk for providing us Example 32 and Witold Marciszewski for references [1,6] as well as helpful discussion on scattered Eberlein compact spaces.

References

1. Alster, K.: Some remarks on Eberlein compacts. Fundam. Math. **104**, 43–46 (1979)
2. Arkhangel'skiĭ, A.V.: An addition theorem for the weight of sets lying in bicompacts (in Russian). Dokl. Akad. Nauk SSSR **126**, 239–241 (1959)
3. Arkhangel'skiĭ, A.V.: Topological Function Spaces. Kluwer, Dordrecht (1992)
4. Arkhangel'skiĭ, A.V.: Spaces of functions in pointwise topology. In: Hart, K.P., Nagata, J., Vaughan, J.E. (eds.) Encyclopedia of General Topology, pp. 131–137. Elsevier, Amsterdam (2004)
5. Arkhangel'skiĭ, A.V., Choban, M.: The extension property of Tychonoff spaces and generalized retracts. C. R. Acad. Bulg. Sci. **41**, 5–7 (1988)
6. Bell, M., Marciszewski, W.: On scattered Eberlein compact spaces. Isr. J. Math. **158**, 217–224 (2007)
7. Bierstedt, D., Bonet, J.: Dual density conditions in (DF)-spaces I. Results Math. **14**, 242–274 (1988)
8. Bierstedt, D., Bonet, J.: Density conditions in Fr échet and (DF)-spaces. Rev. Mat. Complut. **2**, 59–75 (1989)
9. Bierstedt, D., Bonet, J.: Some aspects of the modern theory of Fréchet spaces. RACSAM **97**, 159–188 (2003)
10. El'kin, A.G.: A-sets in complete metric spaces. Dokl. Akad. Nauk SSSR **175**, 517–520 (1967). (**in Russian**)
11. Ferrando, J.C., Gabriyelyan, S., Kąkol, J.: Bounded sets structure of $C_p(X)$ and quasi-(DF)-spaces. Math. Nachr. **292**, 2602–2618 (2019)
12. Ferrando, J.C., Kąkol, J. :Weak compactness and metrizability of Mackey*-bounded sets in Fréchet spaces. Acta. Math. Hung. **157**, 254–268 (2019)
13. Ferrando, J.C., Kąkol, J.: Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces. J. Convex. Anal. **26**, 1337–1346 (2019)
14. Ferrando, J.C., Kąkol, J., López-Pellicer, M.: A characterization of distinguished Fréchet spaces. Math. Nachr. **279**, 1783–1786 (2006)
15. Ferrando, J.C., Kąkol, J., Saxon, S.A.: The dual of the locally convex space $C_p(X)$. Funct. Approx. Comment. Math. **50**, 389–399 (2014)
16. Ferrando, J.C., Kąkol, J., Saxon, S.A.: Characterizing P-spaces X in terms of $C_p(X)$. J. Convex Anal. **22**, 905–915 (2015)
17. Ferrando, J.C., Kąkol, J., Saxon, S.A.: Examples of nondistinguished function spaces $C_p(X)$. J. Convex. Anal. **26**, 1347–1348 (2019)
18. Ferrando, J.C., Kąkol, J., López-Pellicer, M., Saxon, S.A.: Tightness and distinguished Fréchet spaces. J. Math. Anal. Appl. **324**, 862–881 (2006)
19. Ferrando, J.C., López-Pellicer, M., Sánchez Ruiz, L.M.: Metrizable Barrelled Spaces. Longman, Harlow (1995)
20. Gillman, L., Jerison, M.: Rings of Continuous Functions. Van Nostrand, Princeton (1960)
21. Hancell, R.: Descriptive topology. In: Husek, M., Van Mill, J. (eds.) Recent Progress in General Topology, pp. 257–315. North Holland, Amsterdam (1992)
22. Haydon, R.: Three examples in the theory of spaces of continuous functions. C. R. Acad. Sci. Sér. A **276**, 685–687 (1973)
23. Heinrich, S.: Ultrapowers of locally convex spaces and applications I. Math. Nachr. **118**, 285–315 (1984)
24. Horváth, J.: Topological Vector Spaces and Distributions. Addison-Wesley, Reading (1966)
25. Jarchow, H.: Locally Convex Spaces. B. G. Teubner, Stuttgart (1981)
26. Köthe, G.: Topological Vector Spaces I. Springer, Berlin (1983)
27. Marciszewski, W.: A function space $C_p(X)$ not linearly homeomorphic to $C_p(X) \times \mathbb{R}$. Fundam. Math. **153**, 125–140 (1997)
28. McCoy, R.A., Ntantu, I.: Topological Properties of Spaces of Continuous Functions. Springer, Berlin (1980)
29. Michael, E.: $\aleph_0$-spaces. J. Math. Mech. **15**, 983–1002 (1966)
30. Ohta, H.: Special Spaces. In: Hart, K.P., Nagata, J., Vaughan, J.E. (eds.) Encyclopedia of general topology, pp. 76–79. Elsevier, Amsterdam (2004)
31. Peris, A.: Quasinormable spaces and the problem of topologies of Grothendieck. Ann. Acad. Sci. Fenn. Math. **19**, 167–203 (1994)
32. Pérez Carreras, P., Bonet, J.: Barrelled Locally Convex Spaces, North-Holland Mathematics Studies 131. North-Holland, Amsterdam (1987)
33. Pfeffer, W.F.: On some subspaces of Kelly space. Colloq. Math. **35**, 41–46 (1976)
34. Pospíšil, B.: Sur la puissance d'un espace contenant una partie dense de puissance donnée. Časopis pro pěstování Matematiky a Fysiky **67**, 89–96 (1937)

35. Robertson, W.J., Saxon, S.A., Robertson, A.P.: Barrelled spaces and dense vector subspaces. Bull. Aust. Math. Soc. **37**, 383–388 (1988)
36. Saxon, S.A.: The density character of topological vector spaces. In: Lau, A., Tweddle, I. (eds.) Topological Vector Spaces, Algebras and Related Area, vol. 316, pp. 24–36. Longman, Harlow (1994)
37. Saxon, S.A., Sánchez Ruiz, L.M.: Optimal cardinals for metrizable barrelled spaces. J. Lond. Math. Soc. **51**, 137–147 (1995)
38. Saxon, S.A., Tweddle, I.: The fit and flat components of barrelled spaces. Bull. Aust. Math. Soc. **51**, 521–528 (1995)
39. Soukup, L.: Scattered spaces. In: Hart, K.P., Nagata, J., Vaughan, J.E. (eds.) Encyclopedia of General Topology, pp. 350–353353. Elsevier PRNM, Amsterdam (1992)
40. Taskinen, J.: Examples of non-distinguished Fréchet spaces. Ann. Acad. Sci. Fenn. Math. **14**, 75–88 (1989)
41. Tkachenko, M.G.: The notion of o-tightness and C-embedded subspaces of products. Topol. Appl. **15**, 93–98 (1983)
42. Tkachuk, V.V.: A C_p-theorem Problem Book. Special Features of Function Spaces. Springer, Berlin (2014)
43. Tkachuk, V.V.: A C_p-theorem Problem Book. Compactness in Function Spaces. Springer, Berlin (2015)
44. Velichko, N.V.: Weak topology of spaces of continuous functions. Math. Notes **30**, 849–854 (1981)
45. Zenor, P.: Hereditary m-separability and the hereditary m-Lindelöf property in product spaces and function spaces. Fund. Math. **106**, 175–180 (1980)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.