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学 位 の 種 類	博士（理学）
学 位 記 番 号	理学博　第 41 号
学位授与の日付	令和 5 年 3 月 25 日
課程・論文の別	学位規則第 4 条第 1 項該当
学 位 論 文 題 名	The first variational formulae for integral invariants of degree two of the second fundamental form of a map between pseudo-Riemannian manifolds 擬リーマン多様体間の写像の第二基本形式に関する 2 次の積分不変量の第一変分公式（英文）
論 文 審 査 委 員	主査　教授　酒井　高司 委員　准教授　赤穂　まなぶ 委員　准教授　高津　飛鳥 委員　准教授　前田　瞬（千葉大学）

【論文の内容の要旨】

The theory of harmonic maps and biharmonic maps is one of the important fields in differential geometry. Recall that a smooth map between Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional, which is defined by the integration of the square of the norm of the differentiation of a map. By the first variational formula of the energy, then a map is a harmonic map if and only if its tension field vanishes. As a generalization of harmonic maps, Eells and Lemaire [6] introduced the notion of biharmonic map, which is a critical point of the bienergy functional. The bienergy functional is defined as the integration of the square of the norm of the tension field of a map. Jiang [8] showed that a map is biharmonic map if and only if its bitension filed vanishes. By definition, it is clear that a harmonic map is biharmonic. As a higher order energy functional, r-energy functional [9], ES-r-energy functional [3,6], etc. have been introduced, and various researchers have studied them from the viewpoint of variational problems and submanifolds.

On the other hand, in integral geometry, Howard [7] formulated integral invariants of the second fundamental form of a submanifold in a homogeneous space. In his formulation, there are some notable integral invariants of submanifolds. One is integral invariants in the Chern—Federer kinematic formula. These integral invariants played significant roles in differential geometry. For example, Allendoerfer and Weil [2] used these integral invariants to describe the extended Gauss—Bonnet theorem, and this leads to the development of the theory of characteristic classes. Another notable one is the integral invariant defined from a certain invariant homogeneous polynomial of degree two. This invariant polynomial also appears in the definition of the Willmore—Chen invariant, which is a conformal invariant of submanifolds ([4,5]).

In this thesis, we study variational problems for integral invariants, which are defined as integrations of invariant functions of the second fundamental form, of a smooth map between pseudo-Riemannian manifolds. The most important point of this research is that we consider the variational problem for a family of energy functionals, rather than fixing one energy functional.

In Section 2, we recall fundamental notions of induced bundles and pseudo-Riemannian manifolds. In Section 3, with an idea of integral geometry, we introduce integral invariants of a smooth map between pseudo-Riemannian manifolds by using invariant functions of the second fundamental form of the map. In particular, we focus on the family of integral invariants of a map defined from invariant homogeneous polynomials of degree two. Here the family of integral invariants includes the bienergy functional. In Section 4, we derive the first variational formulae for the two energy functionals that are the generator of the family of integral invariants. By the linearity, then we have the first variational formulae for all integral invariants of degree two. Note that it implies an alternative expression of the Euler—Lagrange equation of the bienergy functional. As mentioned above, from the viewpoint of integral geometry, there are two notable polynomials, called the Chern—Federer polynomial and Willmore—Chen polynomial, in the space of invariant homogeneous polynomials of degree two. In Section 5, we discuss some properties of the Chern—Federer energy functional from the viewpoint of variational problems. The Euler—Lagrange equation of an integral invariant of degree two is a fourth order PDE in general, however, we show that the Euler—Lagrange equation of the Chern—Federer energy functional is reduced to a second order PDE. In addition, we describe a symmetry of the Euler—Lagrange equation of the Chern—Federer energy functional comparing with a symmetry of the Chern—Federer polynomial. In Section 6, we give some examples of Chern—Federer submanifolds in Riemannian space forms. Here, a Chern—Federer submanifold is the image of an isometric immersion which is a Chern—Federer map. As a trivial example, we can see that any isometric immersion of a Ricci-flat manifold into a Euclidean space is a Chern—Federer map.

This thesis includes the content of the paper [1] to be published in Osaka Journal of Mathematics.

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